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Early default risk and surrender risk: Impacts on participating life insurance policies ☆

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ABSTRACT

We study the risk-neutral valuation of participating life insurance policies with surrender guarantees when an early default mechanism, forcing an insurance company to be liquidated once a solvency threshold is reached, is imposed by a regulator. The early default regulation affects the policies' value not only directly via changing the policies' payment stream but also indirectly via influencing policyholder's surrender. In this paper, we endogenize surrender risk by assuming a representative policyholder's surrender intensity bounded from below and from above and uncover the impact of the regulation on the policyholder's surrender decision making. A partial differential equation is derived to characterize the price of a participating policy and solved with the finite difference method. We discuss the impacts of the early default regulation and insurance company's reaction to the regulation in terms of its investment strategy on the policyholder's surrender as well as on the contract value, which depend on the policyholder's rationality level.

1. Introduction

A typical participating life insurance policy provides policyholders with a minimum interest rate guarantee and bonus payments upon death and upon survival which are linked to the performance of the insurance company. Usually, additional options are embedded in the policies to increase their attractiveness to the policyholders, among which the most popular one is a surrender option. A surrender option entitles the policyholders to terminate their contract prematurely and to obtain the surrender benefits promised by the insurance company.

The policyholders may not necessarily receive the payments specified in their contract even if they hold it until maturity. If the insurance company does not have enough reserves to pay back its liabilities at the maturity date, the policyholders cannot get more than what remains in the company. To protect the policyholders from collecting too few benefits as the insurance company declares bankruptcy at maturity, regulatory authorities impose early default mechanisms to monitor insurance companies' financial status and close them before it is too late. For example, under Solvency II, the supervisory authority withdraws the authorization of an insurance company when its capital falls below the minimum capital requirement and does not recover within a short period of time, see Solvency II Directive (2009/138/EC). Also, an insurance company supervised by the Swiss Financial Market Supervisory Authority (FINMA) can lose its license when its risk-based capital drops below the lowest threshold specified in the Swiss Solvency Test (SST), see FINMA Circ. 08/44 SST, FINMA (2008). Proceeds from liquidated assets are then paid to stakeholders. Hence, the policyholders also face early default risk of the insurance company accompanied with the early default regulatory intervention.

Both surrender and early default intervention definitely have direct impacts on the fair valuation of participating life insurance

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policies since they change the policies' payment stream. In the existing literature, most studies focus on only one of these two aspects. For example, Andreatta and Corradin (2003), Bacinello (2003), Bauer et al. (2006), Grosen and Jørgensen (2000), and Zaglauer and Bauer (2008) study the fair value of participating life insurance policies with an embedded surrender option but have not considered early default risk triggered by the bad performance of the insurance company, while Bernard et al. (2005), Chen and Suchanecki (2007), Grosen and Jørgensen (2002), and Jørgensen (2001) take into account regulatory intervention in valuing participating policies, but leave out surrender risk. The only work, as far as we are aware, that treats early default risk and surrender risk at the same time is Le Courtois and Nakagawa (2013). In their paper, surrender risk is modeled through a Cox process with an intensity that is correlated to the financial market but is independent of the company's liquidation threshold. However, since the early termination of the insurance company imposed by the regulator reforms the contracts' payment structure for the policyholders, which we consider as the direct impact on the contracts' value, as a response the policyholders may change their surrender behavior. Such an influence of enforced early bankruptcy on policyholders' surrender behavior can be considered as a "by-product" of the regulatory intervention, which in turn affects the contracts' payment stream and correspondingly, the contracts' value. Hence, modeling policyholders' surrender being independent of the regulator's early default intervention is oversimplistic.

The present paper incorporates this by-product effect of the regulatory intervention on policyholders' surrender behavior into analyzing the impact of the early default risk on the fair value of participating life insurance policies. We specify a model which endogenizes policyholders' surrender to value participating policies from the perspective of the insurance company which is monitored by an external regulator. Most literature assumes that policyholders are fully rational, which means that they can terminate their contract at the optimal time so that the surrender option is priced as a pure American-style option, see e.g., Andreatta and Corradin (2003), Bacinello (2005, 2003), and Grosen and Jørgensen (2000, 1997). However, since there is not an active market to monitor the contract values, and if policyholders are not capable of valuing their contract correctly, the surrender option is hardly exercised at the right time. Also due to the lack of an active policy trading market, policyholders, when in urgent liquidity needs, have to surrender their contract at the insurance company and collect the surrender guarantees, which are usually lower than the fair contract value. Empirical evidence which confirms the so called emergency fund hypothesis is found e.g. in Kiesenbauer (2011) and Kuo et al. (2003). Given the limitations, it is more reasonable to consider policyholders as partially rational from a purely financial point of view, which also corresponds to the spirit of Solvency II: While valuing options written in the contracts, realistic assumptions concerning the likelihood that policyholders exercise the options should be used, see Solvency II Directive (2009/138/EC), European Parliament (2015). The approach of modeling policyholders' partial rationality in Li and Szimayer (2014) is adopted in our model. Policyholders' surrender is considered as a randomized event and arrival of the event is assumed to follow a Poisson process with an intensity bounded from below and from above. The lower and upper bounds refer to the minimum surrender rate due to exogenous reasons and the maximum surrender rate due to limited financial rationality, respectively.

Following a safe-side equivalence principle in the actuarial practice, participating life insurance policies are priced at the maximum market-consistent value in our paper, which is derived by choosing surrender intensities within the two bounds in the safe-side scenarios.[1] In contrast to the first-order premium calculation based on deterministic safe-side scenarios, see Christiansen and Steffensen (2013), the safe-side scenarios adopted for pricing in our model are determined throughout the contract term dynamically, taking into account the by-protect effect of the regulator's solvency intervention on policyholders' surrender. In addition to incorporating real surrender practice into valuing contracts as required by Solvency II and treating the surrender risk differently for different policyholders in determining solvency capital as emphasized by CEIOPS,[2] we are able to distinguish the effects of regulator's early default intervention on different policyholders' surrender and their contracts' fair value by assuming different surrender intensity bounds. Moreover, when the regulatory rule changes, the insurance company may react to it by adopting a different investment strategy, which again affects the contracts' value directly and indirectly through its influence on policyholders' surrender behavior. Hence, in the present paper, we also study how the insurance company chooses its investment strategy in face of different regulatory rules, and the impacts of the insurance company's investment strategy on policyholders' surrender and their contract value.

The remainder of the paper is organized as follows: In Section 2 we model the insurance company and introduce the payoff structure of a participating life insurance policy. The early default regulatory framework is specified as well. Besides, both the financial market and the insurance market are modeled with respect to the stochastic processes of the underlying asset, the mortality risk intensity and the surrender risk intensity. In Section 3 we derive the partial differential equation for the price of the participating policy. In Section 4 we analyze the effects of the regulatory framework and the investment strategy on the policyholder's surrender and contract value. Section 5 concludes.

Table 1
Insurance company's balance sheet at t_0.

Assets	Liabilities & Equity
A_0	$L_0 \equiv \alpha A_0$ $E_0 \equiv (1-\alpha)A_0$

2. Model framework

2.1. Company overview

Inspired by the model framework in Briys and de Varenne (1994), we consider a life insurance company which acquires an asset portfolio with initial value A_0 at time $t_0 = 0$ financed by two agents, i.e., a policyholder and an equity holder. The policyholder pays a premium to acquire the initial liability $L_0 = \alpha A_0$ with $\alpha \in (0, 1)$. The rest is levied from the equity holder who acquires $E_0 \equiv (1-\alpha)A_0$ with limited liability. The insurance company's balance sheet at time t_0 is shown in Table 1. The parameter α is called the wealth distribution coefficient in Grosen and Jørgensen (2002).

It is assumed that the insurance company operates in an arbitrage-free and complete financial market over a time interval $[0, T]$, where the time T corresponds to the maturity date of the insurance contract. As the insurance contract matures at T, the insurance company closes and its assets are liquidated and distributed to the stakeholders.[3]

[1] The same scenarios, however, are named as worst-case scenarios in Li and Szimayer (2014), which are conceptually equivalent from a mathematical point of view.

[2] CEIOPS refers to the Committee of European Insurance and Occupational Pensions Supervisors, which was replaced by the European Insurance and Occupational Pensions Authority (EIOPA) since 2011. CEIOPS has pointed out that policyholders' surrender behavior poses a significant risk to insurance companies and the surrender risk should be treated differently for different policyholders. For example, if the policyholders are institutional investors, since they tend to be better informed and react more quickly, the surrender risk can be substantially higher, see CEIOPS (2009).

[3] For simplicity, we assume that the company closes when the contract ends. It is not a strict assumption because it can be considered that assets raised from the

2.2. Participating life insurance policy

By investing in the insurance company at time t_0, the policyholder signs a participating insurance contract which promises him a share of the insurance company's profits in addition to the guaranteed minimum interest rate at the maturity date T. If the policyholder dies before time T, the contract pays death benefits. Additionally, the policyholder can exercise the surrender option embedded in the contract before maturity T and collect surrender benefits from the insurance company. To summarize, the contract promises survival benefits, death benefits and surrender benefits, depending on which event happens first. In any event, the policyholder has a priority claim on the company's assets and the equity holder receives what is left.

As the contract matures at T, the policyholder receives a minimum guaranteed benefit, which is given by compounding the initial liability L_0 with a minimum guaranteed interest rate r_g, i.e., $L_T^{r_g} = L_0 e^{r_g T}$, and a bonus conditional on that the asset value generated by the contribution of the policyholder is enough to cover the minimum guaranteed benefit, i.e., $\alpha A_T \geq L_T^{r_g}$. Suppose δ is the participation rate in the asset surplus. The profits shared with the policyholder are $\delta[\alpha A_T - L_T^{r_g}]^+$. However, it may happen that at time T when the company's assets are liquidated, the assets' value is lower than the value of the minimum guaranteed benefit. In this case, based on the assumptions that the policyholder has a priority claim on the company's assets and the equity holder has limited liability, the policyholder collects what is left, i.e., A_T, and the equity holder walks away with nothing in his hands. To sum up, when the contract survives until maturity T, the policyholder receives survival benefits which take the form[4]

$$\Phi(A_T) = L_T^{r_g} + \delta[\alpha A_T - L_T^{r_g}]^+ - [L_T^{r_g} - A_T]^+. \tag{1}$$

The policyholder may die before the contract matures. We use τ_d to denote the death time of the policyholder aged x at time t_0. At time $\tau_d < T$, the contract pays death benefits to the policyholder. We assume that death benefits have the same payment structure as survival benefits, but with all the components valued at the death time τ_d. We use r_d and δ_d to denote the minimum guaranteed interest rate and the participation rate for calculating the promised minimum guarantee, i.e., $L_{\tau_d}^{r_d} = L_0 e^{r_d \tau_d}$, and the share of the asset surplus, respectively. Then, the death benefits have the following form at time τ_d

$$\Psi(\tau_d, A_{\tau_d}) = L_{\tau_d}^{r_d} + \delta_d[\alpha A_{\tau_d} - L_{\tau_d}^{r_d}]^+ - [L_{\tau_d}^{r_d} - A_{\tau_d}]^+. \tag{2}$$

Furthermore, by exercising the surrender option embedded in the contract, the policyholder can terminate the contract before the maturity date T. We use τ_s to denote the surrender time. Once the surrender option is exercised, the company closes and its assets are liquidated and paid to the policyholder as specified in the contract but not more than the liquidated asset value. We consider the following surrender payment form for the policyholder:

$$S(\tau_s, A_{\tau_s}) = L_{\tau_s}^{r_s} - [L_{\tau_s}^{r_s} - A_{\tau_s}]^+, \tag{3}$$

where $L_{\tau_s}^{r_s} = (1 - \beta_{\tau_s}) L_0 e^{r_s \tau_s}$ is the surrender guarantee when the asset value suffices. Here, r_s is the guaranteed interest rate at surrender if the assets are sufficient and β_{τ_s} is a penalty parameter which penalizes the policyholder for early terminating the contract and is assumed to be a deterministic decreasing function of the time. After the policyholder is paid off, the equity holder receives the rest of the asset value.

policyholder and the equity holder are put in a separate fund, as the contract ends, the fund is closed and assets left in the fund are liquidated and distributed to the stakeholders.

4 The same payment form of maturity benefits can also be found in Briys and de Varenne (1997) and Grosen and Jørgensen (2002).

2.3. Early default mechanism

Now, we introduce early default risk of the insurance company into the model. We consider an external regulator who watches on the insurance company's financial status over its operating time horizon. We abstract from cumbersome bankruptcy rules and procedures applied to insurance companies in practice and assume that the insurance company is on-going until either the external regulator intervenes before T or the insurance contract matures at T. We adopt the regulatory mechanism in Bernard et al. (2005), Grosen and Jørgensen (2002), and Jørgensen (2001) and set up a default-triggering barrier based on the minimum survival guarantee $B_t = \theta L_0 e^{r_g t}$, where θ is a default multiplier. Once the company's asset value drops below the barrier before maturity T, the company is closed by the regulator and its assets are liquidated and distributed to the stakeholders. Accordingly, we define the early default time τ_b as the first time that the asset value drops below the barrier,

$$\tau_b = \inf\{t \mid A_t \leq B_t\}. \tag{4}$$

At time τ_b, the policyholder receives early default benefits, denoted by $\Upsilon(\tau_b, A_{\tau_b})$, which have the lower value of the liquidated assets and the minimum survival guarantee accrued at the guarantee rate r_g up to the early default date

$$\Upsilon(\tau_b, A_{\tau_b}) = \min\{A_{\tau_b}, L_{\tau_b}^{r_g}\}, \tag{5}$$

where $L_{\tau_b}^{r_g} = L_0 e^{r_g \tau_b}$. Accordingly, if the company has the liquidated assets more than the promised minimum survival guarantee, the equity holder obtains what is left after paying off the policyholder; otherwise, the equity holder gets nothing.

The default multiplier θ is set by the regulator, which actually reflects how intensively the regulator monitors the insurance company and how strongly the regulator intends to protect the policyholder. If the regulator believes that the insurance company is inclined to take advantage of the policyholder by running a risky business or is not competent enough to manage its assets, the regulator may set a higher default multiplier to protect the policyholder. This implies that the insurance company must bear a higher early default risk. Otherwise, the regulator will set a lower default multiplier, which allows the insurance company to recover from its temporary bad performance. In our model, we restrict θ to be smaller than $1/\alpha$, which ensures $A_0 > B_0$ so that the insurance company does not default at the initial time t_0 when the contract is just issued to the policyholder.

2.4. Mathematical formulation

In this section we model the financial market and the insurance market mathematically. We fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ reflects the flow of information available on the financial market and the insurance market. We assume that the company invests its total initial assets in traded (risk-free and risky) assets on the financial market, where the risk-free interest rate, denoted by r, is assumed to be deterministic in time. Under the market probability measure $\mathbb{P}$, the company's asset price process A is assumed to be governed by the following stochastic process:

$$dA_t = a(t, A_t) A_t dt + \sigma(t, A_t) A_t dW_t, \quad \forall t \in [0, T]. \tag{6}$$

Here W is a standard Brownian motion under $\mathbb{P}$ and generates the filtration $\mathbb{F}^W = \{\mathcal{F}_t^W\}_{0\leq t\leq T}$. The functions a and $\sigma > 0$ refer to the expected rate of return and the volatility of the asset process, respectively, and both are regular enough to guarantee the unique solution of (6).

As the payoff of the contract depends not only on the asset value itself but also on the occurrence of the death event or the surrender event, we enlarge the filtration $\mathbb{F}^W$ in a minimal way to summarize all the information relevant to the contract valuation. The filtration $\mathbb{F}^W$ is thus enlarged to $\mathbb{G} = \mathbb{F}^W \vee \mathbb{H}$, where $\mathbb{H}$ is jointly generated by the jump processes $H_t = 1_{\{\tau_d \le t\}}$ and $J_t = 1_{\{\tau_s \le t\}}$, i.e., the information about whether the policyholder dies before time t and whether he surrenders the contract before time t, respectively. The hazard rate of the random time τ_d, also called mortality intensity, is denoted by μ and assumed to be a deterministic function of time.[5] Similarly, we call the hazard rate of the random time τ_s surrender intensity and denote it by γ, which needs to be modeled. Based on our arguments in the introduction, we take into account that the policyholder has partial rationality in surrendering the contract and follow the approach adopted in Li and Szimayer (2014) by assuming a bounded surrender intensity. Due to personal reasons which urge the policyholder to surrender his contract prematurely, an exogenous minimum surrender intensity $\underline{\rho}$ is assumed in the case when surrendering is not a financially optimal decision for the policyholder, which is called the exogenous surrender intensity in Li and Szimayer (2014). As surrendering becomes financially optimal, a higher surrender intensity value is assumed for the policyholder, which is called an upper bound of the surrender intensity and denoted by $\overline{\rho}$, with $\overline{\rho} > \underline{\rho}$. The increase in the surrender intensity level, $\overline{\rho} - \underline{\rho}$, reveals the policyholder's sensitivity to the optimal surrendering time. Since in order to terminate the contract at the financially optimal time, the policyholder is expected to frequently update his financial market information and put effort into valuing his contract, the size of the increase in the surrender intensity value as surrendering becomes optimal to the policyholder can be considered as a measure of policyholder's information updating frequency and his valuation effort, which is regarded as the policyholder's financial rationality level and named as the endogenous surrender intensity in Li and Szimayer (2014). The more frequently the policyholder updates and analyzes financial information, the larger the increase in the intensity is, and accordingly, the more financially rational he is. In the case that $\overline{\rho} = \infty$, the policyholder surrenders immediately when it is optimal to do so and together with a zero exogenous surrender rate $\underline{\rho} = 0$, we are back to the case of pricing an American-style contract by solving an optimal stopping problem. The insurer's choice of surrender intensity for pricing the contract is made by comparing the continuation value of the contract and the value of surrender benefits, which are denoted by $v(t, A)$ and $S(t, A)$, respectively, which takes the form of

$$\gamma_t = \begin{cases} \underline{\rho}, & \text{for } S(t, A) < v(t, A) \\ \overline{\rho}, & \text{for } S(t, A) \ge v(t, A). \end{cases} \tag{7}$$

Conditional on the current information available on the financial market and the insurance market, the arrival of the death event, the arrival of the surrender event, and W are independent. Thus, W is a $\mathbb{G}$-martingale, and μ and γ are $\mathbb{G}$-intensities of the random death time τ_d and the random surrender time τ_s, respectively.

In the absence of arbitrage, we use the risk-neutral valuation approach with a martingale measure $\mathbb{Q}$ to price the participating life insurance contract. Under the martingale measure $\mathbb{Q}$ the company's asset process is described by

$$\mathrm{d}A_t = r(t)A_t\mathrm{d}t + \sigma(t, A_t)A_t\,\mathrm{d}W_t^{\mathbb{Q}}, \quad \forall t \in [0, T], \tag{8}$$

where $W^{\mathbb{Q}}$ is a standard Brownian motion. Taking the mortality risk and the surrender risk into consideration, pricing the participating life insurance contract under the martingale measure $\mathbb{Q}$ with μ and γ as $\mathbb{G}$-intensities of the random times τ_d and τ_s, respectively, requires additional justification. Given that the mortality intensity is deterministic, if the pool of policyholders is large enough, the mortality risk is diversifiable for the insurer, and thus μ is the $(\mathbb{Q}, \mathbb{G})$-intensity of the arrival of the death. The surrender intensity specified in (7) corresponds to the safe-side scenario from the insurance company's perspective in our model. As long as we assume that the insurance company does not ask for an extra risk premium above the safe-side surrender intensity when changing from the measure $\mathbb{P}$ to the measure $\mathbb{Q}$,[6] the bounds $\underline{\rho}$ and $\overline{\rho}$ are then still valid under the measure $\mathbb{Q}$, and the surrender intensity specified in (7) corresponds to the $(\mathbb{Q}, \mathbb{G})$-intensity of τ_s on the enlarged market represented by the filtration $\mathbb{G}$.[7] The contract value obtained under the measure $\mathbb{Q}$ with the safe-side surrender intensity γ can be interpreted as the upper price bound of the contract. We address this issue formally in Remark 1.

2.5. Remarks

In the current model framework, we assume that there is only one policyholder participating in the insurance company's investment, which is unrealistic with respect to founding an insurance company. For generalizing our model, a pool of policyholders can be considered. Each policyholder pays a premium for purchasing a participating life insurance policy, which is put in a separate fund. The assets in each separate fund, consisting of a premium and the equity holders' contribution, are invested in traded risky and risk-free securities on the financial market by the insurer. The asset price process follows the stochastic process given in (6). In case a policyholder dies or surrenders before his contract matures, the policyholder receives death benefits or surrender benefits, his fund closes, and the equity holders receive what is left. By generalizing our model in this way, the balance sheet presented in Table 1 actually refers to a snap-shot of the asset and liability situation of a separate fund at the issuing time of a contract t_0. As the contract ends, the fund, but not the company, is closed and the assets left in the fund are liquidated and distributed to the stakeholders. We assume that the funds in the insurance company are monitored separately by an external regulator who intervenes by closing the ones whose financial performance becomes bad.[8] This type of intervention corresponds to the timely intervention by regulatory authorities, which is intended to promote good risk management by insurance companies, see Solvency II Directive (2009/138/EC), European Parliament (2015).

In addition, the participating life insurance policy analyzed in this paper is a point-to-point participating policy, which promises a bonus payment based on the insurer's asset value at maturity T. Of course, a cliquet-style participating policy, which credits dividends over time, i.e., yearly, to the maturity guarantee, could also be considered. For doing so, the model framework in Grosen and Jørgensen (2000), where a policy reserve account and a bonus

[5] In the literature, there are many discussions on stochastic mortality intensity which is more consistent with the reality, see e.g., Bacinello et al. (2010), Biffis et al. (2010), Dahl (2004) and Dahl and Møller (2006). However, the stochastic feature of the mortality intensity does not have too much influence on the contract value, see Li and Szimayer (2011). Hence, we assume a deterministic mortality intensity function for simplicity and focus more on the early default risk and the surrender risk.

[6] Alternatively, a higher market price for the surrender risk may be charged by lowering the lower bound $\underline{\rho}$ and increasing the upper bound $\overline{\rho}$ under the measure $\mathbb{Q}$. In Proposition 2 we show formally that a lower $\underline{\rho}$ and a higher $\overline{\rho}$ lead to a higher contract value.

[7] Confer Li and Szimayer (2014) for a formal explanation of the surrender intensity after the change of measure.

[8] As an alternative, we can also assume that the external regulator checks the company as a whole with the default multiplier θ inserted between the company's global assets and its global guaranteed benefits, and in order to attract policyholders, the same multiplier θ is applied to individual funds by the insurer as a protection provided to policyholders against losing the assets in the funds backing their contract.

reserve account are constructed in the insurer's liability side, can be adopted. Liabilities in the policy reserve then consist of a minimum interest rate guarantee, credited dividends over time, and interest on the credited dividends, which are the promised claim to the policyholder. Depending on how to distribute dividends to the policyholder over time, the promised claim can accumulate in many different ways, see some examples of the so called bonus distribution scheme in Bauer et al. (2010) and Grosen and Jørgensen (2000). As a default-triggering barrier based on those liabilities in the policy reserve is imposed, the insurance company can be closed earlier by the regulator. However, as the same endogenous surrender mechanism is established for a partially rational policyholder as in the present paper, we expect the impact of the early default regulation on the fair value of a cliquet-style participating policy, in particular how it depends on the policyholder's rationality level, is qualitatively the same as the impact on the fair value of a point-to-point participating policy, because it is conceptually equivalent to increasing the minimum guaranteed interest rate of a point-to-point participating policy in our present model. Since our focus is not on designing the payment structure of a participating policy, we choose to value a point-to-point participating contract in this paper, which leaves out discussion on the design of a bonus distribution mechanism and lets us directly focus on the impact of the early default regulation on the fair valuation.

3. Contract valuation

In this section we value the contract by taking both the early default risk and the surrender risk into consideration. Given that the surrender intensity is determined endogenously within our model, it is not possible to derive a closed-form pricing formula. However, by applying the partial differential equation (PDE) approach, we can specify the surrender intensity and the contract value at the same time. Importantly, after introducing the early default mechanism in this paper, the contract payoff to the policyholder is connected to the solvency of the company and has a barrier option property. Thus, we need to distinguish the case where the insurance company is ongoing and the case where the regulator intervenes. It means that in order to value the contract, we need to differentiate between the region where $A_t \leq B_t$ and the region where $A_t > B_t$ for $t \in (0, T)$, which is similar to the barrier option pricing. For $A_t \leq B_t$ at time $t \in (0, T)$, the insurance company must be liquidated and the policyholder only obtains $\Upsilon(t, A_t)$. For $A_t > B_t$, we represent the contract value V_t on $\{t \leq \tau_d \wedge \tau_s \wedge T\}$ by

$$
\begin{aligned}
V_t = {} & \mathbb{1}_{\{t<\tau_d\wedge\tau_s\wedge T\}} v(t, A_t) + \mathbb{1}_{\{t=\tau_d<\tau_s\wedge T\}} \Psi(\tau_d, A_{\tau_d}) \\
& + \mathbb{1}_{\{t=\tau_s<\tau_d\wedge T\}} S(\tau_s, A_{\tau_s}),
\end{aligned} \tag{9}
$$

where v is a suitably differentiable function $v : [0, T] \times \mathbb{R}^+ \to \mathbb{R}_0^+$, $(t, A) \mapsto v(t, A)$, representing the pre-death/surrender value. Then we apply the no-arbitrage pricing condition on the set $\{t < \tau_d \wedge \tau_s \wedge \tau_b \wedge T\}$, being

$$
r(t)V(t, A_t)\mathrm{d}t = \mathbb{E}_{\mathbb{Q}}[\mathrm{d}V_t|\mathcal{G}_t]. \tag{10}
$$

On the set $\{t < \tau_d \wedge \tau_s \wedge \tau_b \wedge T\}$, we compute the differential of V as[9]

$$
\begin{aligned}
\mathrm{d}V_t = {} & \mathrm{d}v(t, A_t) + (\Psi(t, A_t) - v(t, A_t))\mathrm{d}H_t \\
& + (S(t, A_t) - v(t, A_t))\mathrm{d}J_t, \quad \text{for } 0 \leq t < T,
\end{aligned} \tag{11}
$$

where H and J refer to the jump processes with the $\mathbb{Q}$-intensities μ and γ, respectively. A jump in H or J leads to a change in the payment liability either of the amount $\Psi(t, A_t) - v(t, A_t)$ or $S(t, A_t) - v(t, A_t)$. Plugging (11) into (10) and using $V_t = v(t, A_t)$ at time $t < \tau_d \wedge \tau_s \wedge \tau_b \wedge T$, we obtain

$$
\begin{aligned}
r(t)v(t, A_t)\mathrm{d}t = {} & \mathbb{E}_{\mathbb{Q}}[dv(t, A_t)|\mathcal{G}_t] + (\Psi(t, A_t) - v(t, A_t))\mu(t)\mathrm{d}t \\
& + (S(t, A_t) - v(t, A_t))\gamma_t\mathrm{d}t.
\end{aligned} \tag{12}
$$

By applying Ito's lemma to $\mathrm{d}v(t, A_t)$, we have

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\mathrm{d}v(t, A_t)|\mathcal{G}_t] = {} & \mathbb{E}_{\mathbb{Q}}\Bigg[\mathcal{L}v(t, A_t)\mathrm{d}t \\
& + \sigma(t, A_t)A_t\frac{\partial v}{\partial A}(t, A_t)\mathrm{d}W_t^{\mathbb{Q}}\Big|\mathcal{G}_t\Bigg] \\
= {} & \mathcal{L}v(t, A_t)\mathrm{d}t,
\end{aligned} \tag{13}
$$

where $\mathcal{L}v(t, A) = \frac{\partial v}{\partial t}(t, A) + r(t)A\frac{\partial v}{\partial A}(t, A) + \frac{1}{2}\sigma^2(t, A)A^2\frac{\partial^2 v}{\partial A^2}(t, A)$. Then, on the set $\{t < \tau_d \wedge \tau_s \wedge \tau_b \wedge T\}$ we have

$$
\begin{aligned}
& \mathcal{L}v(t, A_t) + \mu(t)\Psi(t, A_t) + \gamma_t S(t, A_t) \\
& - (r(t) + \mu(t) + \gamma_t)v(t, A_t) = 0.
\end{aligned} \tag{14}
$$

We summarize the pricing PDE with the following proposition.

Proposition 1. *For the contract value V described by (9), the pre-death/surrender value v for $(t, A) \in [0, T) \times \mathbb{R}^+$ is the solution of the partial differential equation*

$$
\begin{aligned}
& \mathcal{L}v(t, A_t) + \mu(t)\Psi(t, A_t) + \gamma_t S(t, A_t) \\
& - (r(t) + \mu(t) + \gamma_t)v(t, A_t) = 0\,,
\end{aligned} \tag{15}
$$

where

$$
\gamma_t = \begin{cases} \underline{\rho}\,, & \text{for } S(t, A_t) < v(t, A_t)\,, \\ \overline{\rho}\,, & \text{for } S(t, A_t) \geq v(t, A_t)\,; \end{cases} \tag{16}
$$

subject to the boundary condition

$$
v(t, A_t) = \Upsilon(t, A_t), \ \text{for } t \in [0, T), \ A_t = B_t = \theta L_0 e^{r_g t}, \tag{17}
$$

and the terminal condition

$$
v(T, A_T) = \Phi(A_T), \quad \text{for } A_T \in \mathbb{R}^+. \tag{18}
$$

The integral representation of the solution to the above pricing PDE is shown in Corollary 1 and proved in Appendix A.

Corollary 1. *Suppose the surrender intensity γ is given. The value of the participating policy V can be represented on $\{t < \tau_s \wedge \tau_d \wedge \tau_b \wedge T\}$ by*

$$
\begin{aligned}
V_t = {} & \mathbb{E}_{\mathbb{Q}}\Bigg[\int_t^{\tau_b\wedge T} e^{-\int_t^m (r(u)+\mu(u)+\gamma(u,A_u))\mathrm{d}u}(\mu(m)\Psi(m, A_m) \\
& + \gamma(m, A_m)S(m, A_m))\mathrm{d}m \\
& + 1_{\{\tau_b\geq T\}}\Phi(A_T)e^{-\int_t^T (r(u)+\mu(u)+\gamma(u,A_u))\mathrm{d}u} \\
& + 1_{\{\tau_b<T\}}\Upsilon(\tau_b, A_{\tau_b})e^{-\int_t^{\tau_b} (r(u)+\mu(u)+\gamma(u,A_u))\mathrm{d}u}\Big|\mathcal{G}_t\Bigg].
\end{aligned} \tag{19}
$$

Remark 1. The pricing problem can be formulated as looking for the safe-side scenario of the risk-adjusted surrender intensity γ so that the contract value is maximized under the martingale measure $\mathbb{Q}$ on $\{t < \tau_s \wedge \tau_d \wedge \tau_b \wedge T\}$,

$$
\begin{aligned}
v(t, A) = {} & \sup_{\gamma\in\Gamma(t,A)} \mathbb{E}_{\mathbb{Q}}^{t,A}\Bigg[\int_t^{\tau_b\wedge T} e^{-\int_t^m (r(u)+\mu(u)+\gamma(u,A_u))\mathrm{d}u} \\
& \times (\mu(m)\Psi(m, A_m) + \gamma(m, A_m)S(m, A_m))\mathrm{d}m \\
& + 1_{\{\tau_b\geq T\}}\Phi(A_T)e^{-\int_t^T (r(u)+\mu(u)+\gamma(u,A_u))\mathrm{d}u} \\
& + 1_{\{\tau_b<T\}}\Upsilon(\tau_b, A_{\tau_b})e^{-\int_t^{\tau_b} (r(u)+\mu(u)+\gamma(u,A_u))\mathrm{d}u}\Bigg],
\end{aligned} \tag{20}
$$

[9] Notice that in the region $A_t > B_t$, there will not be early default after the instantaneous time period $\mathrm{d}t$ since the asset process is assumed to be continuous in our model.

Table 2
Parameter specifications.

Market parameters		Contract parameters	
A_0	100	α	0.85
r	0.04	T	10
σ	0.2	δ, δ_d	0.9
A^μ	5.0758×10^{-4}	r_g, r_d, r_s	0.02
B	3.9342×10^{-5}		
c	1.1029		

where $\Gamma(t, A) = \{\gamma : [t, T] \times \mathbb{R}^+ \to \mathbb{R}_0^+, \ (u, A) \mapsto \gamma(u, A) : \underline{\rho} \le \gamma(u, A) \le \overline{\rho}\}$ and $\mathbb{E}_{\mathbb{Q}}^{t,A}$ denotes the expectation conditional on $A_t = A$ under the measure $\mathbb{Q}$. This is a stochastic control problem, which can be solved, according to the theorem of the Hamilton–Jacobi–Bellman equation (confer Yong (1997) and Yong and Zhou (1999)), by dealing with an equivalent problem

$$0 = \sup_{\gamma \in \Gamma(t,A)} \mathcal{L}v(t, A) + \mu(t)\Psi(t, A) + \gamma(t, A)S(t, A) - (r(t) + \mu(t) + \gamma(t, A))v(t, A), \tag{21}$$

subject to $v(t, A) = \Upsilon(t, A)$, for $A = B_t = \theta L_0 e^{r_g t}$, and $v(T, A) = \Phi(A)$, for $A \in \mathbb{R}^+$. γ needs to be optimally controlled: Since in the equation above the part that depends on γ is linear in γ, i.e., $\gamma(t, A)(S(t, A) - v(t, A))$, the solution to the problem is exactly the same as is presented in Eq. (7).

Within a given regulatory framework and under a given investment strategy, i.e., for given θ and σ, we prove that a lower value of $\underline{\rho}$ and a higher value of $\overline{\rho}$ lead to an increase in the contract value, see Proposition 2. This is consistent with our intuition, since a lower $\underline{\rho}$ or a higher $\overline{\rho}$ indicates the increase of the rationality level of the policyholder in the monetary sense and thus increases the contract value. The proof is provided in Appendix B.

Proposition 2. *Suppose the early default mechanism is characterized by the default multiplier θ and the insurance company's investment strategy by σ. Furthermore, suppose that v is the pre-death/surrender value function of the participating policy with bounds of the surrender intensity being $\underline{\rho}$ and $\overline{\rho}$, and that w is the pre-death/surrender value function of the policy with bounds $\underline{\zeta}$ and $\overline{\zeta}$. Assume that $\underline{\zeta} \le \underline{\rho}$ and $\overline{\rho} \le \overline{\zeta}$. Then we have $w(t, A) \ge v(t, A)$, for $(t, A) \in [0, \tau_b \wedge T] \times \mathbb{R}^+$.*

4. Numerical analysis

In this section, we adopt the finite difference method proposed by Zvan et al. (2000, 1996) to numerically solve the PDE with a continuously applied barrier (15) as stated in Proposition 1 and study the impacts of the early default risk and the surrender risk on the fair valuation of the contract as well as on the insurance company's investment strategy. The insurance company is set up with initial asset value $A_0 = 100$ and 85% of the asset value is acquired by the policyholder who buys the participating contract at time t_0, which means that $\alpha = 0.85$. The contract matures in $T = 10$ years and promises the same participation rate $\delta = \delta_d = 0.9$ at maturity and at death.[10] The risk-free interest rate and the volatility of the insurance company's asset process are constant, i.e., $r(t) = 0.04$ and $\sigma(t, A_t) = 0.2$. The volatility provides information about the riskiness of the insurance company's investment strategy. A higher σ indicates a higher risk of the investment strategy while a lower σ implies a more conservative investment strategy.[11] The minimum guaranteed interest rates at survival, at death and at surrender are $r_g = r_d = r_s = 0.02$. As for the mortality intensity, we follow Li and Szimayer (2014) and assume that a deterministic process $\mu(t) = A^\mu + Bc^{x+t}$ for the policyholder aged $x = 40$ at $t_0 = 0$ with $A^\mu = 5.0758 \times 10^{-4}$, $B = 3.9342 \times 10^{-5}$, $c = 1.1029$. Additionally, the penalty parameter takes on the value of the following form:

$$\beta_t = \begin{cases} 0.05, & \text{for } t \le 1, \\ 0.04, & \text{for } 1 < t \le 2, \\ 0.02, & \text{for } 2 < t \le 3, \\ 0.01, & \text{for } 3 < t \le 4, \\ 0, & \text{for } t > 4. \end{cases}$$

The parameters are summarized in Table 2.

The analysis in the following subsections is conducted for a representative policyholder. If a pool of policyholders is considered, under the assumption that the pool is large enough, the surrender intensity of a representative policyholder gives an indication of the proportion of policyholders who will surrender their contract at the portfolio level. The implications for a large pool of policyholders will be summarized in Section 5 to conclude the paper.

4.1. Effect of regulatory framework on contract valuation

In this section, we analyze the effect of the early default risk on the fair valuation of the contract. The magnitude of the early default risk depends on the strictness of the regulatory framework, which in our model is represented by the default multiplier θ specified by the regulator. It indicates how the regulator judges the insurance company's ability to manage its assets. If the regulator is very confident about the expertise of the insurance company and about the financial market, she will tolerate a temporary poor performance of the insurance company more and hence choose a lower default multiplier so that the company has the chance to recover. Otherwise, she will set a higher value to protect the policyholder from not being able to obtain the guaranteed benefits promised by the company. Although a lower (higher) default multiplier is less (more) effective to protect the policyholder from the downside development of the company, it gives the company more (less) chance to recover from its temporary bad performance and pay out more (less) benefits to the policyholder when it recovers. Hence, the level of the default multiplier has great influence on the payoff of the contract and thus on the contract value.

Furthermore, the policyholder takes into account the impact of the protection from the regulator on the payments of his contract and adjusts his surrender behavior accordingly, which indirectly influences the contract value. Intuitively, the policyholder makes his surrender decision based not only on the benefits that are promised by the insurance company but also on the ability of the insurance company to meet its promise. The early default mechanism ensures the ability of the insurance company to meet its promise by imposing a limit on its asset value. A higher default multiplier implies that the policyholder has to worry less about the second issue because he is better protected and will surrender the contract only when the surrender benefits are very attractive to him. On the contrary, if the default multiplier is set lower so that the policyholder would not be protected completely, he must take the default risk of the insurance company seriously into account when implementing his surrender strategy. In this case, the policyholder may be willing to surrender his contract earlier to avoid losing too much of his initial investment.

In Table 3, we present the contract value for different values of the default multiplier θ and different rationality levels represented by $(\underline{\rho}, \overline{\rho})$. In the second column are the contract values in the case when there is no early default mechanism. From the third to the fifth column are the contract values with different levels of

10 Regulators usually require the participation rate to be kept at least at a certain level. In Germany, e.g., it lies at 0.9.

11 We choose a relatively higher volatility level as a benchmark to see the effect of the regulation on the insurance company's incentive in selecting the riskiness of its investment strategy. We will show that an effective regulatory rule leads the insurance company to choose a conservative investment strategy voluntarily.

Table 3
Contract value for different default multipliers θ and different rationality levels represented by $(\underline{\rho}, \overline{\rho})$.

	No early default	With early default		
		$\theta = 0.7$	$\theta = 0.9$	$\theta = 1.1$
(0, 0)	85.6129	86.7559	90.3847	89.3619
(0, 0.03)	86.0357	87.0060	90.4002	89.3619
(0, 0.3)	88.1519	88.4274	90.5220	89.3619
$(0, \infty)$	92.0665	92.0666	92.0683	89.3619
(0.03, 0.03)	81.8548	82.8577	86.5947	87.7341
(0.03, 0.3)	84.2637	84.5350	86.7571	87.7341
$(0.03, \infty)$	88.5460	88.5471	88.5481	87.7341
(0.3, 0.3)	75.4562	75.7302	78.0351	83.4083
$(0.3, \infty)$	80.7500	80.7500	80.7500	83.4086

regulatory strength which are represented by the different values of the default multiplier θ. For example, $\theta = 0.7$ means that the regulator does not allow the insurance company's asset value to drop below 70% of the minimum guarantee. $\theta = 1.1$ indicates that the regulator is more conservative and requires the company's asset value to lie above 110% of the minimum guarantee. Comparing the contract values in columns 2–4 where the early default regulation is first introduced, and then strengthened, the contract value increases gradually for all the types of policyholders. Introducing the early termination rule protects the policyholder from the downside risk of the insurance company's investment and increasing the default multiplier enlarges the protection level. An interesting feature is that the effect of the early termination rule depends not only on the default multiplier θ but also on the rationality level of the policyholder. For example, a policyholder with $(\underline{\rho}, \overline{\rho}) = (0, 0)$ never surrenders his contract, which turns to be a European-type contract.[12] We can see in this case when the policyholder is not competent enough to adopt a rational surrender strategy, an effective early termination rule helps the policyholder improve his financial position. However, the benefits from the regulator's protection become smaller as the policyholder becomes financially more rational. In particular, when the policyholder is able to exercise the surrender option optimally, i.e. $(\underline{\rho}, \overline{\rho}) = (0, \infty)$, the default multiplier does not play a significant role any more.[13] Since the fully rational policyholder can find the optimal surrender strategy anyway, he does not need the protection of the regulator. Hence, the early default regulation protects "financially illiterate" (or "naive") policyholders more than "financially literate" ones, which is more obvious as the regulatory rule gets strengthened further. As the default multiplier θ increases from 0.9 to 1.1, the contract value decreases in some cases, e.g., when $(\underline{\rho}, \overline{\rho}) = (0, \cdot)$, $(\underline{\rho}, \overline{\rho}) = (0.03, \infty)$, among which we can even observe the disadvantage of introducing the early default regulatory rule. For the policyholder with $(\underline{\rho}, \overline{\rho}) = (0, \infty)$ and $(\underline{\rho}, \overline{\rho}) = (0.03, \infty)$, the contract value becomes even lower than when there is no early default risk. As we have mentioned in Section 2.4 that the policyholder with $\overline{\rho} = \infty$ may surrender the contract at any time when it is optimal to do so, irrespective of exogenous reasons, he is able to protect himself from the downside risk of the company's investment. However, enforcing an early termination rule with a very large default multiplier stops him from obtaining more benefits in the favorable development of the insurance company, which actually lowers the contract value. But for the policyholder with $(\underline{\rho}, \overline{\rho}) = (0.03, 0.03)$, $(\underline{\rho}, \overline{\rho}) = (0.03, 0.3)$, and $(\underline{\rho}, \overline{\rho}) = (0.3, 0.3)$, who not only is incapable of surrendering the contract for endogenous reasons, but also needs to liquidate the contract for

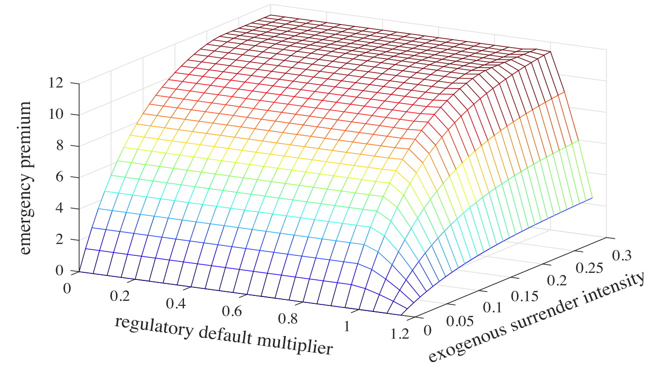


Fig. 1. Emergency premium as a function of the exogenous surrender intensity $\underline{\rho} \in [0, 0.3]$ and the default multiplier $\theta \in [0, 1.15]$.

exogenous reasons, and the policyholder with $(\underline{\rho}, \overline{\rho}) = (0.3, \infty)$, who although is able to surrender the contract at the right time, but has to surrender the contract for personal reasons with a very high probability, the contract value increases again as θ changes from 0.9 to 1.1. So if the regulator is convinced that policyholders on the market are mostly naive, a stricter solvency regulatory rule will be imposed for maximizing the protection effect. In addition, if we take a look at the column of the contract values for $\underline{\rho} = 0$ and $\theta = 1.1$, the contracts have the same value. Since when the default multiplier is so high that the benefits obtained by the policyholder at the liquidation of the insurance company are higher than the surrender benefits, surrendering the contract becomes unattractive, which means that there would also be no endogenous reasons for the policyholder to surrender his contract prematurely. Therefore, if the policyholder does not surrender his contract for exogenous reasons, i.e., $\underline{\rho} = 0$, the contract value stays at the same level as the European-style contract value of 89.6619, no matter how financially literate the policyholder is.

We have discussed in the introduction that due to personal reasons, the policyholder surrenders his contract even though he knows the value of surrender guarantee offered by the insurance company is lower than the contract value. If the policyholder has to liquidate his contract before maturity, which means that exogenous surrender does exist, i.e., $\underline{\rho} > 0$, the contract's fair value should be lower than when no exogenous surrender exists, $\underline{\rho} = 0$. We isolate the impact of such exogenous surrender on the contract's fair value by calculating the decrease in the contract value as $\underline{\rho}$ deviates from 0 while setting $\overline{\rho} = \infty$. The decrease in the contract value actually measures the premium that the insurance company should not have charged the policyholder due to his non-avoidable personal liquidity reasons. We call this premium the emergency premium. In Fig. 1 we present the emergency premium for different values of the default multiplier and different exogenous surrender intensity values. We observe the following trends. First, within the same regulatory framework, the emergency premium becomes larger as the exogenous surrender intensity increases. It implies that as the exogenous surrender intensity increases, the insurance company needs to compensate the policyholder more in terms of lowering contract value in order to make the contract more attractive to the policyholder. Second, the emergency premium increases faster at a lower θ-level while more slowly at a higher θ-level, until the exogenous surrender intensity $\underline{\rho}$ also becomes quite large. This indicates that the value of the emergency premium is more sensitive to the policyholder's exogenous surrender intensity level $\underline{\rho}$ at a lower θ-level, where the protection from the regulator is low and the insurance company needs to assess the policyholder's exogenous surrender rate more precisely in order to ensure enough compensation to the

[12] However, the policyholder would be better off if he terminates his contract and collects the surrender guarantee when the contract value drops below the surrender value. Notice that the contract value increases when $\overline{\rho} > 0$.

[13] The contract values are all around 92 for $(\underline{\rho}, \overline{\rho}) = (0, \infty)$ and $\theta = \{0, 0.7, 0.9\}$.

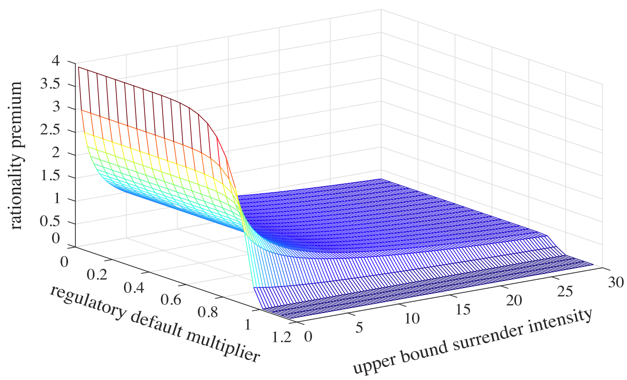


Fig. 2. Rationality premium as a function of the upper bound surrender intensity $\overline{\rho} \in [0.3, 30]$ and the default multiplier $\theta \in [0, 1.15]$.

policyholder. On the contrary, as the intervention by the regulator is enhanced, the probability that the insurance company is closed increases. Liquidation may happen before the policyholder exercises the surrender option due to exogenous reasons. Since the policyholder is not penalized at the liquidation, he may receive more than the surrender guarantee he may otherwise obtain from surrendering his contract.

Similar to the above discussion on the impact of exogenous surrender intensity on the contract value, endogenous surrender intensity also influences the contract value. Since the policyholder has limited information on the financial market and limited knowledge to correctly value the contract on his own, i.e., $\overline{\rho} < \infty$, he may fail to surrender the contract when he should do so, which results in a decrease in the contract value. Similarly, we isolate the impact of limited-rational surrender on the contract's fair value by calculating the decrease in the contract value as the upper bound surrender intensity $\overline{\rho}$ deviates from infinity while setting $\underline{\rho} = 0$. This decrease in the contract value measures the premium that the insurance company should not have charged the policyholder due to his limited information and valuation ability, which we name as the rationality premium. In Fig. 2 we plot the rationality premium as a function of the upper bound surrender intensity $\overline{\rho}$ and the default multiplier θ given $\underline{\rho} = 0$. It is natural to observe that the rationality premium decreases in $\overline{\rho}$ at a given protection level θ settled by the regulator, compare to Proposition 2. Furthermore, for a given value of $\overline{\rho}$, the rationality premium decreases as the intervention level from the regulator is enhanced until the default multiplier θ reaches 1. Intuitively, the intervention by the regulator helps the policyholder terminate the contract prematurely by shutting down the insurance company as its asset value drops below the liquidation threshold, which is often the right time to exercise the surrender option but the policyholder failed to do so. And such a termination of the contract due to the intervention of the regulator does not bring any penalty to the policyholder, which further improves the financial position of the policyholder. Consequently, as the protection level from the regulator is enhanced, the insurance company will charge the policyholder a higher price for the contract, therefore, the compensation to the policyholder is lowered. As the default multiplier θ increases above 1, the rationality premium reduces to 0 for different levels of $\overline{\rho}$, which is consistent with the contract values presented in Table 3. When the default multiplier is so high that the policyholder stops initiating a termination of the contract on his own but fully relies on the regulator's protection, the insurer will not worry about losing the policyholder due to ensuring not-enough-high compensation to the policyholder, but will price the contract by assuming $\overline{\rho} = \infty$, instead.

4.2. Effect of insurance company's investment strategy on contract valuation

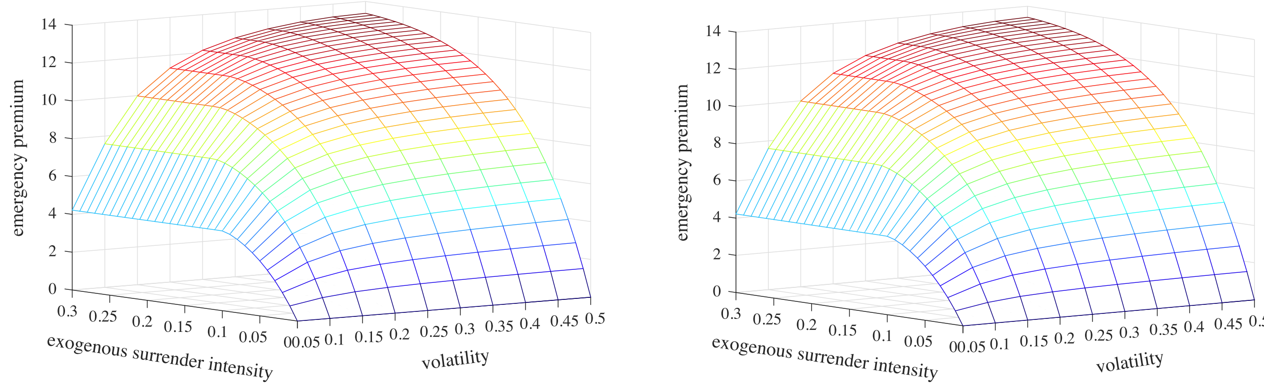
In this section we analyze the effect of the insurance company's investment strategy on the contract valuation and discuss the insurance company's risk-shifting incentives with and without the early default intervention by the regulator. The investment strategy is represented by the volatility σ of the underlying asset A. The higher the volatility σ, the higher the risk that the insurance company has entered into. In Table 4 we present the contract value for different values of the volatility σ and different rationality levels represented by $(\underline{\rho}, \overline{\rho})$. The early default multiplier θ is set to be 0.9.[14]

The effects of the company's investment strategy on the contract valuation are different for different regulatory frameworks. For the no early default case, we observe three tendencies, which depend on the policyholder's rationality level. When $(\underline{\rho}, \overline{\rho}) = (0, \infty)$ or $(0.03, \infty)$, the policyholder is considered to be financially rational because the policyholder will exercise the surrender option once his contract value is higher than the surrender value. The increase of the underlying asset risk also implies the potentially higher expected rate of return, which a rational policyholder can exactly capture. Hence, the contract value increases with the underlying asset risk. For $(\underline{\rho}, \overline{\rho}) = (0, 0)$, $(0, 0.03)$, $(0, 0.3)$ or $(0.03, 0.3)$, which indicates either low exogenous surrender intensity or relatively but not enough high endogenous surrender intensity, we observe first the increase and then the decrease in the contract value as σ increases. Intuitively, when the underlying asset risk increases but still stays at a lower level, the downside risk is still limited and the optimal surrender intensity during the contract's life time stays anyway at a lower level. However, the chance of participating in the favorable development of the asset value increases. Hence, overall the contract value increases slightly when σ increases from 0.1 to 0.2. As the asset risk increases further, the downside risk could be so high that it is necessary to check more frequently whether to surrender the contract or not. A lower endogenous surrender intensity in this case would then lead to a lower contract value. When $(\underline{\rho}, \overline{\rho}) = (0.03, 0.03)$ or $(0.3, 0.3)$, the endogenous surrender intensity is zero and the policyholder surrenders his contract only for exogenous reasons, which are not related to the contract value at all. A higher asset risk requires a more rapid and correct response to the changing market conditions. When the policyholder is not willing to do so, the contract value for the policyholder will decrease with the increase of the volatility σ. As the early default mechanism is implemented by the regulator, the contract value increases as the volatility σ increases in most cases, except when the probability that the policyholder surrenders his contract due to exogenous reasons is relatively large, i.e., $\underline{\rho} = 0.3$. From Table 4 we observe that the contract value decreases and stays constant with σ for $(\underline{\rho}, \overline{\rho}) = (0.3, 0.3)$ and $(\underline{\rho}, \overline{\rho}) = (0.3, \infty)$, respectively. In these cases, it happens that the policyholder surrenders his contract even when the asset value increases, which deprives him of the chance to participate in the asset appreciation. Since the policyholder is protected by the regulator through the early default barrier, the potential downside risk of the insurance company's investment is limited while the potential participation in the favorable asset performance is still possible. As long as the policyholder is not rushing to liquidate his contract, he can benefit more from the regulator's protection as the riskiness of the investment strategy increases and his contract value increases accordingly.

14 We have also studied the cases with $\theta = 0.7$ and 1.1. However, we have not found any qualitative differences in the effect of the volatility σ on the contract valuation and hence do not present all the results here.

Table 4
Contract value for different investment strategies represented by σ and different rationality levels represented by $(\underline{\rho}, \overline{\rho})$.

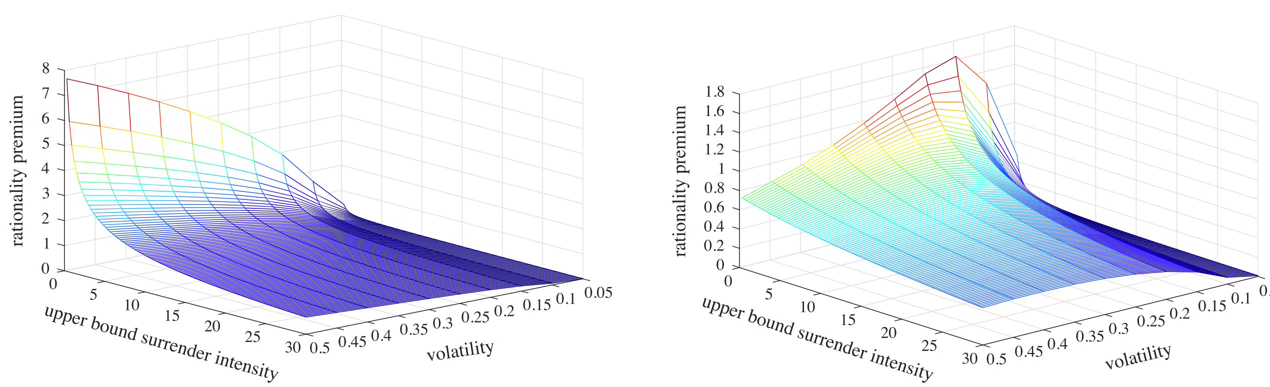
	No early default			With early default ($\theta = 0.9$)		
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
$(0, 0)$	85.3375	85.6129	84.7097	86.4174	90.3847	92.1587
$(0, 0.03)$	85.5733	86.0357	85.2479	86.5119	90.4002	92.1627
$(0, 0.3)$	86.7154	88.1519	87.9810	87.0526	90.5220	92.1967
$(0, \infty)$	88.3433	92.0665	93.3809	88.3435	92.0683	93.3880
$(0.03, 0.03)$	82.8200	81.8548	79.7099	83.7439	86.5947	87.9297
$(0.03, 0.3)$	84.0271	84.2637	83.0330	84.3370	86.7571	87.9840
$(0.03, \infty)$	85.5407	88.5460	89.6152	85.5414	88.5481	89.6212
$(0.3, 0.3)$	78.2577	75.4562	71.5569	78.4950	78.0351	77.8066
$(0.3, \infty)$	80.7500	80.7500	80.7500	80.7500	80.7500	80.7500



(a) No early default.

(b) $\theta = 0.9$.

Fig. 3. Emergency premium as a function of the exogenous surrender intensity $\underline{\rho} \in [0, 0.3]$ and the volatility $\sigma \in [0.05, 0.5]$ with and without the early default intervention.



(a) No early default.

(b) $\theta = 0.9$.

Fig. 4. Rationality premium as a function of the upper bound surrender intensity $\overline{\rho} \in [0.3, 30]$ and the volatility $\sigma \in [0.05, 0.5]$ with and without the early default intervention.

Similar to Section 4.1, we present in Fig. 3 the emergency premium under different investment strategies (adopted by the insurance company) in both the case with no early default intervention by the regulator, see Fig. 3(a), and the case with the early default mechanism, see Fig. 3(b). We see that, for a given rationality level $(\underline{\rho}, \overline{\rho})$, emergency premium increases with the volatility of the underlying asset in both cases. As the investment risk of the insurance company increases with a larger volatility σ, the probability that the policyholder sells his contract due to exogenous reasons back to the insurance company which has been experiencing financial difficulties becomes higher. This indicates that increasing the riskiness of the investment generally does harm to the policyholder who is likely to cash out of his contract when personal liquidity difficulties occur. Hence, the insurance company needs to lower the contract price more in order to attract the policyholder when its investment risk increases, which happens in both cases with and without the early default regulation. Moreover, since the regulatory intervention, as a substitute of an exogenous surrender of the policyholder, helps the policyholder close his contract prematurely without bringing him any penalty, the insurance company accordingly lowers its compensation to the policyholder.

We present the rationality premium depending on the investment strategy and the endogenous surrender intensity in Fig. 4. For a given value of $\overline{\rho}$, the rationality premium increases monotonically with the riskiness of the investment strategy σ when there is no early default mechanism. The rationality premium is much higher when the upper bound surrender intensity $\overline{\rho}$ is low. Unlike a fully rational policyholder who can track the financial performance of the insurance company and act optimally to maximize his financial benefits, a partially rational policyholder faces the risk of mistakenly holding a contract whose value is lower than the value of the surrender guarantee. This risk increases as the company's asset value becomes more volatile and is reflected by the increasing rationality premium with respect to σ. However, when the early default mechanism is imposed, the rationality premium first increases and then decreases with σ. The decreasing effect can be explained by the protection from the early default mechanism, which as a remedy for the policyholder's "insufficient" surrendering leads the insurance company to lower its compensation to the policyholder.

Due to the influence of the policyholder's rationality level and the regulatory framework on the contract value, as a response, the insurance company may change its investment strategy. We assume that the insurance company performs in the interest of the equity holder. Since the contract value can be regarded as the market value of the insurance company's liabilities when the insurance company is ongoing, the objective of the company, maximizing the residual value for the equity holder, is thus to minimize the value of the policyholder's policy. From Table 4 we can infer which investment strategy the insurance company tends to adopt. If there is no early default regulatory rule and the rationality level of the policyholder is very high, the insurance company prefers a low-risk investment strategy. This gives us two implications. First, if the policyholder is rational enough to surrender his contract, the regulator, who aims at inducing the insurance company to avoid too risky investments, does not need to interfere in the business of the insurance company. Second, looking back into the history, insurance companies have not always taken conservative investment strategies. Although there are many reasons for them not to do so. For example, the market interest rate was too low in the past and the insurance companies had to invest more riskily to achieve higher excess return so as to meet their payment obligations. An aspect that we can infer from our study is that the insurance company has actually assumed that the policyholder will not always act optimally. Considering this, it is then inappropriate to price the surrender option as a pure American-style option as it is often assumed in the literature, since the policy tends to be overpriced under this assumption which is unfair for the policyholder. Moreover, as the insurance company prices the contract by knowingly assuming an unrealistic higher rationality level and leading us to think that it will adopt an investment strategy with low risk under its assumption, the company actually has the incentive to increase the riskiness of its investment strategy afterwards. It implies that the policyholder not only pays an unfair price, but also gets hurt later as the company deviates from its presumed low-risk investment since his contract's value decreases with the riskiness of the company's investment strategy. This problem will be avoided most likely as the early default regulation is introduced. As we can read out from Table 4, the company prefers a low-risk investment strategy in all cases but one when the early default regulatory rule is present.

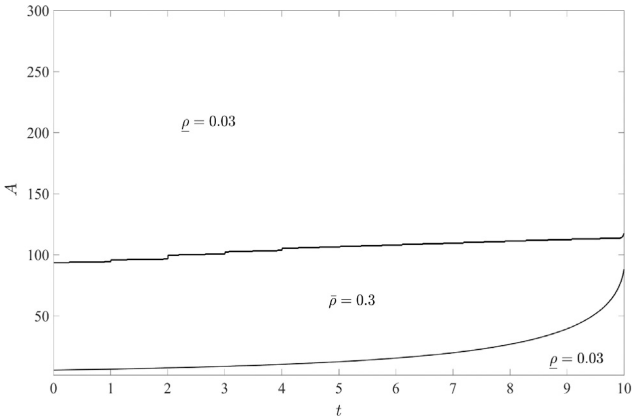
4.3. *Effects of regulatory framework and insurance company's investment strategy on policyholder's surrender*

To demonstrate the effect of the regulatory framework on the policyholder's surrender, we depict in Fig. 5 the separating surrender boundaries depending on the company's asset value A over the ten-year insuring period for the policyholder with the rationality level $(\underline{\rho}, \overline{\rho}) = (0.03, 0.3)$. The region, where the policyholder surrenders the contract for exogenous reasons, is marked with $\underline{\rho} = 0.03$, and the region, where the policyholder surrenders the contract for endogenous reasons, is marked with $\overline{\rho} = 0.3$.[15] When the early default regulation is enforced, part of the surrender regions will be replaced by the early default region in Fig. 5.

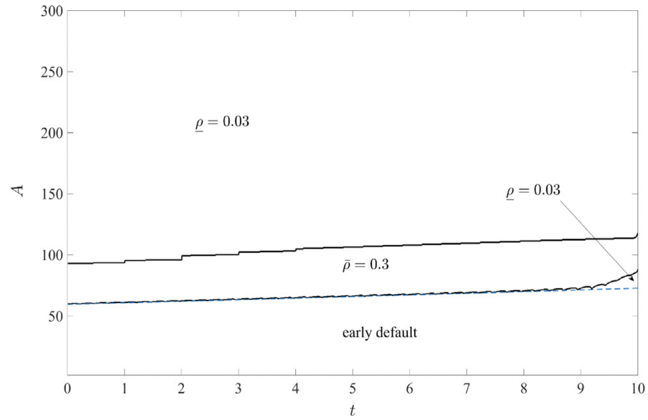
We begin with the graph for the case where there is no early default regulatory rule, see Fig. 5(a). Here we can observe three regions. When the asset price A is relatively high, the policyholder surrenders only for exogenous reasons, because participation in the insurance company's favorable asset performance is very attractive, which is, according to the contract design, only possible when the policyholder holds the contract until death or until maturity. The region in the middle of the graph corresponds to the case when the policyholder surrenders his contract for endogenous reasons. In this low-asset-value region, where participating in the company's asset appreciation is not very likely, the policyholder is sensitive to potential decreases in the company's asset value since he does not want to risk losing the minimum guarantee (or part of it) which he can obtain by closing his contract early enough. It implies that in this region the probability that the policyholder surrenders the contract increases mainly due to the reason that the policyholder wants to protect himself from the potential downside risk of the company's investment when it is still not too late to do so. However, when the asset price is very low, there would be again only exogenous surrender. In the beginning of the insuring period, since the asset value A is so low that the potential downside risk of the company's investment is naturally limited, and early surrendering carries penalty on the minimum guarantee, the policyholder would rather stay in the contract and wait for the company to recover if he does not have other exogenous surrender reasons. As time t approaches maturity, the lowest separating boundary in Fig. 5(a) moves slowly upwards, and accordingly the region where the policyholder surrenders only for exogenous reasons expands. Intuitively, compared to terminating the policy and collecting the minimum guarantee (precisely, part of it since the company's asset value is not enough high to cover the minimum guarantee) shortly before maturity, the policyholder prefers to wait a bit longer to see whether he is lucky to participate in company's asset appreciation at maturity. If the regulator intervenes, the company is closed by the regulator when its asset value is lower than the prespecified default-triggering barrier, see the early default region in the bottom of Fig. 5(b)–(d). As the default multiplier θ increases, we see that the region with $\overline{\rho} = 0.3$ where the policyholder surrenders for endogenous reasons is more and more replaced by the early default. Instead of letting the policyholder carry out the financially optimal surrender on his own, the regulator intervenes and closes the insurance company as the company's asset value deteriorates. When $\theta = 1.1$ the policyholder only surrenders the contract for exogenous reasons. This is consistent with the results presented in Fig. 2, where the rationality premium reduces to 0 when $\theta = 1.1$.

In Figs. 6 and 7 we present the separating boundaries of the policyholder's surrender behavior as the volatility σ increases from 0.1 to 0.3 when there is no early default risk and when there is early default risk, respectively, assuming the rationality level of the policyholder is $(\underline{\rho}, \overline{\rho}) = (0.03, 0.3)$. First, when $\sigma = 0.1$ and $\sigma = 0.3$, we see that the region with $\overline{\rho} = 0.3$ is larger in the case where there is no early default risk than in the case where there is early default risk, which is the same as in the case of $\sigma = 0.2$, see Fig. 5. Since the policyholder feels protected by the early default regulation imposed by the regulator, closure of the contract is more likely carried out by the regulator instead of the policyholder himself, which holds for every asset risk level σ.

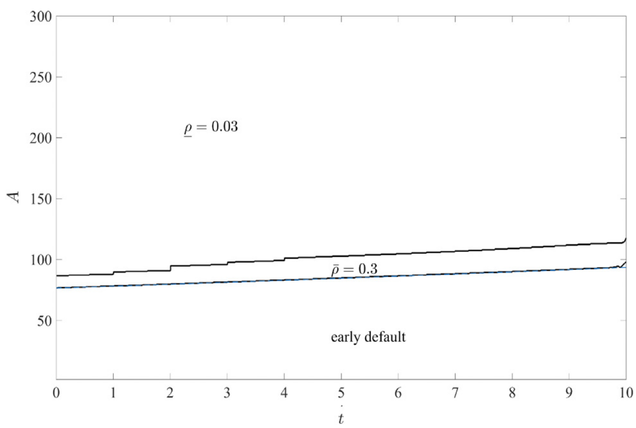
[15] Since surrender intensity has only the value of 0.03 as higher asset values, i.e., $A > 300$, are included in the figure, and accordingly the top region where the policyholder surrenders only for exogenous reasons simply just expands, we cut off the company's asset value to 300 in Fig. 5 in order to see clearly how the separating boundaries change as an early default regulatory rule is imposed and further strengthened.



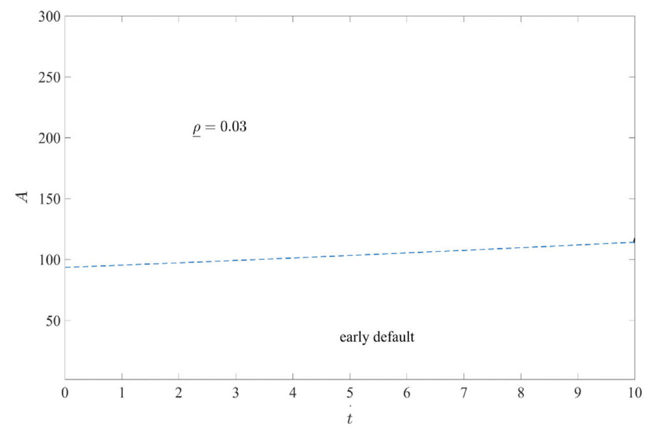
(a) No early default.



(b) Default multiplier $\theta = 0.7$.



(c) Default multiplier $\theta = 0.9$.



(d) Default multiplier $\theta = 1.1$.

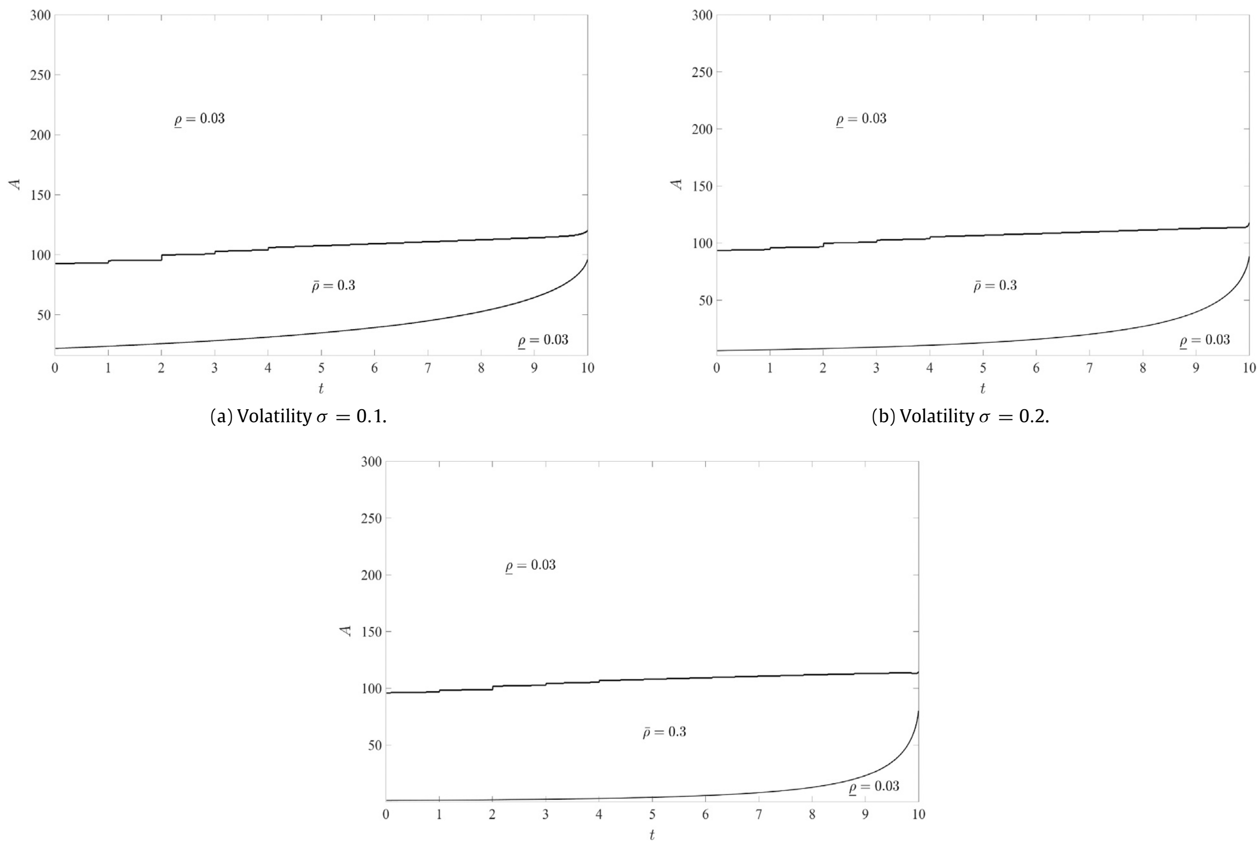
Fig. 5. Separating surrender boundaries both in the case of no early default intervention and in the case of the early default mechanism with $\theta = 0.7$, 0.9, and 1.1.

Second, as volatility σ increases the region with $\overline{\rho} = 0.3$ expands when there is no early default regulation, while it shrinks when the regulator specifies an early termination barrier during the life time of the contract. It indicates that the policyholder can become more sensitive and exercise the surrender option more likely for endogenous reasons as the financial world becomes more volatile when no early termination mechanism is specified by the regulator. However, when the early termination mechanism is introduced, the policyholder may more likely rely on the protection from the regulator as the financial world becomes more volatile so that the region $\overline{\rho} = 0.3$ shrinks as σ increases, and at the same time, the regulator's early intervention becomes more intensive.

5. Conclusion

In this paper we study the impacts of early default risk and surrender risk on participating life insurance policies. Early default of the insurance company is triggered once its asset value touches a prespecified liquidation threshold. Surrender risk is represented by a surrender intensity which is bounded from below and from above, accounting for the bounded rationality of a representative policyholder in making his surrender decision. The lower bound refers to the policyholder's surrender intensity for exogenous reasons while the upper bound is taken if the surrender value is higher than the contract value. Since the regulator's early default intervention reforms the contract's payment structure, it influences the policyholder's surrender behavior, which consequently affects the contract value. We incorporate such influence into the contract valuation by endogenizing the policyholder's surrender in this paper. We derive the pricing partial differential equation which characterizes the contract value and solve it numerically with the finite difference method. Based on the numerical examples, we analyze the influence of early default risk, given that the insurance company's investment strategy is known, on the policyholder's surrender behavior and consequently on the contract value. Furthermore, we discuss the insurance company's reaction to the regulator's intervention in terms of its investment strategy and analyze the influence of the investment strategy on the policyholder's surrender as well as on the contract value given the regulatory rule prescribed by the regulator.

The analysis for a representative policyholder can be transferred to a large pool of policyholders. Many implications can be drawn from our analysis. First, solvency regulation and policyholders' rational surrender are substitutes. If policyholders are able to surrender their participating policy optimally, it is not necessary for the regulator to set an early default regulatory rule to monitor the insurance company. In this case, the company is actually monitored by these financially literate policyholders themselves and regulator's intervention is rather redundant. However, since policyholders are mostly not rational enough to make financially optimal decisions, an early default regulatory rule can protect them. The less financially rational the policyholders are, the more they can benefit from regulator's early default intervention. Second, as one rationality level is adopted for pricing by the insurance company, financially more literate policyholders will be undercharged while financially less literate ones will be overcharged in the absence of regulator's early termination. Since financially less literate policyholders can benefit more from regulator's intervention than financially more literate ones, this inequality in the



(a) Volatility $\sigma = 0.1$.

(b) Volatility $\sigma = 0.2$.

(c) Volatility $\sigma = 0.3$.

Fig. 6. Separating surrender boundaries for different investment strategies represented by σ when there is no early default risk.

financial position of policyholders with different rationality levels can be reduced by imposing an early default regulatory rule by the regulator. If the regulator believes that policyholders in the risk pool are mostly financially illiterate, she would prescribe a quite strict regulatory rule for maximizing their financial benefits, which however could have contrary effects if the policyholders are actually rational enough to make optimal surrender decisions on their own since too strict regulation can also decrease the policyholder's contract value. Third, without the introduction of the regulatory framework, we are not clear about the insurance company's investment risk preferences. We find that the equity holder prefers to adopt a less risky investment strategy if policyholders are able to surrender their contract optimally. Since the equity holder knows that policyholders are most of the time not financially rational enough to make optimal surrender decisions and there are always exogenous reasons for them to surrender their contract prematurely, the equity holder actually tends to invest more riskily. However, when the early default barrier is set, an increase in the riskiness of the investment strategy will generally have a positive effect on the contract value. The equity holder will then have the incentive to reduce the riskiness of the investment, which is independent of the policyholders' rationality level. This result is consistent with the goal of the regulator.

Appendix A. Proof of Corollary 1

We follow the proof of Theorem 2.1 in Freidlin (1985) for a similar Dirichlet problem.

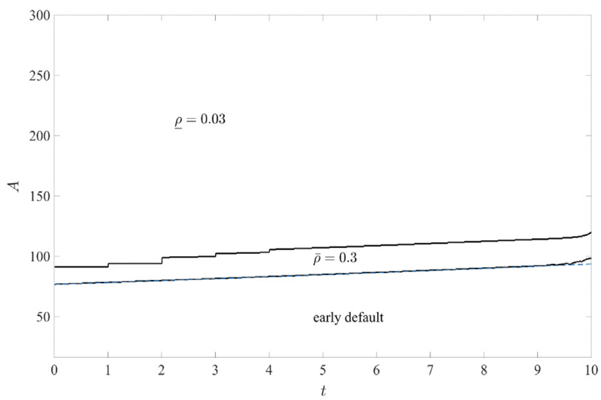
Define

$$\begin{aligned} g(t,A_t) &:= \mu(t)\Psi(t,A_t) + \gamma(t,A_t)S(t,A_t), \\ c(t,A_t) &:= r(t) + \mu(t) + \gamma(t,A_t), \\ Y_t^A &:= -\int_0^t c(z,A_z)\mathrm{d}z, \\ U_t^A &:= v(t,A_t)e^{Y_t^A}. \end{aligned}$$

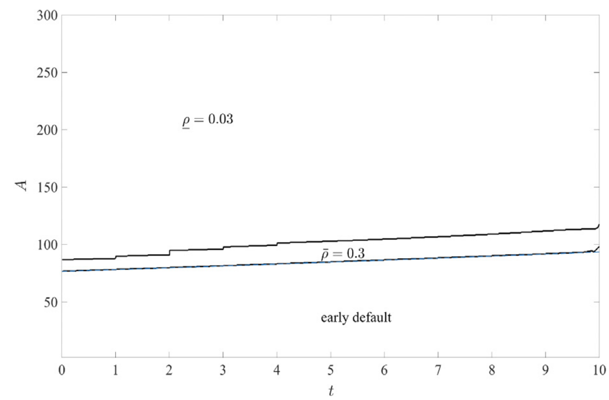
According to Ito's lemma, we obtain, for all $t < m < \tau_b \wedge T$ where $t < \tau_d \wedge \tau_s \wedge \tau_b \wedge T$, the stochastic differential equation of U_m^A as

$$\begin{aligned} \mathrm{d}U_m^A &= \frac{\partial U_m^A}{\partial m}\mathrm{d}m + \frac{\partial U_m^A}{\partial A_m}\mathrm{d}A_m + \frac{1}{2}\frac{\partial^2 U_m^A}{\partial A_m^2}\mathrm{d}A_m^2 \\ &= \left[\frac{\partial v}{\partial m}(m,A_m)e^{Y_m^A} - v(m,A_m)e^{Y_m^A}c(m,A_m)\right]\mathrm{d}m \\ &\quad + \frac{\partial v}{\partial A_m}(m,A_m)e^{Y_m^A}\left(r(m)A_m\mathrm{d}m + \sigma(m,A_m)A_m\mathrm{d}W_m\right) \\ &\quad + \frac{1}{2}\frac{\partial^2 v}{\partial A_m^2}(m,A_m)e^{Y_m^A}\sigma^2(m,A_m)A_m^2\mathrm{d}m \\ &= e^{Y_m^A}\left(\mathcal{L}v(m,A_m) - c(m,A_m)v(m,A_m)\right)\mathrm{d}m \\ &\quad + e^{Y_m^A}\frac{\partial v}{\partial A_m}(m,A_m)\sigma(m,A_m)A_m\mathrm{d}W_m \\ &= e^{Y_m^A}(-g(m,A_m))\mathrm{d}m + e^{Y_m^A}\frac{\partial v}{\partial A_m}(m,A_m)\sigma(m,A_m)A_m\mathrm{d}W_m. \end{aligned}$$

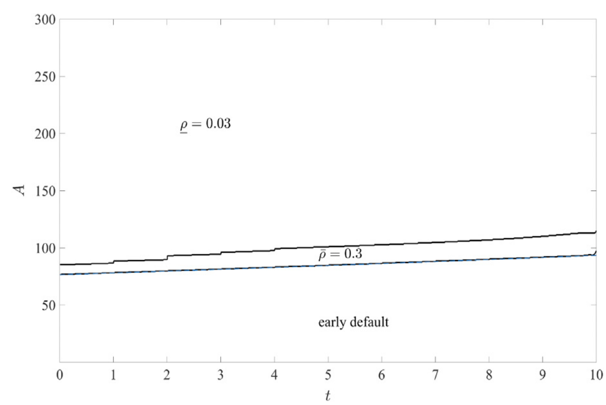
The last equation follows from Eq. (15) in Proposition 1.



(a) Volatility $\sigma = 0.1$.



(b) Volatility $\sigma = 0.2$.



(c) Volatility $\sigma = 0.3$.

Fig. 7. Separating surrender boundaries for different investment strategies represented by σ when there is early default risk.

Integrate both sides of the above equation from t to $\tau_b \wedge T$ and take the expectation on both sides. Under the assumption that

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_b\wedge T} \|e^{Y_m^A}\frac{\partial v}{\partial A_m}(m, A_m)\sigma(m, A_m)A_m\|^2 \mathrm{d}m\right] < \infty$$

which ensures

$$\mathbb{E}_{\mathbb{Q}}\left[\int_t^{\tau_b\wedge T} e^{Y_m^A}\frac{\partial v}{\partial A_m}(m, A_m)\sigma(m, A_m)A_m \mathrm{d}W_m\middle|\mathcal{G}_t\right] = 0,$$

we obtain

$$\mathbb{E}_{\mathbb{Q}}[U^A_{\tau_b\wedge T}|\mathcal{G}_t] = U_t^A - \mathbb{E}_{\mathbb{Q}}\left[\int_t^{\tau_b\wedge T} e^{Y_m^A} g(m, A_m)\mathrm{d}m\middle|\mathcal{G}_t\right],$$

and thus

$$v(t, A_t) = \mathbb{E}_{\mathbb{Q}}\left[\int_t^{\tau_b\wedge T} e^{Y_m^A - Y_t^A} g(m, A_m)\mathrm{d}m + e^{Y^A_{\tau_b\wedge T} - Y_t^A} v(\tau_b \wedge T, A_{\tau_b\wedge T})\middle|\mathcal{G}_t\right].$$

Since

$$\begin{aligned} e^{Y^A_{\tau_b\wedge T} - Y_t^A} v(\tau_b \wedge T, A_{\tau_b\wedge T}) &= 1_{\{\tau_b<T\}} e^{Y^A_{\tau_b} - Y_t^A} v(\tau_b, A_{\tau_b}) \\ &\quad + 1_{\{\tau_b\geq T\}} e^{Y^A_T - Y_t^A} v(T, A_T) \\ &= 1_{\{\tau_b<T\}} e^{Y^A_{\tau_b} - Y_t^A} \Upsilon(\tau_b, A_{\tau_b}) \\ &\quad + 1_{\{\tau_b\geq T\}} e^{Y^A_T - Y_t^A} \Phi(A_T), \end{aligned}$$

where the last equation results from the boundary conditions in Proposition 1, we obtain, by substituting $g(\cdot,\cdot)$, $c(\cdot,\cdot)$, and Y with their original forms,

$$\begin{aligned} V_t = \mathbb{E}_{\mathbb{Q}}\Bigg[&\int_t^{\tau_b\wedge T} e^{-\int_t^m (r(u)+\mu(u)+\gamma(u,A_u))\mathrm{d}u}(\mu(m)\Psi(m, A_m) \\ &+ \gamma(m, A_m)S(m, A_m))\mathrm{d}m \\ &+ 1_{\{\tau_b\geq T\}}\Phi(A_T)e^{-\int_t^T (r(u)+\mu(u)+\gamma(u,A_u))\mathrm{d}u} \\ &+ 1_{\{\tau_b<T\}}\Upsilon(\tau_b, A_{\tau_b})e^{-\int_t^{\tau_b} (r(u)+\mu(u)+\gamma(u,A_u))\mathrm{d}u}\Bigg|\mathcal{G}_t\Bigg] \end{aligned}$$

Corollary 1 is therefore proved.

Appendix B. Proof of Proposition 2

The pre-death/surrender value function v is the solution of the PDE (15) with terminal condition $v(T, A_T) = \Phi(A_T)$ and boundary condition $v(t, A_t) = \Upsilon(t, A_t)$ for $A_t \leq B_t$, and bounds for the surrender intensity $\underline{\rho}$ and $\overline{\rho}$. The pre-death/surrender value function w is the solution of the same PDE (15) with identical terminal condition $w(T, A_T) = \Phi(A_T)$ and boundary condition $w(t, A_t) = \Upsilon(t, A_t)$ for $A_t \leq B_t$ but different bounds $\underline{\zeta}$ and $\overline{\zeta}$. Assume that $\underline{\zeta} \leq \underline{\rho}$ and $\overline{\rho} \leq \overline{\zeta}$. Now define $z = w - v$. It follows directly that $z(T, A_T) = w(T, A_T) - v(T, A_T) = \Phi(A_T) - \Phi(A_T) = 0$ and $z(t, A_t) = \Upsilon(t, A_t) - \Upsilon(t, A_t) = 0$ for $A_t \leq B_t$. To obtain the dynamics of z take the difference of the PDEs describing w and v, i.e.:

$$\begin{aligned} 0 = &\mathcal{L}w(t, A_t) + \mu(t)\Psi(t, A_t) + \gamma^w(t, A_t)S(t, A_t) \\ &- (r(t) + \mu(t) + \gamma^w(t, A_t))\, w(t, A_t) \\ &- (\mathcal{L}v(t, A_t) + \mu(t)\Psi(t, A_t) + \gamma^v(t, A_t)S(t, A_t) \end{aligned}$$

$$
\begin{aligned}
&-(r(t)+\mu(t)+\gamma^{v}(t,A_t))\,v(t,A_t))\\
&=\mathcal{L}z(t,A_t)+(\gamma^{w}(t,A_t)-\gamma^{v}(t,A_t))(S(t,A_t)-w(t,A_t))\\
&\quad-(r(t)+\mu(t)+\gamma^{v}(t,A_t))\,z(t,A_t),
\end{aligned}
$$

where γ^v and γ^w, respectively, are given by (7) using the appropriate bounds. Similar to the proof of Corollary 1, we obtain the integral representation of z as follows:

$$
\begin{aligned}
z(t,A)=\mathbb{E}_{\mathbb{Q}}^{t,A}\Bigg[\int_t^{\tau_b\wedge T} & e^{-\int_t^m(r(u)+\mu(u)+\gamma^v(u,A_u))\,\mathrm{d}u}(\gamma^w(m,A_m)\\
&-\gamma^v(m,A_m))(S(m,A_m)\\
&-w(m,A_m))\mathrm{d}m\bigg|\mathcal{G}_t\Bigg],
\end{aligned}
$$

where $\mathbb{E}_{\mathbb{Q}}^{t,A}$ denotes the expectation conditioned on $A_t=A$. From the definition of γ^w in (7) and the assumption $\overline{\zeta}\geq\overline{\rho}$ we see that if $(S-w)\geq 0$ we have $\gamma^w=\overline{\zeta}\geq\overline{\rho}\geq\gamma^v$ and thus $(\gamma^w-\gamma^v)\geq 0$. On the other hand, if $(S-w)<0$ then $\gamma^w=\underline{\zeta}$. By assumption we have $\underline{\zeta}\leq\underline{\rho}$ and thus $\gamma^w\leq\underline{\rho}\leq\gamma^v$, or, $(\gamma^w-\gamma^v)\leq 0$. Thus, we see that the integrand in the above equation is nonnegative and therefore $z\geq 0$. Since $z=w-v$ we obtain $w\geq v$.

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