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# Dynamics and structure of social networks from a systems and control viewpoint: A survey of Roberto Tempo's contributions

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ABSTRACT

In this paper, we provide an overview of recent works on dynamics of social networks and distributed algorithms for their exploration, contributed by Dr. Roberto Tempo (1956–2017) and his colleagues. These works, based on the recent achievements in multi-agent systems theory and distributed randomized algorithms, contribute in bridging the gap between the two classical sciences of Social Network Analysis and Systems and Control. The topics covered by this survey include distributed algorithms for analysis of complex social networks and novel dynamic models, describing opinion formation processes in such networks.



## 1. Introduction

Originating from studies on sociometry [1], *Social Network Analysis* (SNA) has grown into an interdisciplinary theory, being an indispensable part of modern Network Science. SNA employs numerous mathematical tools, coming from graph theory, algorithm theory, probability theory and statistics. Whereas the important role of "cybernetical" (system, control and information-theoretic) methods in social and behavioral sciences has been foreseen by Wiener [2], the fruitful interaction among SNA and systems theory has started quite recently with introducing distributed algorithms for analysis of social networks and novel dynamical models, describing processes over such networks (e.g. evolution of individual opinions, attitudes and beliefs under social influence). The rapidly developing subfield of control theory, studying social networks by system-theoretic methods, is very young and still has no name. One of the pioneers of this new area, Roberto Tempo was very passionate about filling the gap between control and societal studies. Among his last works there were tutorial papers [3,4] making an attempt to systematize the most mature results and achievements of "social systems theory", opening this novel area to a broad interdisciplinary research community. In this survey, we focus on two Tempo's key achievements, namely, development of *distributed algorithms* for network analysis (in particular, computing *centrality measures* [5–7]), works on the Friedkin–Johnsen model of opinion formation [8–10] and recent important extensions of this model [11–14].

## 2. Historical notes

Tempo's interest in social networks was initially motivated by his works on distributed randomized algorithms. After being one of the leading researchers in the area of control systems analysis and design, pursuing the so-called *robust* paradigm (aimed at guaranteeing system performance in all uncertain conditions), in the late eighties Tempo was one of the first to recognize the intrinsic limitations and drawbacks of this approach, and became one of the first scholars to propose the new paradigm of probabilistic robustness. This approach introduced a probabilistic description of the uncertainty, and proposed algorithms for assessing performance *in probability*.

A randomized algorithm is simply "an algorithm where some steps are based on a random choice". In the context of uncertain systems, this choice corresponds to a random selection of the uncertainty. For instance, a system's performance can be guaranteed in a probabilistic way by designing a controller which performs in the desired way for a prescribed number of randomly generated scenarios. The importance of randomized methods quickly become evident, and they became a key tool of probabilistic robust control.

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This research is summarized in [15], which develops and discusses several randomized algorithms for analysis and design of systems affected by deterministic and stochastic uncertainty, with particular emphasis on iterative randomized techniques.

Then, Tempo's interests rapidly extended to other fields where the use of randomization plays a crucial role, in particular those based on the so-called Las Vegas methods (see [16]), and on the implementation of these algorithms in a distributed manner. This motivated the study of networks, and in particular his works on PageRank recalled in Section 3.

Numerous analogies between distributed algorithms and social dynamics, together with his encounter with key players in the field of social networks, were the key drivers of Tempo's work in this latter field, which is recalled in Section 4.

## 3. Distributed algorithms for network analysis

In the last decades, the interest on the study of networks has constantly grown in various disciplines, including computer science, mathematics, physics, engineering science, economics, and social sciences. The notion of network refers to a structure defined by entities and connections among them. These entities can represent people, but may also be groups, organizations, nation states, web sites, or scholarly publications. Mathematically, the complex networks arising in these fields can be represented by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, consisting of a set of nodes $\mathcal{V}$ and a set of connections $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ that reflect the dependency, influence or similarity relations.

In social network analysis, the identification of the most relevant entities in a network is a problem that attracted much attention. The relevance can be defined in several ways according to the specific context and application, leading to different notions of *centrality* (a node's importance) measure.

The simplest definition of centrality is represented by the *degree*, i.e. the number of neighbors of a node $v \in \mathcal{V}$. More precisely, the out-degree of $v \in \mathcal{V}$ is defined by

$$n_v = |\{w \in \mathcal{V} : (v, w) \in \mathcal{E}\}|,$$

where $|\mathcal{X}|$ denotes the cardinality of the set $\mathcal{X}$. In social systems, the degree can provide a measure of the immediate risk of a node of catching or spreading some information. However, since this notion of centrality is a local definition that is independent of the rest of the network, in many cases it is not able to capture the role of this node in connection to the others. The most significant definitions of centrality that involve the entire network are, to mention just a few, *closeness, betweenness*, and *eigenvector centrality*.

In a connected network, the closeness centrality of a node $v \in \mathcal{V}$ is defined as the inverse sum of the distances from a node to all other nodes in the graph. More precisely, one can consider

$$c_v = \frac{1}{\sum_{w \in \mathcal{V} \setminus \{v\}} d_{vw}}$$

where $d_{vw}$ is the distance between node $v \in \mathcal{V}$ and node $w \in \mathcal{V}$, that can be defined as the length of the shortest path between $v$ and $w$, but other distances may also be considered [17,18]. According to a closeness centrality, a node is more central if it is closer to most of the other nodes [19]. In social networks closeness can be interpreted as a measure of how long it will take to spread information from $v$ to all other nodes in the network.

Betweenness, instead, is defined as follows, see e.g. [17,18]

$$b_v = \sum_{j,k \in \mathcal{V}, j \neq k \neq v} \frac{|S_v(j,k)|}{|S(j,k)|}$$

where $S(j, k)$ denotes the set of shortest paths from $j$ to $k$, and $S_v(j, k)$ the set of shortest paths from $j$ to $k$ that contain the node $v$. In other terms, node with a high betweenness can be thought as a bridge between two groups of vertices within the network, since many paths in different groups must pass through this node.

Eigenvector centrality uses the entries of the leading eigenvector $x^\star$ of a suitable weighted adjacency matrix, call it $A$, associated to the network to assign the relevance of the nodes. We thus have

$$\lambda x^\star = A x^\star$$

where $\lambda$ is the leading eigenvalue of $A$. Examples include Bonacich centrality [20,21], Katz centrality [22], and the PageRank centrality, suggested in [23] to rank websites in the Google search engine results, and belonging in fact to a broad class of centrality measures, introduced by Friedkin [5,10].

PageRank can be considered as a natural extension of the degree centrality, taking into account not only the number of nodes that are connected, but, in a certain sense, the *quality* of these connections.

The PageRank performs ranking of the nodes following the idea that connections with high-ranking nodes must contribute more to the ranking of the node than those with low-scoring nodes. This is an important feature in the estimation of the social power in a group, since connections with influential actors will make you more central than connections to non-influential agents.

In Fig. 1a–d the values of different centrality measures are computed and compared for the Karate Club network [24]; even in small-size networks, one notices the evident difference in the node ranking. Tempo's works on centrality measures focused on low-complex, robust and distributed algorithms for their computation. For PageRank computation he also outlined the connections with other problems of interest for the systems and control community, including consensus of multi-agent systems [25] and aggregation-based techniques [26]. Moreover, the contribution in [27] showed the applicability of the PageRank algorithms to ranking control journals, comparing it to the scientometric journal ratings. We begin by quickly reviewing the basic rudiments on PageRank computation. Afterwards, we will summarize the main contributions of Tempo on algorithms for its distributed computation.

### 3.1. PageRank computation

The PageRank, introduced by Brin and Page in [23], is based on the definition of a suitable Markov model describing the random surfing of the world wide web, called *teleportation model*, and is independent of the semantic content or the view count of web pages. We briefly summarize the main notation and the mathematical formulation of the problem.

A network consisting of $n$ web pages [23] is considered and described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the set of vertices correspond to the web pages and the edges are the links between two web pages. More precisely, we draw the edge $(i, j) \in \mathcal{E}$, if page $i$ has an outgoing link to page $j$. The idea is that a surfer begins at a random web page and follows a random walk, by choosing a hyperlink with equal probability. The probabilities of jumping between the web pages can be encoded in the so-called *hyperlink matrix* obtained as follows Let $\mathcal{N}_i = \{h \in \mathcal{V} : (i, h) \in \mathcal{E}\}$ be the set of nodes linked by $i$, and $n_i = |\mathcal{N}_i|$, for each node $i \in \mathcal{V}$, and $A \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ the matrix such that

$$A_{ij} = \begin{cases} 1/n_j & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise.} \end{cases}$$

In order to overcome the problem of *dangling nodes*, i.e. webpages with no outgoing hyperlinks, it is generally assumed that the surfer jumps to some random webpage, chosen at random according to

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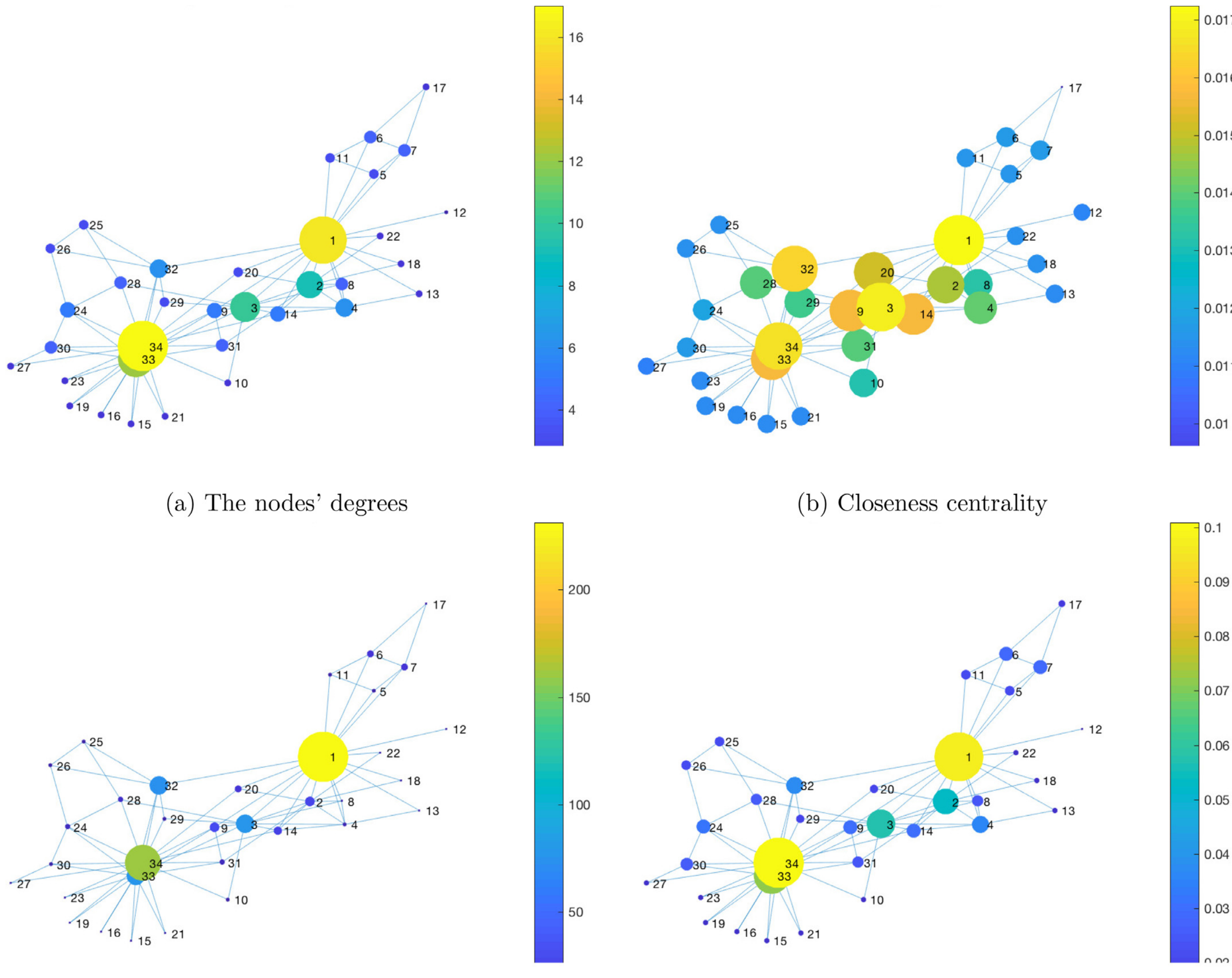

(a) The nodes' degrees (b) Closeness centrality

(c) Betweenness centrality (d) PageRank centrality

**Fig. 1.** Different types of centrality measures for the graph of Karate Club.

a prescribed dangling node distribution [28]. With this modification, the columns of $A$ corresponding to dangling nodes are equal to some known distribution vector. It should be noticed that with this modification the entries of matrix $A$ are nonnegative and $A$ is a stochastic matrix, i.e. its columns sum up to one, $\sum_{i\in\mathcal{V}} A_{ij} = 1$.

The teleportation model considers the possibility that the random surfer may randomly jump from the currently visited webpage to some other webpage, not directly connected to the current one, with a uniform probability. Fixed $m \in (0, 1)$, the transition probability matrix is given by

$$M := (1-m)A + \frac{m}{n}\mathbb{1}_n\mathbb{1}_n^\top \qquad (1)$$

obtained as the a convex combination of the original hyperlink matrix $A$ and the matrix $1/n\mathbb{1}_n\mathbb{1}_n^\top$, where $\mathbb{1}_n$ stands for the $n$-dimensional column vector containing all ones. In the pioneering algorithm devised by Google [23], the parameter of the convex combination was set to $m = 0.15$. As the surfer proceeds in this random walk from node to node, he visits some nodes more often than others; intuitively, these are nodes with many links coming in from other frequently visited nodes. The assumption behind the PageRank computation is that pages that are visited more often in this random walk are more relevant.

Then, the PageRank of graph $\mathcal{G}$ is defined as the vector $x^\star_{\text{pgr}}$ such that

$$Mx^\star_{\text{pgr}} = x^\star_{\text{pgr}}, \qquad \sum_{i=1}^{n} x^\star_{\text{pgr}_i} = 1.$$

Since the matrix $M$ is a primitive matrix (see Appendix A for a formal definition), the theory of Markov chains ensures that the PageRank vector can be computed through the recursion

$$x(k+1) = Mx(k) = (1-m)Ax(k) + \frac{m}{n}\mathbb{1}_n, \qquad (2)$$

with the initial condition satisfying $\mathbb{1}_n^\top x(0) = 1$, (i.e., $x(0)$ stochastic vector).

It should be noticed that that the teleportation model ensures that matrix $(1-m)A$ has eigenvalues inside the open unit circle $\{z \in \mathbb{C} : |z| < 1\}$ (such matrices are also called *Schur stable*), and thus guarantees the convergence of the sequence in (2) to the vector

$$x^\star_{\text{pgr}} = (I - (1-m)A)^{-1}\frac{m}{n}\mathbb{1}_n. \qquad (3)$$

*Example [PageRank computation]* [27]. Let us consider a network consisting of six nodes, as depicted in Fig. 2.

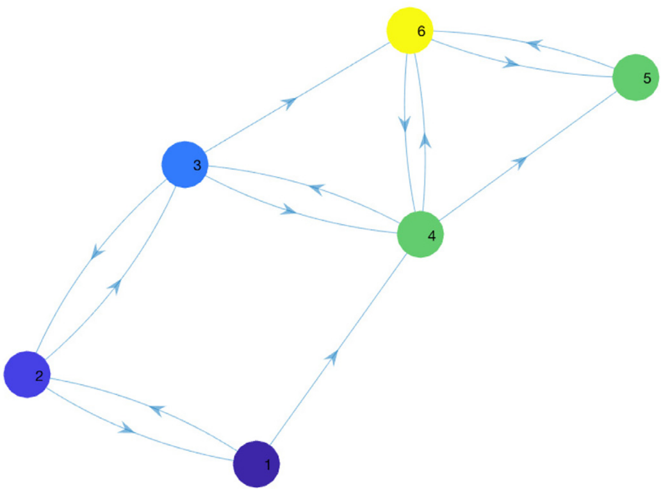


**Fig. 2.** Example of a network of six nodes.

The hyperlink matrix $A$ is given by the following column stochastic matrix

$$A = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/3 & 0 & 1/2 \\ 0 & 0 & 1/3 & 1/3 & 1 & 0 \end{bmatrix}$$

and the corresponding transition matrix for teleportation model is obtained from (1) with $m = 0.15$

$$M = \begin{bmatrix} 0.025 & 0.450 & 0.025 & 0.025 & 0.025 & 0.025 \\ 0.45 & 0.025 & 0.308 & 0.025 & 0.025 & 0.025 \\ 0.025 & 0.45 & 0.025 & 1/3 & 0.025 & 0.025 \\ 0.45 & 0.025 & 0.308 & 0.025 & 0.025 & 0.45 \\ 0.025 & 0.025 & 0.025 & 0.308 & 0.025 & 0.45 \\ 0.025 & 0.025 & 1/3 & 1/3 & 1 & 0.025 \end{bmatrix}.$$

The PageRank vector is given by

$$x^\star_{\text{pgr}} = \big(0.0614, 0.0857, 0.122, 0.214, 0.214, 0.302\big)^\top. \quad (4)$$

As discussed, a classical interpretation of the PageRank is that "a node is important if it is linked by important nodes." This is clear by looking at nodes 2 and 5 in Fig. 2: they both have two incoming nodes, but the PageRank of node 5 is higher than that of node 2, since node 2 has an incoming link from a page with high PageRank (node 6).

The PageRank, as well as the most significant definitions of centrality in social networks, are functions of the full network. Consequently, in order to compute it for a given node, information on all other nodes of the network is necessary, and algorithms require a large number of synchronous operations at each time step. To give an order of the computational effort to compute the PageRank, just think that, today, the world wide web has reached over 1 billion of websites, the Facebook network can boast 2.07 billion monthly active Facebook users with 16 percent increase year over year, and Twitter has 328 million monthly active users. In contrast with classical works focused on centralized algorithms based on the iterations in (2), Tempo was active in looking for distributed and iterative randomized algorithms, involving only local information and possibly avoiding synchronous updates.

### *3.2. Distributed randomized approach*

Distributed randomized methods for PageRank computation have been studied in several papers (see e.g. [27] and references therein) in order to reduce requirements for PageRank computation and to avoid synchronized updates.

In this section, we describe a "gossip" algorithm, originally proposed by Tempo and his coauthors in [29], in which only one edge is randomly selected at each time. More precisely, each node $i \in \mathcal{V}$ holds a pair of states $(x_i, \bar{x}_i)$. At each time $k \in \mathbb{N}$, an edge $\theta(k)$ is selected uniformly at random among all possible edges in $\mathcal{E}$. Then, the algorithm performs the following computations:

$$x(k+1) = (1-r)A_{\theta}(k)x(k) + \frac{r}{n}\mathbb{1}_n, \quad (5)$$

$$\bar{x}_i(k+1) = \frac{kx_i(k) + x_i(k+1)}{k+1} \quad \forall i \in \mathcal{V}, \quad (6)$$

where $r \in (0, 1)$ is a design parameter that replaces the parameter $m = 0.15$ used in the power method, $A_{\theta}(k)$ is a random matrix which depends on the outcome of $\theta(k) = (i, j)$ and is defined as

$$A_{\theta}(k) = I + \frac{1}{n_i}(e_j e_i^\top - e_i e_i^\top)$$

where $n_i$ is the number of links outgoing from node $i$, and $e_i$ denotes the vector with all zeros and a one in the $i$-th entry. More precisely, $A_{\theta}(k)$ is the *distributed hyperlink matrix* which is uniformly distributed over the set of matrices $\{I + \frac{1}{n_i}(e_j e_i^\top - e_i e_i^\top) : (i, j) \in \mathcal{E}\}$. It should be noticed that, since the conditional distribution of the future states are independent of the past values, the sequence of estimations $\{x(k)\}_{k\in\mathbb{N}}$ is a Markov process.

In [29] it is shown that the introduction of the randomization induces persistent oscillations in the estimation of the PageRank computation and the system in (5) and (6) fails to converge in a deterministic sense. However, these oscillations are not chaotic, but in fact appear to be *ergodic*.

**Definition 1.** A random sequence $\{\xi(k)\}_{k\in\mathbb{N}}$ is said to be almost sure *ergodic* if there exists a random variable $\xi^*$ such that almost surely

$$\lim_{k\to\infty} \bar{\xi}(k) = \mathbb{E}[\xi^*], \quad \bar{\xi}(k) = \frac{1}{k}\sum_{j=1}^{k} \xi(j).$$

In other words, an ergodic random process converges *on average* to some steady value, no matter if the averaging is over random argument (expectation) or over the time.

The ergodicity of the process $\{x(k)\}$ has been analytically proved in [29]. Obviously, the expected dynamics evolves through

$$\mathbb{E}[x(k+1)] = (1-r)\left(\left(1 - \frac{1}{|\mathcal{E}|}\right)I + \frac{1}{|\mathcal{E}|}A\right)\mathbb{E}[x(k)] + \frac{r}{n}\mathbb{1}_n,$$

is stable, and, when $r = m/(m - |\mathcal{E}|m + |\mathcal{E}|)$, converges to the PageRank vector

$$\lim_{k\to\infty} \mathbb{E}[x(k)] = x^\star_{\text{pgr}}.$$

The ergodicity is implied by the following theorem. *[29]*

**Theorem 1.** *Let*

$$r = \frac{m}{m - |\mathcal{E}|m + |\mathcal{E}|}$$

*and consider the dynamics (5) and (6) with $x(0)$ stochastic vector. Then, the time-averaged sequence $\{\bar{x}(k)\}_{\mathbb{Z}_+}$ converges to the PageRank vector*

$$\lim_{k\to\infty} \bar{x}(k) = x^\star_{\text{pgr}}$$

*almost surely and in mean square norm, moreover,*

$$\mathbb{E}\Big[\|\bar{x}(k) - x^\star_{\text{pgr}}\|_2^2\Big] \le \frac{C}{k}, \quad (7)$$

*where the constant $C > 0$ depends on the graph and on m.*

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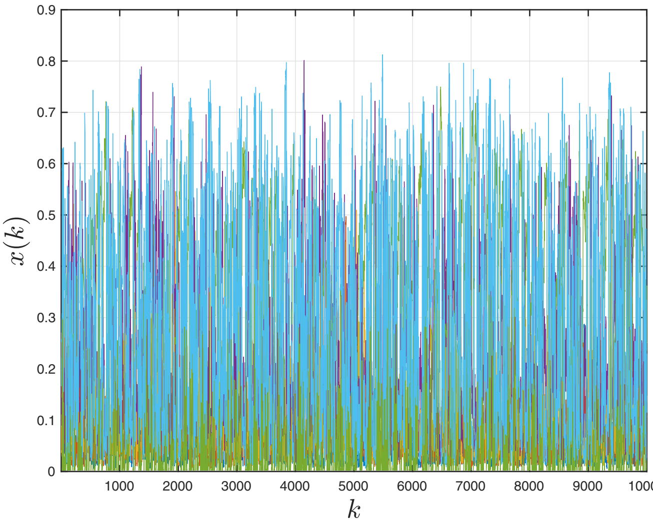


**Fig. 3.** Distributed randomized approach: the sequence of estimations $x(k)$ fails to converge in a deterministic sense.

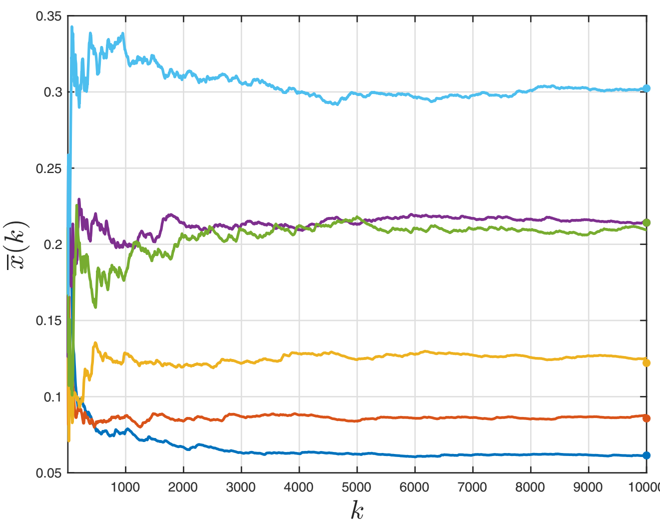


**Fig. 4.** Distributed randomized approach: a sample evolution of $\bar{x}(k)$ time-averages. It should be noticed the convergence to the PageRank values.

The result in (7) points out that the rate of convergence of time-averages is only polynomial, in contrast to the exponential convergence for the centralized computation (2).

Other choices for gossip distributed hyperlink matrices can be adopted. For example, in [30] a node $\theta(k) = v$ is sampled from the uniform distribution of $\mathcal{V}$ and the $A_\theta(k)$ uses only the column $v$ of matrix $A$. Also in that case similar results can be obtained and the algorithm needs a time averaging smoothing procedure to guarantee the convergence to the PageRank vector.

*Example*. Let us consider now the network consisting of six nodes, shown in Fig. 2. In Figs. 3 and 4 the evolution of the sequences $x(k)$ and $\bar{x}(k)$, computed by the randomized algorithm, are shown as a function of time $k$, respectively. It can be noticed that the sequence of estimations $x(k)$ does not converge. However, as to be expected from theory, the time-averages $\bar{x}(k)$ converge to the PageRank vector $x^\star_{\text{pgr}}$ given in (4). As discussed before, the average-based approach discussed above only guarantees a linear rate of convergence. Moreover, knowledge of the network size $n$ is required to implement the algorithm. To overcome these drawbacks, in [31] a different randomized method for PageRank computation has been proposed by Tempo and coworkers. A notable feature of this algorithm is that it does not require to know $n$, which is estimated at every node, and that it is asyncronous (i.e. clock-free). Moreover, the algorithm is proved to converge almost surely and, when the network size is known, with exponential rate.

In particular, the algorithm in [31] reformulates the PageRank problem as the following least-squares problem

$$x^\star_{\text{pgr}} = \arg\min_x \|Hx - y\|^2,$$

with $H := (I - (1-m)A)$ and $y := \frac{m}{n}\mathbb{1}_n$. Clearly, the solution of this optimization problem is exactly the PageRank given in (3). The proposed randomized iterative algorithm selects at each step $k$ a random node, indexed as $\theta(k)$, and incrementally updates the PageRank estimate $x(k)$ via the following *fusion algorithm*

$$x(k+1) = x(k) - \frac{1}{n} H^\top_{\theta(k)} (y - H_{\theta(k)} x)^2 \tag{8}$$

where $H_{\theta(k)}$ is the $\theta(k)$-th row of matrix $H$. Note that this algorithm is distributed by construction, since $H_{\theta(k)}$ only contains information from the neighbors with outgoing links to node $\theta(k)$, which is obviously known to node $\theta(k)$. The key technical feature of the algorithm proposed in [31] is the way the random signal $\theta(k)$ is designed. Indeed, a random walk to update is proposed, based on a mechanism which recalls the random-surfer model. Finally, the work also considers the interesting case of time-varying networks.

### 3.3. A web aggregation approach

In this section, we briefly illustrate a distributed algorithm for PageRank computation based on a web aggregation approach proposed by Tempo and coauthors in [32]. The main goal is to design an algorithm with low complexity and low communication requirements for PageRank computation. The procedure provides an approximation of the values of PageRank with a guaranteed error bound. The algorithm exploits the observation that (a) many links in the networks are intrahost, i.e. many web pages are connected within the same domains, as it happens for websites of several organizations, universities and companies; (b) the network is sparse, in the sense that just few links appear between different cluster servers.

We can summarize the procedure in [32] as a *clustering step*, a *global computation step*, and a *local computation step*. In the clustering step the set of nodes in the network are partitioned into groups in order to obtain an aggregated graph with a reduced number of nodes. It is natural to group the web pages within the same server or domain in a single one. This operation can be performed locally and in a decentralized way. We refer the reader to [33,34] for other techniques for the clustering step that are based on communities detection.

We now define a measure of the quality of the approximation of the PageRank value we are going to obtain. Let $n_v^{\text{out}}$ be the number of external outgoing link of node $v$, i.e. links to pages outside of its own domain or directory and define $\delta_v = n_v^{\text{out}}/n_v$. Now consider all $v \in \mathcal{V}$ such that $\delta_v \le \delta$ where $\delta$ is some small parameter. Nodes with many external links for which $\delta_v > \delta$ are considered as single groups consisting of only one member. The parameter $\delta$ determines the number of nodes in the aggregated graph and, consequently, will influence the accuracy and the efficiency of the algorithm.

Let us denote by $g$ the number of groups and $g_1$ the number of single groups, and with $g_i$ the number of nodes in each cluster. Without loss of generality, we can reorder the entries of $x^\star_{\text{pgr}}$ in such way that the first $g_1$ elements corresponds to pages in group 1, the following $g_2$ to the values of PageRank corresponding to nodes in cluster 2, and so on.

Let now $\widetilde{x}^\star = [(\widetilde{x}_1^\star)^\top, (\widetilde{x}_2^\star)^\top]^\top$ where $x_1^\star \in \mathbb{R}^g$ whose $i$-th entry correspond to the sum of the pages in cluster $i$ and is called the *aggregated PageRank* and $\widetilde{x}_2^\star \in \mathbb{R}^{|\mathcal{V}|}$ is a vector where each entry is the difference between a page value and the average value of the group members. It can be shown that there exists an invertible matrix $V = [V_1^\top, V_2^\top]^\top$ such that

$$\widetilde{x}^\star = V x_{\text{pgr}}^\star$$

or, equivalently,

$$\begin{bmatrix} \widetilde{x}_1^\star \\ \widetilde{x}_2^\star \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} x_{\text{pgr}}^\star.$$

In these new coordinates, using the relation in (2), the PageRank relation can be rewritten as

$$\begin{bmatrix} \widetilde{x}_1^\star \\ \widetilde{x}_2^\star \end{bmatrix} = (1-m)\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}\begin{bmatrix} \widetilde{x}_1^\star \\ \widetilde{x}_2^\star \end{bmatrix} + \frac{m}{n}\begin{bmatrix} V_1 \mathbb{1}_n \\ 0 \end{bmatrix}$$

where the contribution of internal links, i.e. links within the same cluster, appears only in $\widetilde{A}_{21}$ and $\widetilde{A}_{22}$. Moreover, such construction guarantees that $\widetilde{A}_{11}$ is stochastic and the elements in $\widetilde{A}_{12}$ are small in magnitude, see [32] for details. These properties suggest a method to approximate the values of $\widetilde{x}$ through the vector $\widetilde{x}'$ satisfying

$$\begin{bmatrix} \widetilde{x}_1' \\ \widetilde{x}_2' \end{bmatrix} = (1-m)\begin{bmatrix} \widetilde{A}_{11} & 0 \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}\begin{bmatrix} \widetilde{x}_1' \\ \widetilde{x}_2' \end{bmatrix} + \frac{m}{n}\begin{bmatrix} V_1 \mathbb{1}_n \\ 0 \end{bmatrix}$$

Then the approximated PageRank $x'_{\text{pgr}}$ is obtained by the transformation

$$x'_{\text{pgr}} = V^{-1}\widetilde{x}'$$

The error in the approximation is examined in [32] and is related to the level of sparsity $\delta$. Formally, the following result holds.

**Theorem 2.** *[32] For a given tolerance $\epsilon \in (0, 1)$, an aggregation with level of sparsity*

$$\delta \le \frac{m\epsilon}{4(1-m)(1+\epsilon)}$$

*is sufficient to guarantee $\|x_{\text{pgr}}^\star - x'_{\text{pgr}}\|_1 \le \epsilon$.*

This result points out that if the network can be aggregated in a sparse reduced graph (with small $\delta$) then a good approximation of the PageRank can be obtained by the proposed method. Moreover $x'_{\text{pgr}}$ can be computed in a distributed way with a low-order algorithm (see [32] for details of implementation).

*Example* [27]. Let us consider a network consisting of six nodes, depicted in Fig. 2. The set of nodes is partitioned into three groups $\widetilde{V}_1 = \{1, 2\}$, $\widetilde{V}_2 = \{3\}$, and $\widetilde{V}_3 = \{4, 5, 6\}$. It should be noticed that the resulting aggregated graph is sparse in the sense that the number of external links pointing to other groups is small except for node 3 in the original network that is considered as a single group. In fact, we have $\delta_1 = 1/2, \delta_2 = 1/2, \delta_3 = 1, \delta_4 = 1/4, \delta_5 = 0, \delta_6 = 0$. Then the performance measure is given by $\delta = 1/2$. In this case the difference between PageRank vector and the approximation obtained with the Web-aggregated approach is $\|x_{\text{pgr}}^\star - x'_{\text{pgr}}\|_1 \le 0.0188$. □

### 3.4. Distributed computation of other centrality measures

The importance of deriving distributed algorithms for the computation of other centrality measures besides PageRank, such as degree, closeness and betweenness centralities, is also extensively discussed by Tempo and coauthors in [31].

In particular, the paper proposes finite-time convergent algorithms that compute the closeness centrality of a directed graph in a distributed way. These algorithms have been extended to the computation of betweenness centrality of an oriented *tree*, based on the idea of partitioning the network into multi-levels of neighbors. The algorithm takes specific advantage of the fact that a tree does not contain any loop, and therefore every pair of nodes has at most one shortest path.

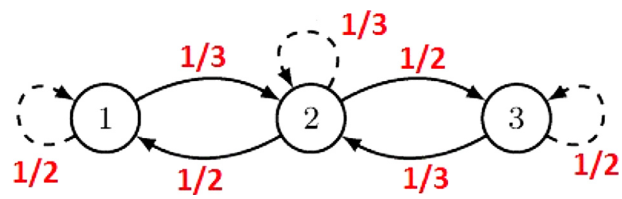


**Fig. 5.** An example of the French model with $n = 3$ agents.

## 4. Evolution of opinions and belief systems

In the literature on social simulation, the term “opinion” stands for an individual’s *cognitive orientation* towards some object [10] (e.g. issue, event, action or another individual), represented by a scalar or vector quantity. Opinions can stand e.g. for signed attitudes [35–37], subjective certainties of belief [13,38] or subjective probabilities [39,40]. Mathematically, an opinion is a scalar or vector variable associated with an individual, which can attain values in a finite set of alternatives (e.g. vote for or against a bill) or a *continuum* (e.g. some interval).

Modeling the evolution of opinion under social influence remains a challenging problem; numerous mathematical models, proposed in recent years, capture only some important traits of dynamics observed in real social groups. For detailed overview of the relevant models, the reader is referred to surveys [3,10,41–43]. In this survey, we consider only one type of models, taking their origin in the seminal work of French [44], Harary [45] and DeGroot [39], and later developed by Friedkin and Johnsen [8]. Tempo and his colleagues have obtained fundamental results, regarding the dynamics of the Friedkin-Johnsen model and its extensions. These results are discussed in the next subsections.

### 4.1. The French–DeGroot model and social power

In his seminal work [44], French studied the following simple model of opinion evolution. Consider a group of individuals (called social actors, or *agents*), being in one-to-one correspondence with nodes of some graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The arc connecting node $i$ to node $j$ corresponds to social influence of individual $i$ on individual $j$ (using terminology from [44], $i$ “has social power over” $j$). This means that opinion of individual $i$ is displayed to individual $j$ and affects his/her opinion at each stage of the opinion iteration.

The process of opinion formation is described by a simple procedure of iterative averaging. At each step $k = 0, 1, 2, \ldots,$ an individual updates his/her opinion to the *mean value* of all opinions, exposed to him/her. For instance, consider a graph of social influence in Fig. 5. Denoting the opinions of individual $i \in \{1, 2, 3\}$ by $x_i(k)$, the vector of opinions evolves in accordance with the equations

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}. \tag{9}$$

As shown in Fig. 5, each of the entries can be treated as a *weight* of the correspondent arc, endowing thus $\mathcal{G}$ with the structure of a weighted (valued) graph.

A more general model of opinion dynamics, proposed by DeGroot [39] under the name of “iterative opinion pooling” (see

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also [46,47]) replaces (9) by the more general equation

$$x(k+1) = Wx(k) \tag{10}$$

$$x_i(k+1) = \sum_j w_{ij} x_j(k) \quad \forall i. \tag{11}$$

where $x(k) \in \mathbb{R}^n$ stands for the vector of opinions and $W = (w_{ij})_{i,j=1}^n$ is a *row-stochastic* matrix (that is, $w_{ij} \geq 0$ and $\sum_j w_{ij} = 1$). The entry $w_{ij}$ may be treated as an *influence weight* individual $i$ assigns to individual $j$. If $w_{ij} = 0$, then the $j$th agent's opinion does not influence agent $i$'s opinion directly (although an indirect influence through a chain of other opinions may exist). Another extreme case is $w_{ij} = 1$ (and thus $w_{ik} = 0 \, \forall k \neq j$), where agent $i$ fully relies on the opinion of agent $j$ in the sense that $x_i(k+1) = x_j(k)$. If $w_{ii} = 1$, agent $i$ is said to be *stubborn* since its opinion remains unchanged $x_i(k) \equiv x_i(0)$, being unaffected by the others' opinions. Obviously, the French's model is a special case of DeGroot's model (10), where each agent *uniformly* distributes influence between itself and its neighbors in the influence graph (equivalently, in each row of $W$ all non-zero entries are equal).

DeGroot's opinion pooling model (10) has been proposed as an algorithm for reaching an *agreement* (consensus) in expert community; the main concern of the work [39] was the convergence of the opinions to a common value independent of the initial condition

$$\lim_{k\to\infty} x_1(k) = \ldots = \lim_{k\to\infty} x_n(k) \quad \forall x(0).$$

It can be shown that consensus in this sense is established if and only if the sequence $W^k$ converges to the rank one matrix with identical rows

$$W^k \xrightarrow[k\to\infty]{} \mathbb{1}_n p_\infty^\top. \tag{12}$$

Since $W$ is stochastic, the vector $p_\infty$ is nonnegative with $p_\infty^\top \mathbb{1}_n = 1$. The equality (12) also implies that $p_\infty$ is a left eigenvector of $W$ since

$$\mathbb{1}_n p_\infty^\top W \overset{(12)}{=} \lim_{k\to\infty} W^{k+1} = \lim_{k\to\infty} W^k \overset{(12)}{=} \mathbb{1}_n p_\infty^\top.$$

Matrices satisfying the consensus condition (12) are said to be *fully regular* [48] or SIA (stochastic indecomposable aperiodic) [49]; such matrices have been long studied in matrix analysis and the Markov chain theory [48,50,51]. Considering $W$ as a matrix of transient probabilities in a Markov chain with $n$ states, the vector of probability distribution $p(k) \in \mathbb{R}^n$ obeys the equation "dual" to (10)

$$p(k+1)^\top = p(k)^\top W \iff p(k)^\top = p(0)^\top W^k \, \forall k. \tag{13}$$

Condition (12) implies that the Markov chain is ergodic, "forgetting" its initial distribution and converging to the stationary distribution $p_\infty$.

A necessary and sufficient spectral criterion for the full regularity of $W$ (that is, consensus in (10)) is as follows [48]: the matrix $W$ has no eigenvalues on the complex unit circle $\{z \in \mathbb{C} : |z| = 1\}$ except for $z = 1$, and the eigenvalue $z = 1$ is simple. A sufficient condition for this is the matrix's *primitivity* (see Appendix A, that is, positivity of all entries of $W^k$ for $k$ being large [39] (primitivity is equivalent to aperiodicity and irreducibility of $W$, see [48]). Necessary and sufficient conditions for consensus are surveyed in [3]. In the special case of French's model with interaction graph $\mathcal{G}$, these conditions boil down to the existence of a "root" node in the graph, from which all other nodes are accessible by walks (this condition is known also as quasi-strong connectivity of the graph and is equivalent to the existence of a directed spanning tree); the relevant criteria were obtained by Harary [45,52].

It should be noticed that consensus phenomenon was not the main concern of the original work [44], focused on the phenomenon of *social power*. The probability distribution $p_\infty$ from (12) has the following relation with the consensus opinion of the group. In accordance with (12), one has

$$x(k+1) = W^k x(0) \xrightarrow[k\to\infty]{} (p_\infty^\top x(0)) \mathbb{1}_n. \tag{14}$$

Hence the element $p_{\infty i}$ can thus be treated as the weight of its initial opinion $x_i(0)$ in the final opinion of the group, measuring the individual's *social power*. The greater this weight is, the more influential is the $i$th individual's opinion. A more detailed discussion of social power and social influence mechanism is provided in [44,53]. The social power may be considered as a *centrality measure* on the nodes of the interaction graph, similar in flavor to the Bonacich or *eigenvector centrality* discussed in Section 3. Usually centrality measures are introduced as functions of the graph topology [54] while their relations to dynamical processes over graphs are not well studied. French's model of social power disclose a *dynamic* mechanism of centrality measure and gives an efficient way to compute it.

*Example*. Consider the French model with $n = 3$ agents (9), corresponding to the graph in Fig. 5. One can expect that the "central" node 2 corresponds to the most influential agent in the group. This is confirmed by a straightforward computation: solving the system of equations $p_\infty^\top = p_\infty^\top W$ and $p_\infty^\top \mathbb{1}_3 = 1$, one obtains the vector of social powers $p_\infty^\top = (\frac{2}{7}, \frac{3}{7}, \frac{2}{7})$.

### 4.2. The Friedkin–Johnsen model and PageRank

Whereas the French–DeGroot model naturally describes consensus of opinions in social groups, the behavior of real communities is more complex. Instead of reaching consensus, individual behaviors may exhibit bimodal polarization or other forms of clustering. This motivated Abelson [35,55] to formulate the following problem, known nowadays as the problem of *community cleavage* [10] or Abelson's *diversity puzzle* [56]: to disclose the reasons, preventing social actors to reach consensus. The informal formulation of the diversity puzzle given in [35] was as follows [35]: "Since universal ultimate agreement is an ubiquitous outcome of a very broad class of mathematical models, we are naturally led to inquire what on earth one must assume in order to generate the bimodal outcome of community cleavage studies".

It may seem paradoxical that mathematical models explaining the irregular clustering behavior of opinions have appeared later than models of consensus. Tempo and his colleagues have contributed to the development of one important class of opinion cleavage models, originating from works of Friedkin and Johnsen [8,9,57].

As has been already mentioned, the French–DeGroot model (10) admits the presence of *stubborn* individuals with maximal self-confidence $w_{ii} = 1$, closed to the interpersonal influence and retaining constant opinions $x_i(k) \equiv x_i(0)$. Obviously, in presence of more than one stubborn individual the community usually fails to reach consensus (unless the opinions of all stubborn individuals coincide). The Friedkin–Johnsen model (henceforth referred to as the FJ model) is a natural extension of the French-DeGroot model with stubborn individuals, allowing some individuals to be "partially" stubborn, being "anchored" in some sense at the initial opinion and still susceptible to social influence.

Similar to DeGroot's dynamics (10), the FJ model employs a stochastic matrix of influence weights $W$. Along with $W$, a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is introduced, where $0 \leq \lambda_i \leq 1$ describes the *susceptibility* of agent $i$ to the social influence. The vector of the agents' opinions evolves in accordance with

$$x(k+1) = \Lambda W x(k) + (I - \Lambda)x(0)$$

$$x_i(k+1) = \lambda_i \sum_{j=1}^{n} w_{ij} x_j(k) + (1 - \lambda_i)x_i(0). \tag{15}$$

An individual who is not susceptible to social influence ($\lambda_i = 0$) is stubborn and retains its opinion unchanged $x_i(k) \equiv x_i(0)$. A maximally susceptible agent ($\lambda_i = 1$) assimilates the others' opinions in accordance with DeGroot's rule (11). When all agents are maximally susceptible $\Lambda = I$, the FJ model (15) turns into the French–DeGroot model (10). An individual with $0 < \lambda_i < 1$ participates in the process of opinion pooling, however, factors his/her initial opinion into any opinion iteration. Such an "anchorage" of some agents at their initial opinions is explained by ongoing effect of external factors that had influenced the social group before the process of opinion formation started [8]; the vector of initial opinions stores the group's memory on these external factors. Without loss of generality, one may assume that $\lambda_{ii} = 0$ whenever $w_{ii} = 1$ (using induction on $k$, it can be easily shown that $w_{ii} = 1$ implies stubbornness $x_i(k) \equiv x_i(0)$ independent of the choice of $\lambda_{ii}$). It is often assumed [8], furthermore, that the coupling condition $\lambda_{ii} = 1 - w_{ii}\,\forall i$ holds, in other words, the more self-confident individual is, the less is his/her susceptibility to the others' opinions.

*Example* [3]. Compare the behavior of opinions in the FJ model (15) with $n = 4$ agents and the following matrix of influence weights [8]

$$W = \begin{bmatrix} 0.220 & 0.120 & 0.360 & 0.300 \\ 0.147 & 0.215 & 0.344 & 0.294 \\ 0 & 0 & 1 & 0 \\ 0.090 & 0.178 & 0.446 & 0.286 \end{bmatrix}. \tag{16}$$

We put $x(0) = [-1, -0.2, 0.6, 1]^\top$ and consider the evolution of opinions for three different matrices $\Lambda$ (Fig. 6): $\Lambda = I$, $\Lambda = I - \mathrm{diag}(W)$ and $\Lambda = \mathrm{diag}(1, 0, 0, 1)$. In all cases agent 3 is stubborn. In the first case the model (15) reduces to the French-DeGroot model, and the opinions reach consensus (Fig. 6a). In the second case (Fig. 6b) agents 1,2,4 move their opinions towards the stubborn agent 3's opinion, however, the visible disagreement of their opinions is observed. In the third case (Fig. 6c) agents 2 and 3 are stubborn, and the remaining agents 1 and 4 converge to different yet very close opinions, lying between the opinions of the stubborn agents.

Whereas models of opinion formation have been proposed in abundance, the FJ model is among a few models that have been validated experimentally [8,9,58,59]. In Tempo's works, some important mathematical properties of the model have been established, concerned with *convergence* of the opinion vector. In experiments with real social groups, it has been observed that individual's opinions usually stabilize at some steady values [8]. These final opinions are much easier to measure than the intermediate opinion trajectory. A natural question thus arises which conditions ensure the existence of

$$x(\infty) = \lim_{k \to \infty} x(k)$$

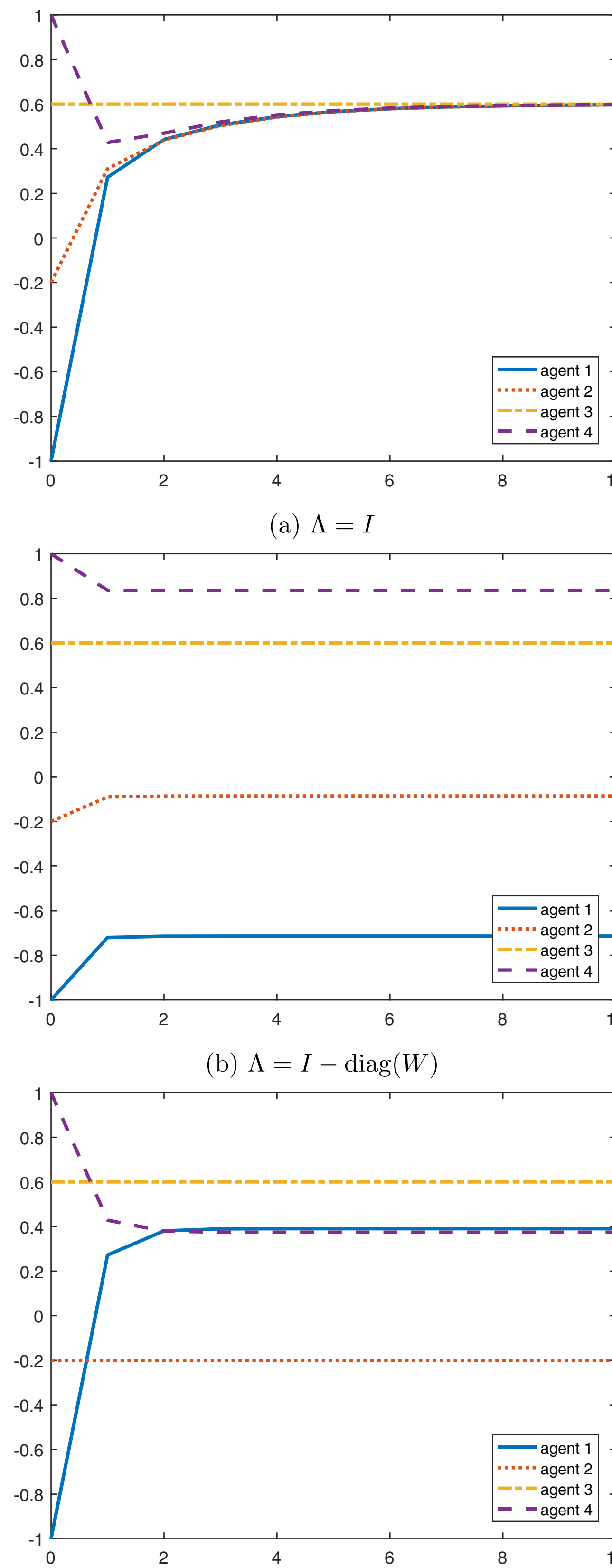
under any initial condition $x(0)$ and how "generic" is this convergence property of the FJ model.

Denoting $u = x(0)$, the equation (15) can be considered as a dynamical system with static input

$$x(k+1) = \Lambda W x(k) + (I - \Lambda)u. \tag{17}$$

It can be easily shown that the vector of opinions converges if system (17) is asymptotically stable, i.e. matrix $\Lambda W$ is Schur stable. In such a situation, the steady opinion of the group can be easily found

$$x(\infty) = Vu, \quad V = (I - \Lambda W)^{-1}(I - \Lambda). \tag{18}$$




(a) $\Lambda = I$

(b) $\Lambda = I - \mathrm{diag}(W)$

(c) $\Lambda = \mathrm{diag}(1, 0, 0, 1)$

**Fig. 6.** Opinion dynamics for $W$ from (16) and different $\Lambda$.

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It can be shown [3,10] that $V$ is a stochastic matrix. Although for a small-size group the stability can be tested in a straightforward way (by e.g. computing the spectrum of the matrix $\Lambda W$), for large-scale social networks such a procedure can be troublesome. Also, it is interesting to understand the role of each matrix $\Lambda$, $W$ in the stability criterion. In Tempo's works, the following convenient test for stability has been obtained.

We say that a matrix $A = (a_{ij})_{i,j\in\mathcal{V}}$ is *adapted* to the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if the graph's arcs and non-zero entries of $W$ are in one-to-one correspondence:

$$(j, i) \in \mathcal{E} \iff a_{ij} \neq 0.$$

Obviously, the influence matrix $W = (w_{ij})_{i,j=1}^n$ is adapted to the only graph with the set of nodes $\mathcal{V} = \{1, 2, \ldots, n\}$, henceforth denoted $\mathcal{G}[W]$. The arc $j\mapsto i$ in this graph corresponds to the influence of agent $j$ on agent $i$ (since $w_{ij} > 0$, agent $i$ allocates a positive influence weight to $j$). In other words, the directions of arcs have the same meaning as in the original French's model [44]. A directed walk $j\mapsto j_1\mapsto\cdots\mapsto j_k\mapsto i$ corresponds to an *indirect* influence of agent $j$ on agent $i$.

**Theorem 3.** *The matrix $\Lambda W$ is Schur stable if and only if the subset of nodes $\mathcal{V}_0 = \{j : \lambda_{jj} < 1\}$ is connected to all other nodes in $\mathcal{G}[W]$ by directed walks. In other words, any individual is either partially stubborn (has $\lambda_{ii} < 1$) or influenced (directly or indirectly) by some partially stubborn individual.*

Theorem 3 has been proved[2] in [14], the sufficiency part under the coupling condition $\lambda_{ii} = 1 - w_{ii}$ has been first proved in [11].

The convergence of opinions, in general, does not require stability. For instance, as it has been discussed, the French–DeGroot model (10) ($\Lambda = I_n$) can converge (and even reach consensus), whereas the corresponding matrix $\Lambda W = W$ has eigenvalue $z = 1$ (obviously, $W\mathbb{1}_n = \mathbb{1}_n$). In fact, any unstable FJ model contains the French–DeGroot model in the following sense.

**Theorem 4.** *[14] Renumbering the agents, any unstable FJ model (15) can be decomposed as follows*

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \Lambda^{11}W^{11} & \Lambda^{11}W^{12} \\ 0 & W^{22} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} I-\Lambda^{11} \\ 0 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

*where the matrix $W^{22}$ is stochastic and $\Lambda^{11}W^{11}$ is Schur stable. The following conditions are equivalent:*

1. *for any initial condition $x(0)$, the vector of opinions has a limit;*
2. *there exists the limit $\lim\limits_{k\to\infty} W_{22}^k$;*
3. *there exists the limit $\lim\limits_{k\to\infty} (\Lambda W)^k$.*

It should be noticed that the equivalence of 3) and 1) has been stated in Friedkin's work [10], however, the first complete proof has been published in [14]. The group of agents corresponding to the subvector $x_2(k)$ is "closed" in the sense that the opinions of the remaining individuals do not affect its behavior. The vector of initial opinions also does not *directly* influence the behavior of the opinion vector $x_2(k)$; in this sense, individuals "forget" the external factors that influenced the group before the start of opinion formation process; for this reason in [14] such agents have been called "oblivious".

The concept of social power, introduced for the French–DeGroot model, can be extended to the model (15). The corresponding centrality measure appears to be a natural extension of PageRank, discussed in the previous sections. Consider an asymptotically stable model (15), whose steady opinion vector is given by (18) ($u = x(0)$). Recall that the definition of French's social power assumed consensus of opinions $x_1(\infty) = \cdots = x_n(\infty)$; the social power of agent $i$ is defined as the weight of its initial opinion $x_i(0)$ in this final opinion of the group. In his work [5], Friedkin proposed a generalization of social power of agent $i$ as the *mean weight* of its initial opinion in determining group members' final opinions [10]. Mathematically, these mean influence weights are elements of the non-negative vector $c = n^{-1}V^\top\mathbb{1}_n$, which satisfies the following equality

$$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i(\infty) = \frac{1}{n}\mathbb{1}_n^\top Vx(0) = c^\top x(0). \tag{19}$$

Following [5,10], we call $c_i$ the *influence centrality* of agent $i$. Since $V$ is stochastic, $c^\top\mathbb{1}_n = 1$. Similar to French's social power, the influence centrality is generated by an opinion formation mechanism, relating thus the properties of the influence graph and special dynamics over this graph. Suppose now that $W = A^\top$, where $A$ is the hyperlink matrix from Section 3.1, and $\Lambda = (1-m)I_n$. Then

$$c = n^{-1}V^\top\mathbb{1}_n = (I-(1-m)A)^{-1}\frac{m}{n}\mathbb{1}_n,$$

that is, $c$ coincides with the PageRank vector. In this sense, the classical PageRank appears to be a *special case* of the more general Friedkin's centrality. Further relations between the PageRank and FJ model can be found in [3,60,61]. It can also be shown [3] that as $m \to 1$, the PageRank converges to the French's social power (provided that it exists).

### 4.3. From opinions to belief systems

Both French–DeGroot (10) and FJ (15) models can be extended to multidimensional opinions, that are conveniently represented by *row vectors* $x_i = (x_{i1}, \ldots, x_{im})$. Here $m \geq 1$ stands for the number of topics (issues) the agents discuss. Stacking these row vectors on top of each other, one obtains the $n \times m$ opinion matrix $X(k) = (x_{ij}(k))$. The multidimensional extensions of (10), (15) are, respectively,

$$X(k+1) = WX(k) \tag{20}$$

$$X(k+1) = \Lambda WX(k) + (I-\Lambda)X(0). \tag{21}$$

These multidimensional extensions inherit all properties of the original scalar-valued models, discussed in the previous subsection. A closer look at them shows that in fact the dynamics of opinions matrices $X(k)$ can be decomposed into $m$ scalar French-DeGroot or FJ models, describing the evolution of different dimensions of the opinion, i.e. the column vectors $x_{*s} = (x_{1s}, \ldots, x_{ns})^\top$, $s = 1, \ldots, m$. Such a model is natural if the multidimensional opinion represents an individual's positions on several topics that are logically *independent*, and hence their dynamics can be decoupled.

The situation where agents communicate on several *mutually dependent* (entangled) topics is much more challenging. Mathematical modeling of opinion formation with interdependent topics is in its infancy, and only a few simplified models have been proposed to describe them. Among the pioneering works in this direction were the papers [13,14], co-authored by Tempo, where an extension of the FJ model with multiple interdependent topics has been proposed. Alternative models have been advocated in [62] (dealing with an extension of the Deffuant–Weisbuch model) and [63] (dealing with a novel "energy-based" approach to opinion dynamics modeling).

A multidimensional opinion, whose scalar elements represent an individual's positions on several interrelated issues, can be used to describe a *belief system* [13]. The notion of a belief system has been introduced by Converse who defined it as a "configuration

[2] Note that partially stubborn agents in [14] are called stubborn, and stubborn in our sense are said to be totally stubborn.

of ideas and attitudes in which the elements are bound together by some form of constraint or functional interdependence" [64]. The existence of logical dependencies imply that change in an attitude to some issue implies changes in the attitudes to the dependent issues in order to avoid tensions, caused by logical inconsistences. The within-individual (introspective) tension-resolving process, studied in cognitive dissonance and cognitive consistency theory [65,66], is believed to be an automatic process of the human brain and maintains a coherent system of attitudes and beliefs.

Modeling of logical dependencies between beliefs and tension-resolving introspective processes is a challenging problem, and the relevant mathematical models can hardly be simple. In the works [13,14] a simplified two-stage opinion formation model has been advocated, where the convex mechanism of iterative opinion pooling (being the base for the French-DeGroot and FJ models) is combined with the additional "integration" process, mimicking the real tension-resolving process and introducing coupling (entanglement) between the individual beliefs. The opinions considered as *row* vectors $x_{i*} = (x_{i1}, \dots, x_{im})$ evolve as follows

$$\begin{aligned} y_{i*}(k+1) &= \sum_{j=1}^{n} w_{ij} x_{j*}(k) \\ x_{i*}(k+1) &= \lambda_{ii} y_{i*}(k+1) C^\top + (1-\lambda_{ii}) x_{i*}(0). \end{aligned} \tag{22}$$

The intermediate opinion vector $y_{i*}(k+1)$ resulting from averaging the displayed opinions; the updated opinion of individual $i$ is based on the modified vector $y_{i*}C^\top$ (being the results of the "integration" process) and, as in the usual FJ model, the initial opinion $x_i(0)$. The "integration" part of the opinion evolution is determined by an $m \times m$ matrix $C$, referred in [14] to as the matrix of *multi-issues dependence structure* (MiDS). The MiDS matrix need not be stochastic, nor even nonnegative (its negative entries may stand for "repulsion" between the topics), however, usually the opinions vary in some predefined interval (e.g. certainties of belief, or subjective probabilities, stay in [0,1]), and the invariance of this interval imposes a restriction on the choice of $C$.

The dynamics (22) can be rewritten as follows

$$x_{i*}(k+1) = \lambda_{ii} \sum_{j=1}^{n} w_{ij} \left( x_{j*}(k) C^\top \right) + (1-\lambda_{ii}) x_{i*}(0), \tag{23}$$

in other words, one may equally suppose that the "integration" step precedes the exchange and averaging of opinions rather than follows them. The convenient matrix-form representation of (22),(23) is [13]

$$X(k+1) = \Lambda W X(k) C^\top + (I-\Lambda) X(0), \tag{24}$$

where $X(k)$ is the $n \times m$ matrix, composed of $n$ row opinion vectors similar to the models (20) and (21).

*Example.* The following "academic" example from [14] demonstrates that the presence of MiDS matrix, introducing even a "weak" coupling among the topic, can substantially change the dynamics of the opinion vector. Consider first two mutually dependent topics, e.g. the attitude to fish as a part of diet and the attitude to salmon, measured from −100% (maximally negative attitude) to +100% (maximally positive attitude). As discussed in [14], "If the influence process changes individuals' attitudes toward fish, say promoting fish as a healthy part of a diet, then the door is opened for influences on salmon as a part of this diet. If, on the other hand, the influence process changes individuals' attitudes against fish, say warning that fish are now contaminated by toxic chemicals, then the door is closed for influences on salmon as part of this diet." Consider a group of $n = 4$ individuals with the matrix of influence weights (16) and $\lambda_{ii} = 1 - w_{ii}$. Suppose that the initial matrix of attitudes is given by

$$X(0) = \begin{bmatrix} 25 & 25 \\ 25 & 15 \\ 75 & -50 \\ 85 & 5 \end{bmatrix}.$$

Agents 1 and 2 have modest positive liking for fish and salmon; the third (totally stubborn) agent has a strong liking for fish, but dislikes salmon; the agent 4 has a strong liking for fish and a weak positive liking for salmon. Assuming the independence between two topics (the matrix $C$ is trivial $C_1 = I_2$, the final matrix of opinions [14] is as follows

$$X_1(\infty) = \begin{bmatrix} 60 & -19.3 \\ 60 & -21.5 \\ 75 & -50.0 \\ 75 & -23.2 \end{bmatrix},$$

in other words, under the influence of the stubborn individual (agent 3) the group's attitude to salmon becomes negative, whereas the attitude to fish as a part of diet is quite positive. Suppose now that the MiDS matrix, describing influence between two topics, is the following

$$C = C_2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}.$$

Then the matrix of final attitudes is

$$X_2(\infty) = \begin{bmatrix} 39.2 & 12 \\ 39 & 10.1 \\ 75 & -50.0 \\ 56 & 5.3 \end{bmatrix}.$$

A moderate dependence between two topics dramatically changes the outcome of opinion formation process. All agents, except for the stubborn individual 3, retain moderately positive attitudes to salmon as a part of diet, at the same time, their attutides to fish in general become much less positive than in the case of independent topics. The trajectories of opinions are illustrated in Fig. 7.

A more interesting example has been considered in [13] and studies the 1992–2003 fluctuations of the U.S. population's certainties of belief on truth statements involved in the decision to invade Iraq. The corresponding statements, presented in Colin Powell's speech to UN Security Council in 2003, are as follows: (A) Saddam Hussein has a stockpile of weapons of mass destruction; (B) These weapons of mass destruction are real and present dangers to the region and to the world; (C) A preemptive invasion of Iraq would be a just war. It was conjectured in [13] that Powell's speech presented a logic structure, described by the rank 1 matrix

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

in which high certainty of belief on statement 1 implies high certainty of belief on statements 2 and 3 and, conversely, the low certainty of belief on statement 1 implies low levels of belief on statements 2 and 3. The model (21) with $\Lambda = I_n$ (all individuals are open-minded), $C$ just introduced and a random large-scale matrix $W$, simulated in [13], explains the strong majority support of the preemptive invasion in the immediate aftermath of Colin Powell's speech and strong majority of the public believed that the Iraq War was based on incorrect assumptions upon the failure to find the weapons of mass destruction, witnessed by the actual opinion polls. We do not include the corresponding results of simulation and refer the interested readers to [13].

Although the dynamics of the model (24) may seem very complicated, its stability conditions remain the same for any MiDS matrix with the spectral radius $\rho(C) \le 1$ (this holds, in particular, when $C$ is stochastic or substochastic).

**Theorem 5.** *[14] Assume that $\Lambda W$ is a Schur stable matrix (that is, the usual FJ model(15) is asymptotically stable) and $\rho(C) \le 1$. Then the model (24) is also asymptotically stable. The same holds in a more general situation where $\rho(\Lambda W)\rho(C) < 1$. The matrix of steady opinions $X(\infty)$ is the unique solution to*

$$X(\infty) = \Lambda W X(\infty) C^\top + (I - \Lambda) X(0).$$

Introducing the *row vectorization* of the matrix $X$, that is, the column ($nm$)-dimensional vector

$$\tilde{x} = (x_{11}, \ldots, x_{1m}, x_{21}, \ldots, x_{2m}, \ldots, x_{n1}, \ldots, x_{nm})^\top,$$

the model (24) can be rewritten as follows [14]

$$\tilde{x}(k+1) = (\Lambda W \otimes C)\tilde{x}(k) + ((I - \Lambda) \otimes I_m)\tilde{x}(0),$$

and the vector of final opinions can be found explicitly

$$\tilde{x}(\infty) = (I_{nm} - \Lambda W \otimes C)^{-1}((I - \Lambda) \otimes I_m)\tilde{x}(0). \tag{25}$$

Here $A \otimes B$ denotes the *Kronecker* (or tensor) product of two matrices $A$, $B$, see Appendix A for a definition.

### 4.4. Gossip-based asynchronous interactions

A visible disadvantage of the FJ model is the assumption on synchronous interactions among the agents, being too simplistic even for small groups interacting face-to-face. This was clearly realized by Friedkin and Johsen, admitting that “interpersonal influences do not occur in the simultaneous way and there are complex sequences of interpersonal influences in a group” [8]. In other words, real social groups are featured by asynchronous “ad hoc” interactions among individuals. At the same time, extensive experiments with real social group [8,58,59] have demonstrated the predictive power of the FJ model (15) and revealed the strong correspondence between the predicted final opinion vector (18) and actual outcome of the opinion formation process. A natural question thus arises: can a modification of the FJ model be designed that converges to the same steady opinion vector (18) yet allows *asynchronous* interactions among the agents?

In the works of Tempo and his colleagues [11,14,29,67], gossip-based asynchronous counterparts of the FJ model and its multidimensional extension have been proposed that converge *on average* to the same vectors of final opinions as the original models. The gossiping interactions lead to randomness in the model: at each stage of the opinion evolution, a pair of individuals (corresponding to one of nodes in the interaction graph) meets and discuss some issues, upon which one of the opinions is modified. Whereas the idea of gossip-based communication is not very new (see e.g. [68,69]), the effects of gossiping in social systems have not been well studied.[3]

Here we consider the most general gossip-based counterpart of a *stable* model (15), introduced in [14]. Let $\mathcal{G}[W] = (\mathcal{V}, \mathcal{E})$ be the influence graph, corresponding to the matrix $W$, $\Gamma^1 = \Lambda W$ and $\Gamma^2 = (I - \Lambda)W$. We use $\gamma_{ij}^1, \gamma_{ij}^2$ to denote the entries of the matrices $\Gamma^1$, $\Gamma^2$. Similarly to gossip algorithms for PageRank computation, at each step an arc is randomly sampled with the uniform distribution from the interaction graph $\mathcal{G}[W]$. If this arc is ($i$, $j$), then the $i$th agent updates its opinion in accordance with

$$x_i(k+1) = (1 - \gamma_{ij}^1 - \gamma_{ij}^2)x_i(k) + \gamma_{ij}^1 x_j(k) + \gamma_{ij}^2 x_i(0). \tag{26}$$

[3] Besides the models from [11,14,29,67], only a few models of opinion formation admit gossip-based interactions, the most known of them is the Deffuant–Weisbuch model [62,70–72] and novel models of opinion evolution over signed networks [73,74].

The opinions of other individuals (including agent $j$) remain unchanged

$$x_l(k+1) = x_l(k) \quad \forall l \ne i. \tag{27}$$

In some sense, the entries $\gamma_{ij}^2$ determine the level of “anchorage” at the initial opinion; notice however that, unlike the FJ model (15), these coefficient depend on both $\Lambda$ and $W$. The following theorem proved in [14], shows that the gossip-based algorithm (26),(27) converges *on average* (similarly to the PageRank algorithm, values $x_i(k)$ do not converge and typically exhibit oscillatory behavior).

**Theorem 6.** *Assume that $\rho(\Lambda W) < 1$. Let $\Gamma^1 = \Lambda W$ and $\Gamma^2 = (I - \Lambda)W$. Then, the limit $x_* = \lim\limits_{k\to\infty} \mathcal{E}x(k)$ exists and equals to the final opinion (18). Furthermore, the random process $x(k)$ is almost sure and $L^p$ ergodic, i.e. $\bar{x}(k) \to x_*$ with probability 1 and $\mathcal{E}\|\bar{x}(k) - x_*\|^p \xrightarrow[k\to\infty]{} 0$, where*

$$\bar{x}(k) := \frac{1}{k+1}\sum_{l=0}^{k} x(l).$$

*The aforementioned statements remain valid, replacing $\Gamma^2 = (I - \Lambda)W$ by any matrix, such that $0 \le \gamma_{ij}^2 \le 1 - \gamma_{ij}^1$, $\sum_{j=1}^n \gamma_{ij}^2 = 1 - \lambda_{ii}$ and $\gamma_{ij}^2 = 0$ as $(i, j) \notin \mathcal{E}$.*

A similar gossip-based counterpart exists for the model (24), where for the randomly chosen arc ($i$, $j$) one has

$$\begin{aligned} x_{i*}(k+1) &= (1 - \gamma_{ij}^1 - \gamma_{ij}^2)x_{i*}(k) + \gamma_{ij}^1 x_{j*}(k)C^\top + +\gamma_{ij}^2 x_{i*}(0), \\ x_{l*}(k+1) &= x_{l*}(k) \quad \forall l \ne i. \end{aligned} \tag{28}$$

Theorem 6 remains valid for the model (28) (the corresponding multidimensional averaged opinions $\bar{x}_{i*}$ converge to the rows of final opinion matrix $X(\infty)$, or, equivalently, subvectors of the vector (25)).

Fig. 8 illustrates the behavior of the gossip-based counterpart of the model from Fig. 7. As one notices, the convergence of averages is rather slow, whereas the random opinions themselves do not converge, exhibiting oscillatory behavior.

### 4.5. Identification of social dynamics

Motivated by recent breakthroughs in systems and control approaches to analyze how individuals’ opinions evolve and reach consensus/disagreement, Tempo and his coauthors recognized that a crucial point in the analysis of social systems is to perform model identification and validation.

Although the identification of structures and dynamics in social networks using experimental data is a very active area in statistics, physics and signal processing [75–77], this problem has received relatively less attention in the control community so far. The main challenge consists in setting theoretical foundations for the estimation of the parameters using large-scale data, e.g. $W$, $\Lambda$ and $C$ in (22). This requires the development of sophisticated mathematical tools to deal with partial information and high-dimensional data. Two different approaches, known as *finite-horizon* and *infinite horizon* identification procedures, can be adopted [14].

In the finite-horizon approach the opinions are observed after $T$ rounds of conversation and, if enough observations are available, then $W$, $\Lambda$ and $C$ can be estimated as the matrices that best fit the dynamics for the considered $T$ time steps. This method requires the knowledge of the time-sequence for observations made and may involve a large amount of data. Moreover, the system is usually updated with an unknown interaction rate and the interaction times between agents are not observable in most scenarios, as in [76].

The infinite-horizon identification procedure is applicable only to stable models and performs parameter estimation starting from

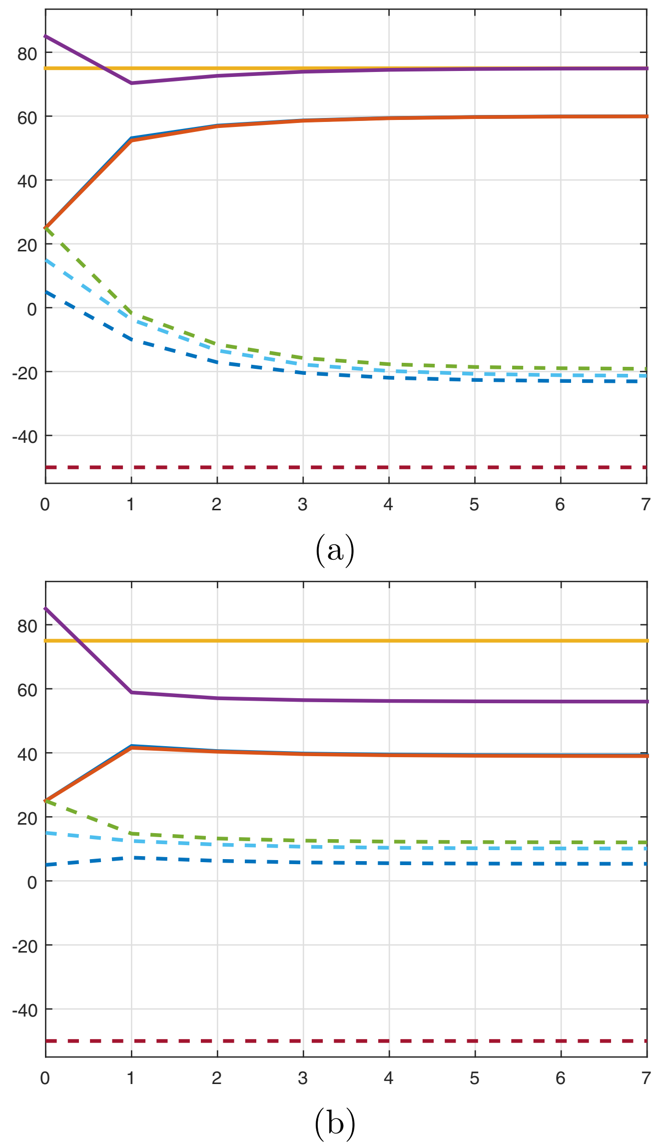


**Fig. 7.** Dynamics of opinions in the model (24): independent topics (a) vs. dependent topics (b). ©2017IEEE. Reprinted, with permission, from [14].

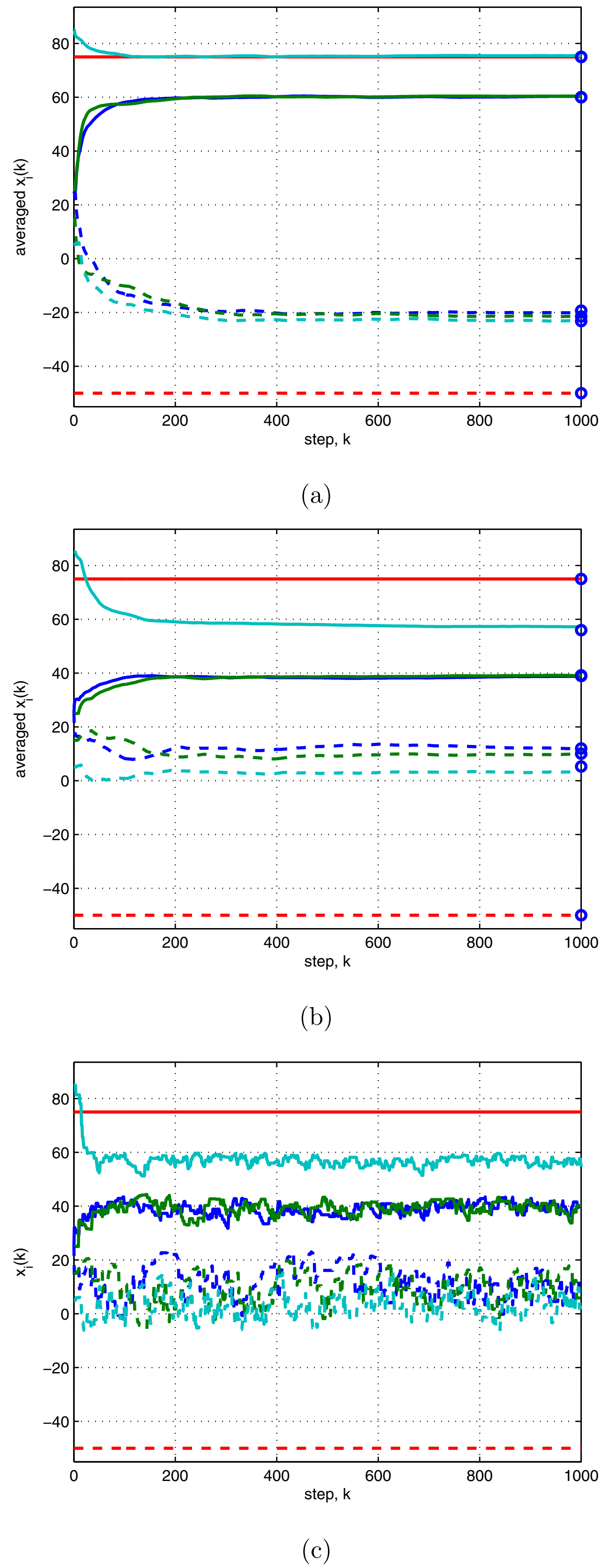


**Fig. 8.** Gossip-based counterparts of opinion dynamics in Fig. 7: (a) averaged opinions, independent topics; (b) averaged opinions, dependent topics and (c) opinions without averaging (dependent topics) exhibit no convergence. ©2017IEEE. Reprinted, with permission, from [14].

observations of the initial and final opinions' profile only. In [78,79], Tempo and coauthors adopt this second approach to estimate the influence matrix $W$ under the assumption that the matrix is sparse (i.e. agents are influenced by few friends), using tools borrowed from compressed sensing theory. In particular, under suitable assumptions, they derive theoretical conditions to guarantee that the estimation problem is well posed and sufficient requirements on the number of observation ensuring perfect recovery.

## 5. Conclusions and acknowledgements

In this paper, we overview the recent achievements of Tempo and his coauthors (Er-Wei Bai, Ming Cao, Paolo Frasca, Noah Friedkin, Hideaki Ishii, Sergey Parsegov, Li Qiu, and Keyou You) in SNA and modeling of social dynamics. Focusing on Tempo's works, we deliberately avoid many related works on social dynamics and distributed algorithms; the most "mature" of the recent achievements in the field are summarized in the tutorial papers [3,4] and other recent surveys [10,42,43,67,80].

Another important work that has to be mentioned is the vision paper [81], giving an overview of the new trends and future directions of control theory. An active and highly respected member of systems and control community, Tempo was invited to contribute several sections to this report and, in particular, he formulated several open problems lying at the frontier between control theory and SNA [81, Section 5.20] and concerned with the dynamics of "techno-social" networks, where social actors interact via online social media. One of the toughest problems is to create and examine rigorously novel dynamics models, describing asynchronous and spontaneous interactions among the agents discussing several interrelated topics. Modeling real interactions in social media, one has to take into account the "temporal" structure of the network, where both nodes and arcs can continuously appear and disappear. Many problems of the classical SNA (e.g. centrality computation and community detection) remain unsolved for such temporal graphs, and new efficient algorithms are needed to cope with these problems. Another important class of problems is concerned with resilience to malicious attacks on the social media and "techno-social" spamming, when individuals, or machines, are fictitiously broadcasting artificial connections.

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## Appendix A. Matrix analysis

Unless otherwise stated, a $n$-dimensional vector $x = (x_i)_{i=1}^n$ is treated as a column of height $n$; the set of all $n$-dimensional real vectors is denoted $\mathbb{R}^n$. More generally, one may consider a $m \times n$ matrix $A = (a_{ij})$ with $m$ rows and $n$ columns (here $i = 1, \dots, m$ and $j = 1, \dots, n$); the set of such matrices with real entries $a_{ij}$ is denoted $\mathbb{R}^{m\times n}$. Given a matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, we employ $A^\top$ to denote its *transpose* $A^\top = (a_{ji}) \in \mathbb{R}^{n\times m}$. For instance, a transpose to a single row ($1 \times n$ matrix) is a column

$$(x_1, \dots, x_n)^\top = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A matrix with equal numbers of rows and columns is said to be *square*. We use $\mathbb{1}_n$ to denote the $n$-dimensional column of ones, and $I_n$ is the *identity* $n \times n$ matrix

$$\mathbb{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

For the basics of matrix operations (e.g. their addition, subtraction and multiplication of matrices) and the key concepts of a square matrix's *determinant* (denoted $\det A$) and *eigenvalues* (spectrum points), we refer the reader to any standard textbook on matrices or linear algebra, e.g. [82]. We only remind that the product of $m \times n$ matrix $A$ and $n \times p$ matrix $B$ is a $(m \times p)$ matrix, in particular, for two column vectors $x, y \in \mathbb{R}^n$ the product $x^\top y$ is a scalar ($1 \times 1$ matrix), whereas $xy^\top$ is a square $n \times n$ matrix.

A square matrix is said to be *diagonal* if its off-diagonal entries are zeros. Given a sequence of scalars $\alpha_1, \dots \alpha_n \in \mathbb{R}$, $\mathrm{diag}(\alpha_1, \dots, \alpha_n)$ stands for the matrix

$$\begin{bmatrix} \alpha_1 & 0 & & 0 \\ 0 & \alpha_2 & & \\ \vdots & & \ddots & \\ 0 & 0 & & \alpha_n \end{bmatrix}$$

Given a square matrix $A = (a_{ij})_{i,j=1}^n$, let $\mathrm{diag}\, A = \mathrm{diag}(a_{11}, a_{22}, \dots, a_{nn}) \in \mathbb{R}^{n\times n}$.

Given a pair of matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, their Kronecker (or tensor) product is defined [51] by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}$$

(the matrix is obtained by combining the blocks $a_{ij}B$ into a single matrix). For instance,

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 2 & 3 \\ 6 & 9 \end{bmatrix}$$

Given a $n \times n$ matrix $A$ with eigenvalues $\lambda_1, \dots, \lambda_n$ (that can be *complex* numbers), the maximal modulus of the eigenvalues $\rho(A) = \max |\lambda_i|$ is said to be the *spectral radius* of the matrix. The matrix $A$ is *Schur stable* if $\rho(A) < 1$, which appears to be equivalent to the relation $\lim_{n\to\infty} A^n = 0$.

A matrix $A = (a_{ij})$ with nonnegative numbers $a_{ij} \geq 0$ is *stochastic* if $\sum_{j=1}^n a_{ij} = 1\ \forall i$ and *substochastic* if $\sum_{j=1}^n a_{ij} \leq 1\ \forall i$. For a substochastic matrix $A$, $\rho(A) \leq 1$ as implied e.g. by the Gershgorin disc theorem [48]; if $A$ is stochastic, then $\rho(A) = 1$ (hence a stochastic matrix cannot be Schur stable).

We call a stochastic matrix $M$ *primitive* if $M^n$ consists of strictly positive entries for some power $n \geq 1$ (in particular, a matrix with positive entries is primitive). It can be shown [48,83] that the primitive matrix is *regular* in the sense that a limit $M_* = \lim_{n\to\infty} M^n$. Moreover, the rows of $M_*$ are *identical*, and hence a primitive matrix is in fact *fully regular* [48] or SIA [49].

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