Computers & Industrial Engineering 115 (2018) 543–558

Contents lists available at ScienceDirect

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

Fuzzy risk analysis in poultry farming using a new similarity measure on generalized fuzzy numbers

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ARTICLE INFO

Keywords:
Similarity measure
Value
Ambiguity
Radius of gyration
Risk analysis
Poultry farming

ABSTRACT

Similarity measure of fuzzy numbers plays an important role in the risk analysis problem. Generally, it is tool, which gives lingustic term to the risk obtained. In recent times, a vast numbers of literature are evident on application of similarity measure in risk analysis. It has been observed that the existing similarity measure on fuzzy numbers have numerous drawbacks and limitations. Hence, a robust method of similarity measure is necessary. With this point of view, a new method to measure the degree of similarity between fuzzy numbers has been proposed. The method has been discussed based on the concept of value, ambiguity, radius of gyration point, geometric distance and the height of fuzzy numbers. The concept of value and ambiguity have never been used in similarity measure of fuzzy numbers. However, the inclusion of these concepts value and ambiguity contributed in many ways in overcoming the limitations and drawbacks of the existing similarity measures. The out-performance of the proposed method is illustrated by comparing with existing methods of similarity measure. Further the proposed method is effectively applied in risk analysis of poultry farming.

1. Introduction

Fuzzy risk analysis has become very popular in recent times as the knowledge of expressing imprecise quantity in terms of fuzzy numbers has emerged. Most of the time similarity measure between fuzzy numbers is used in the risk analysis problem and other decision making problem. The similarity measures are defined on the different characteristic of the fuzzy number such as geometric distance, center of gravity (COG), area, radius of gyration (ROG) etc. Further, these measures are being generalized for use in different types of fuzzy numbers. It has been observed that the existing similarity measures on fuzzy numbers bear various limitations and drawbacks.

A review of some of the existing methods to measure the degree of similarity reveals various limitations and drawbacks. Chen (1996) defined a similarity measure based on the geometric distance. This definition does not carry the information about the shape of the fuzzy numbers such as triangular, trapezoidal, etc. Hence, in many circumstances this method fails to give a proper degree of similarity between fuzzy numbers. Hsieh and Chen (1999) proposed a similarity measure between two fuzzy numbers using graded mean integration representation distance. This method has no contribution from heights and shapes of the fuzzy numbers. Hence, the method is confined to normal fuzzy numbers. As like Hsieh and Chen's method Lee's (2002) method is just confined to normal fuzzy numbers. As such, it is not going to give correct similarity between fuzzy numbers having different heights and shapes. So far, the information about the heights is missing in the similarity measures. Hence, Chen and Chen (2001) developed a similarity measure for generalized fuzzy number (GFN) using the concept of the COG. Although this method seems to outperform in many situations, yet drawbacks are obtained in some situations as discussed in the Section 3. Replacing Chen and Chen's COG by ROG, Yong, Wenkang, Feng, and Qi (2004) proposed a new similarity measure and applied in pattern recognition problems. The method seems very promising. However, it fails to give proper similarity between crisp-valued fuzzy numbers. Wei and Chen (2009) proposed a measure based on the geometric distance and the perimeter of the fuzzy numbers. However, the method fails to give proper similarity between fuzzy numbers depicting similar shape located at different positions. Xu, Shang, Qian, and Shu (2010) again used the COG and the geometric distance in measuring the degree of similarity between GFNs. Although the method is based on GFNs yet it fails to measure similarity between fuzzy numbers depicting similar shape with different heights. Hejazi, Doostparast, and Hosseini (2011) used the concept of geometric distance, perimeter, area and height to discuss the degree of similarity. However, the drawbacks are pointed out by Patra and Mondal (2015). Recent study of similarity based on area, geometric distance and height

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https://doi.org/10.1016/j.cie.2017.11.001
Received 21 April 2017; Received in revised form 3 October 2017; Accepted 1 November 2017
0360-8352/

Nomenclature

A_ω	fuzzy number with height ω
A_{ω_1,ω_2}	fuzzy number with left height ω_1 and right height ω_2
μ_A	membership function fuzzy number A
$\mathrm{ar}(A)$	area of the fuzzy number A
$\mathrm{Amb}(A)$	ambiguity of the fuzzy number A
$\mathrm{Val}(A)$	value of the fuzzy number A
I_x	moment of inertia with respect to x-axis
I_y	moment of inertia with respect to y-axis
r_x	radius of gyration point with respect to x-axis
r_y	radius of gyration point with respect to y-axis
(x_A^*,y_A^*)	center of gravity point of the fuzzy number A
$P(A)$	perimeter of the fuzzy number A
(r_x^A,r_y^A)	radius of gyration point of the fuzzy number A
$S(A,B)$	similarity measure between fuzzy numbers A and B

was done by Patra and Mondal. In many situations, its drawbacks are obtained as discussed in Section 3. Moreover, a very recent study by Khorshidi and Nikfalazar (2017) clearly criticized the study by Patra and Mondal pointing out its drawbacks. Khorshidi and Nikfalazar (2017) in 2017 developed a modified method to measure the degree of similarity. This method is based on the existing concepts such as geometric distance, COG, areas, perimeters and heights of the GFNs. This method seems to outperform in many situations. Eventually, its drawback has been obtained as discussed in Section 3.

As mentioned earlier, similarity measure is often used in the risk analysis problem. Schmucke (1984) first introduced the fuzzy risk analysis in production system using the parameters probability of failure and severity of loss. Different researchers have proposed different methods at different times for the risk analysis problem. Most of the times, due to its nature the parameters involved in those risk analysis problems are expressed as linguistic terms. Kangari and Riggs (1989) proposed a method of risk analysis using linguistic terms. Some of the studies involving risk analysis are Chen (1996), Chen and Chen (2001, 2003, 2007), Tang and Chi (2005), Wang and Elhag (2006) etc. Even in recent years some study focused the idea on risk analysis expressing the linguistic terms in terms of interval-valued fuzzy numbers (Gorzalczany, 1987; Guijun & Xiaoping, 1998; Hong & Lee, 2002; Wang & Li, 1999).

This study conveys that a proper and efficient method to measure the degree of similarity is lacking. Hence, a robust method of similarity measure of GFNs has been proposed. The proposed method is based on the concepts of geometric distance, value, ambiguity, area and heights of the GFN. The proposed method seems to outperform in all situations as discussed by the numerical examples in the comparative study in Section 5. The method has been discussed using the concept of GFN with different left heights and right heights. The method is not just confined to GFN with different left heights and right heights, but also can handle all types of fuzzy numbers. Further, effort has been made to apply the proposed similarity measure in the risk analysis problem. A real-life problem of risk analysis in poultry farming has been demonstrated. The parameters probability of failure and severity of loss are expressed by linguistic terms. Under the assumed parameters the total risk of probability of failure using the proposed similarity measure turn out to be 'Fairly low'. Hence, under such circumstances a farmer can successfully establish a poultry farm for self-employment.

The rest of the paper is organized as follows. Section 2 introduces the basic definitions of GFN and also related definitions to the discussions. Section 3 refers to a brief review of the existing method of similarity measure and also the limitations and drawbacks are pointed out. Section 4 proposes a new similarity measure of GFN. Also, its properties and main characteristic are discussed. In Section 5 a comparative study through numerical examples to highlight the advantages of the proposed similarity measure has been performed. In Section 6 a risk analysis on real-life problem on poultry farming has been performed. Finally, in Section 7 conclusions and main features of the proposed method are highlighted.

2. Definitions and notations

In this section, brief review of some concepts of GFN with different left height and right height are put forwarded.

Definition 2.1. If X is a collection of objects, then a fuzzy set A in X is a set of ordered pairs:

$$A = \{(x,\mu_A(x)): x \in X, \mu_A: X \to [0,1]\}. \tag{1}$$

Definition 2.2 (*Basu, 2005*). A null set is denoted by Φ, and is that fuzzy set for which the membership grade for each element is zero. Thus,

$$\Phi = \{(x,\mu_A(x)): x \in X, \mu_A(x) = 0\}. \tag{2}$$

Definition 2.3. A fuzzy number A_ω is an ordered pair $(\underline{A}_\omega(r),\overline{A}_\omega(r))$ of functions $\underline{A}_\omega(r)$ and $\overline{A}_\omega(r), 0 \leqslant r \leqslant \omega$, satisfying the following properties:

(1) $\underline{A}_\omega(r)$ is a bounded monotonic increasing left continuous function over the interval $[0,\omega]$,
(2) $\overline{A}_\omega(r)$ is a bounded monotonic decreasing left continuous function over the interval $[0,\omega]$,

where $0 \leqslant \omega \leqslant 1$ is the height.

Consider a trapezoidal fuzzy number $A_\omega = (a_1,a_2,a_3,a_4;\omega)$ with height ω, then the membership function is defined as

$$\mu_{A_\omega}(x) = \begin{cases} \frac{\omega(x-a_1)}{a_2-a_1}, & \text{if } a_1 \leqslant x \leqslant a_2,\\ \omega, & \text{if } a_2 \leqslant x \leqslant a_3,\\ \frac{\omega(a_4-x)}{a_4-a_3}, & \text{if } a_3 \leqslant x \leqslant a_4,\\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where ω is the height of the fuzzy number. Then, the functions $\underline{A}_\omega(r)$ and $\overline{A}_\omega(r)$ are defined as

$$\underline{A}_\omega(r) = a_1 + \frac{r}{\omega}(a_2-a_1), \overline{A}_\omega(r) = a_4 - \frac{r}{\omega}(a_4-a_3). \tag{4}$$

respectively. If $\omega = 1$, then the fuzzy number A_ω is called as normal fuzzy number otherwise non-normal fuzzy number. If $a_2 = a_3$, then it is non-normal triangular fuzzy number.

Chen, Munif, Chen, Liu, and Kuo (2012) first proposed the concept of GFN with different left heights and right heights. Later its parametric form has been defined by Chutia and Chutia (2017). Let A_{ω_1,ω_2} is represented by $A_{\omega_1,\omega_2} = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$ on the real line $\mathbb{R}$ is called a GFN with different left heights and right heights, where a_1,a_2,a_3 and a_4 are real values, ω_1 is called the left height and ω_2 is called the right height of it where $\omega_1 \in [0,1]$ and $\omega_2 \in [0,1]$. If $\omega_1 = \omega_2 = 1$, then the GFN A_{ω_1,ω_2} reduces to a normal trapezoidal fuzzy number. If $0 \leqslant \omega_1 = \omega_2 \leqslant 1$, then the fuzzy number A_{ω_1,ω_2} is simply a GFN proposed by Chen (1985).

Definition 2.4. A GFN A_{ω_1,ω_2} with different heights ω_1 and ω_2 for $0 \leqslant r \leqslant \max(\omega_1,\omega_2)$ is represented as follows:

(1) if $\omega_1 < \omega_2$, then

$$p(r) = \begin{cases} [\underline{A}_{\omega_1,\omega_2}(r),\overline{A}_{\omega_1,\omega_2}(r)], & \text{if } 0 \leqslant r \leqslant \omega_1,\\ [\widetilde{A}_{\omega_1,\omega_2}(r),\overline{A}_{\omega_1,\omega_2}(r)], & \text{if } \omega_1 \leqslant r \leqslant \omega_2, \end{cases} \tag{5}$$

(2) if $\omega_1 > \omega_2$, then

$$p(r) = \begin{cases} [\underline{A}_{\omega_1,\omega_2}(r), \overline{A}_{\omega_1,\omega_2}(r)], & \text{if } 0 \leqslant r \leqslant \omega_2, \\ [\underline{A}_{\omega_1,\omega_2}(r), \widetilde{A}_{\omega_1,\omega_2}(r)], & \text{if } \omega_2 \leqslant r \leqslant \omega_1, \end{cases} \tag{6}$$

where $0 \leqslant \max(\omega_1,\omega_2) \leqslant 1$ and the functions $\underline{A}_{\omega_1,\omega_2}(r), \overline{A}_{\omega_1,\omega_2}(r)$ and $\widetilde{A}_{\omega_1,\omega_2}(r)$ satisfies the following properties:

(a) $\underline{A}_{\omega_1,\omega_2}(r)$ is a bounded monotonic increasing (non-decreasing) left continuous function over $[0,\omega_1]$,
(b) $\overline{A}_{\omega_1,\omega_2}(r)$ is a bounded monotonic decreasing (non-increasing) left continuous function over $[0,\omega_2]$,
(c) $\widetilde{A}_{\omega_1,\omega_2}(r)$ is a bounded and monotonic increasing or decreasing left continuous function over .

This definition is an generalized one on which proper substitution of $\omega_1 = \omega_2 = \omega$ (say) will reduce to GFN A_ω (Eqs. (4)) with equal heights. Consider a GFN $A_{\omega_1,\omega_2} = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$ with unequal heights, then the membership function as shown in Fig. 1 is given by

$$\mu_{A_{\omega_1,\omega_2}}(x) = \begin{cases} \frac{\omega_1(x-a_1)}{a_2-a_1}, & \text{if } a_1 \leqslant x \leqslant a_2; \\ \frac{\omega_1(a_3-a_2)+(\omega_2-\omega_1)(x-a_2)}{a_3-a_2}, & \text{if } a_2 \leqslant x \leqslant a_3; \\ \frac{\omega_2(a_4-x)}{a_4-a_3}, & \text{if } a_3 \leqslant x \leqslant a_4; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

where $0 \leqslant \omega_1 \leqslant 1, 0 \leqslant \omega_2 \leqslant 1$ and ω_1 may not be equal to ω_2. If $\omega_1 = \omega_2$, then the fuzzy number reduces to non-normal fuzzy number. Thus the parametric form of the GFN A_{ω_1,ω_2} with different left height and right height for $0 \leqslant r \leqslant 1$ is defined as follows:

(1) If $\omega_1 < \omega_2$, then

$$p(r) = \begin{cases} \left[\frac{a_1\omega_1 + r(a_2-a_1)}{\omega_1}, \frac{a_4\omega_2 + r(a_3-a_4)}{\omega_2}\right], & \text{if } 0 \leqslant r \leqslant \omega_1; \\ \left[\frac{a_2(\omega_2-\omega_1)+(r-\omega_1)(a_3-a_2)}{\omega_2-\omega_1}, \frac{a_4\omega_2 + r(a_3-a_4)}{\omega_2}\right], & \text{if } \omega_1 \leqslant r \leqslant \omega_2. \end{cases} \tag{8}$$

(2) If $\omega_1 > \omega_2$, then

$$p(r) = \begin{cases} \left[\frac{a_1\omega_1 + r(a_2-a_1)}{\omega_1}, \frac{a_4\omega_2 + r(a_3-a_4)}{\omega_2}\right], & \text{if } 0 \leqslant r \leqslant \omega_2; \\ \left[\frac{a_1\omega_1 + r(a_2-a_1)}{\omega_1}, \frac{a_2(\omega_2-\omega_1)+(r-\omega_1)(a_3-a_2)}{\omega_2-\omega_1}\right], & \text{if } \omega_2 \leqslant r \leqslant \omega_1. \end{cases} \tag{9}$$

Definition 2.5. Area under the membership function of a GFN $A_{\omega_1,\omega_2} = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$, described by the membership function (7) with different left heights and right heights, is defined as the cardinality of it. It is given by the integral

$$\mathrm{ar}(A_{\omega_1,\omega_2}) = \int_{a_1}^{a_4} \mu_{A_{\omega_1,\omega_2}}(x)dx = \frac{1}{2}\{\omega_2(a_4-a_2) + \omega_1(a_3-a_1)\} \tag{10}$$

Definition 2.6 (Delgado, Vila, and Voxman, 1998). Let a fuzzy number A_ω denoted by the ordered pair $(\underline{A}_\omega(r), \overline{A}_\omega(r))$ and s: $[0,1] \to [0,1]$ be a reducing function. Then, the ambiguity of A_ω with respect to s is defined as

$$\mathrm{Amb}(A_\omega) = \int_0^1 s(r)(\overline{A}_\omega(r) - \underline{A}_\omega(r))dr. \tag{11}$$

Definition 2.7 (Delgado et al., 1998). Let a fuzzy number A_ω denoted by the ordered pair $(\underline{A}_\omega(r), \overline{A}_\omega(r))$ and s: $[0,1] \to [0,1]$ be a reducing function. Then, the value of A_ω with respect to s is defined as

$$\mathrm{Val}(A_\omega) = \int_0^1 s(r)(\overline{A}_\omega(r) + \underline{A}_\omega(r))dr. \tag{12}$$

Hence, the definition of the value and the ambiguity of an arbitrary fuzzy number with different heights with respect to the parametric forms given in Eqs. (5) and (6) are as follows:

Definition 2.8. Let A_{ω_1,ω_2} be an arbitrary fuzzy number with different heights ω_1 and ω_2, then the value with respect to the reducing function s is defined as

(1) If $\omega_1 < \omega_2$, then

$$\mathrm{Val}(A_{\omega_1,\omega_2}) = \int_0^{\omega_1} s(r)(\overline{A}_{\omega_1,\omega_2}(r) + \underline{A}_{\omega_1,\omega_2}(r))dr + \int_{\omega_1}^{\omega_2} s(r)(\overline{A}_{\omega_1,\omega_2}(r) + \widetilde{A}_{\omega_1,\omega_2}(r))dr. \tag{13}$$

(2) If $\omega_1 > \omega_2$, then

$$\mathrm{Val}(A_{\omega_1,\omega_2}) = \int_0^{\omega_2} s(r)(\overline{A}_{\omega_1,\omega_2}(r) + \underline{A}_{\omega_1,\omega_2}(r))dr + \int_{\omega_2}^{\omega_1} s(r)(\widetilde{A}_{\omega_1,\omega_2}(r) + \underline{A}_{\omega_1,\omega_2}(r))dr. \tag{14}$$

(3) If $\omega_1 = \omega_2 = \omega$, then A_{ω_1,ω_2} would reduce to A_ω; hence,

$$\mathrm{Val}(A_\omega) = \int_0^{\omega} s(r)(\overline{A}_\omega(r) + \underline{A}_\omega(r))dr. \tag{15}$$

Definition 2.9. Let A_{ω_1,ω_2} be an arbitrary fuzzy number with different heights ω_1 and ω_2, then the ambiguity with respect to the reducing function s is defined as

(1) If $\omega_1 < \omega_2$, then

$$\mathrm{Amb}(A_{\omega_1,\omega_2}) = \int_0^{\omega_1} s(r)(\overline{A}_{\omega_1,\omega_2}(r) - \underline{A}_{\omega_1,\omega_2}(r))dr + \int_{\omega_1}^{\omega_2} s(r)(\overline{A}_{\omega_1,\omega_2}(r) - \widetilde{A}_{\omega_1,\omega_2}(r))dr. \tag{16}$$

(2) If $\omega_1 > \omega_2$, then

$$\mathrm{Amb}(A_{\omega_1,\omega_2}) = \int_0^{\omega_2} s(r)(\overline{A}_{\omega_1,\omega_2}(r) - \underline{A}_{\omega_1,\omega_2}(r))dr + \int_{\omega_2}^{\omega_1} s(r)(\widetilde{A}_{\omega_1,\omega_2}(r) - \underline{A}_{\omega_1,\omega_2}(r))dr. \tag{17}$$

(3) If $\omega_1 = \omega_2 = \omega$, then A_{ω_1,ω_2} would reduce to A_ω; hence,

$$\mathrm{Amb}(A_\omega) = \int_0^{\omega} s(r)(\overline{A}_\omega(r) - \underline{A}_\omega(r))dr \tag{18}$$

Definition 2.10. Assume that $A_{\omega_1,\omega_2} = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$ and $B_{\omega_1',\omega_2'} = (b_1,b_2,b_3,b_4;\omega_1',\omega_2')$ are two GFNs where $a_i,b_i,i = 1,2,3,4$ are real values and $0 \leqslant \omega_1,\omega_2 \leqslant 1, 0 \leqslant \omega_1',\omega_2' \leqslant 1$. The arithmetic operation for these GFNs are defined as below:

(1) Addition of fuzzy numbers $\oplus$:

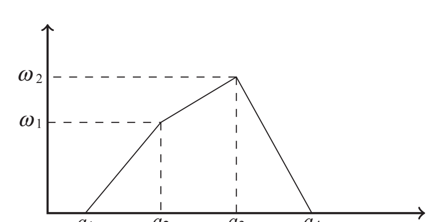

Fig. 1. Graphical representation of GFN with different left height and right height.

$$A_{\omega_1,\omega_2} \oplus B_{\omega_1',\omega_2'} = (a_1,a_2,a_3,a_4;\omega_1,\omega_2) \oplus (b_1,b_2,b_3,b_4;\omega_1',\omega_2'),$$
$$= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4; \min(\omega_1,\omega_2'), \min(\omega_1,\omega_2')),$$

where a_i and b_i for $i = 1,2,3,4$ are any real numbers.

(2) Subtraction of fuzzy numbers $\ominus$:

$$A_{\omega_1,\omega_2} \ominus B_{\omega_1',\omega_2'} = (a_1,a_2,a_3,a_4;\omega_1,\omega_2) \ominus (b_1,b_2,b_3,b_4;\omega_1',\omega_2'),$$
$$= (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1; \min(\omega_1,\omega_2'), \min(\omega_1,\omega_2')),$$

where a_i and b_i for $i = 1,2,3,4$ are any real numbers.

(3) Multiplication of fuzzy numbers $\otimes$:

$$A_{\omega_1,\omega_2} \otimes B_{\omega_1',\omega_2'} = (a_1,a_2,a_3,a_4;\omega_1,\omega_2) \otimes (b_1,b_2,b_3,b_4;\omega_1',\omega_2'),$$
$$= (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4; \min(\omega_1,\omega_2'), \min(\omega_1,\omega_2')),$$

where a_i and b_i for $i = 1,2,3,4$ are any real numbers.

(4) Division of fuzzy numbers $\varnothing$:

$$A_{\omega_1,\omega_2} \varnothing B_{\omega_1',\omega_2'} = (a_1,a_2,a_3,a_4;\omega_1,\omega_2) \varnothing (b_1,b_2,b_3,b_4;\omega_1',\omega_2'),$$
$$= (a_1 \div b_4, a_2 \div b_3, a_3 \div b_2, a_4 \div b_1; \min(\omega_1,\omega_2'), \min(\omega_1,\omega_2')),$$

where a_i for $i = 1,2,3,4$ are any positive real numbers and b_i for $i = 1,2,3,4$ are any non-zero positive real numbers.

Definition 2.11 (*Yong et al., 2004*). Consider an element dR of coordinates x and y of an area R located in the xy-plane as shown in Fig. 2. Then, the moment of inertia of the area R with respect to the x and y axis are defined, respectively, as

$$I_x = \int_R y^2 dR \tag{19}$$

and

$$I_y = \int_R x^2 dR. \tag{20}$$

Definition 2.12 (*Yong et al., 2004*). The ROG point of an area R with respect to x and y axis r_x and r_y are given by the relations $I_x = r_x^2 R$ and $I_y = r_y^2 R$ respectively. Then, r_x and r_y are defined, respectively, as

$$r_x = \sqrt{\frac{I_x}{R}}, \tag{21}$$

and

$$r_y = \sqrt{\frac{I_y}{R}}. \tag{22}$$

Note 1. Hereafter, the heights of fuzzy number will not be denoted as suffices in the fuzzy number. Hence, it will be represented just by the notation $A = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$ which will be well understood as fuzzy number with left height ω_1 and right height ω_2. Also GFN $A = (a_1,a_2,a_3,a_4;\omega)$ will be understood as fuzzy number with height ω.

3. Review of some existing similarity measures between fuzzy numbers and its limitations

In this section, a brief review some of the existing similarity measures of fuzzy numbers have been forwarded. It has been observed that the similarity measures available in the literature have some drawbacks and limitations. Hence, the drawbacks and limitations of different methods are also discussed through numerical examples.

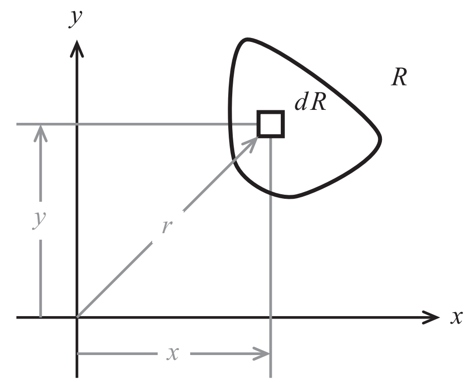

Fig. 2. The moment of inertia of an area R.

3.1. Chen's similarity measures between fuzzy numbers

Consider two GFNs given by $A = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$ and $B = (b_1,b_2,b_3,b_4;\omega_1',\omega_2')$, then Chen (1996) defined the degree of similarity $S(A,B)$ between the fuzzy numbers A and B as follows:

$$S(A,B) = 1 - \frac{\Sigma_{i=1}^{4}|a_i - b_i|}{4}. \tag{23}$$

where $0 \leqslant S(A,B) \leqslant 1$.

Consider the fuzzy numbers $A = (0.2,0.3,0.4,0.5;1.0,1.0)$, $B = (0.5,0.65,0.65,0.8;1.0,1.0)$ and $C = (0.5,0.6,0.7,0.8;1.0,1.0)$ as shown in Fig. 3(a) and (b). Then, the similarity measures are given as below.

$$S(A,B) = \left(1 - \tfrac{1}{4}(0.30 + 0.35 + 0.25 + 0.30)\right) = 0.7$$
$$S(A,C) = \left(1 - \tfrac{1}{4}(0.30 + 0.30 + 0.30 + 0.30)\right) = 0.7$$

Now, the similarity measure for the sets (A,B) and (A,C) are the same, but one can see from Fig. 3(a) and (b) that the similarity measures between these two sets cannot be same. Hence, there are drawbacks in the method proposed by Chen (1996). This drawback might have been overcome by the inclusion of the area difference between the fuzzy numbers in the sets. This will be further discussed in the later section in comparison to the proposed method.

3.2. Hsieh and Chen's similarity measures between fuzzy numbers

Consider the GFNs given by $A = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$ and $B = (b_1,b_2,b_3,b_4;\omega_1',\omega_2')$, then the similarity measure of these GFNs using graded mean integration representation distance was forwarded by Hsieh and Chen (1999). This measure follows as

$$S(A,B) = \frac{1}{1 + d(A,B)}, \tag{24}$$

where $d(A,B) = |R(A) - R(B)|$ and

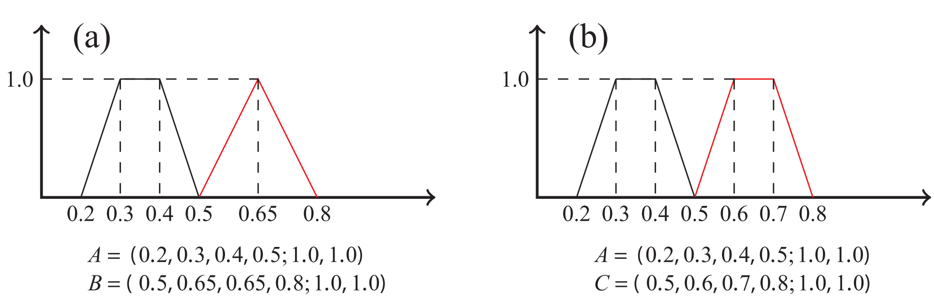

Fig. 3. Graphical representation of different sets of GFNs A,B and C.

$$R(A) = \frac{a_1 + 2(a_2 + a_3) + a_4}{6}, \tag{25}$$

$$R(B) = \frac{b_1 + 2(b_2 + b_3) + b_4}{6}. \tag{26}$$

Consider the sets of GFNs (A,B) and (A,C) shown in Fig. 3(a) and (b). Then, $S(A,B) = 0.769$ and $S(A,C) = 0.769$ which are again same. As discussed earlier similarity measure between these two sets cannot be same which is evident from the graphical representation in Fig. 3(a) and (b). Hence, there are drawbacks in this method.

3.3. Lee's similarity measures between fuzzy numbers

Lee (2002) proposed a similarity measure between two GFNs $A = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$ and $B = (b_1,b_2,b_3,b_4;\omega_1',\omega_2')$ as

$$S(A,B) = 1 - \frac{\|A - B\|_{l_p}}{\|U\|} \times 4^{-\frac{1}{p}}, \tag{27}$$

where

$$\|A - B\|_{l_p} = \left(\sum_{i=1}^{4} (|a_i - b_i|)^p\right)^{\frac{1}{p}} \tag{28}$$

and $\|U\| = \max(U) - \min(U)$, and U is the universe of discourse.

Consider the sets of GFNs (A,B) and (A,C) as shown in Fig. 3(a) and (b). As discussed earlier the similarity measure between these sets cannot be same, but by Lee's method (Lee, 2002) $S(A,B) = 0.50$ and $S(A,C) = 0.50$. Consider another set of GFN $P = Q = (1.0,1.0,1.0,1.0;1.0,1.0)$ as shown in Fig. 4 which are two identical real numbers, infact crisp-valued fuzzy numbers. According to Lee's method (Lee, 2002) $S(P,Q) = \infty$. Hence, it can be seen that this method cannot properly describe the similarity degree between two identical crisp-valued fuzzy numbers. Intuitively, the similarity measure between P and Q should be 1.

In the above discussed method it is being observed that the formula $S(A,B)$ is independent of the heights of the fuzzy numbers. Hence, those limitations arises. To overcome such limitations following are the methods that had been developed.

3.4. Chen and Chen's similarity measures between fuzzy numbers

Chen and Chen (2001) developed a similarity measure between GFN with different heights through the concept of COG. If $A = (a_1,a_2,a_3,a_4;\omega_A)$ is a GFN with height $0 \leqslant \omega_A \leqslant 1$, then COG point (x_A^*,y_A^*) is defined as follows:

$$y_A^* = \begin{cases} \dfrac{\omega_A \times \left(\frac{a_3 - a_2}{a_4 - a_1} + 2\right)}{6}, & \text{if } a_4 \neq a_1 \text{ and } 0 < \omega_A \leqslant 1; \\ \dfrac{\omega_A}{2}, & \text{if } a_4 = a_1 \text{ and } 0 < \omega_A \leqslant 1, \end{cases} \tag{29}$$

$$x_A^* = \frac{y_A^* \times (a_3 + a_2) + (a_4 + a_1)(\omega_A - y_A^*)}{2\omega_A}. \tag{30}$$

Then, the degree of similarity measure $S(A,B)$ between the GFNs $A = (a_1,a_2,a_3,a_4;\omega_A)$ and $B = (b_1,b_2,b_3,b_4;\omega_B)$ and with different heights ω_A and ω_B is defined as

$$S(A,B) = \left(1 - \frac{1}{4}\sum_{i=1}^{4} |a_i - b_i|\right)(1 - |x_A^* - x_B^*|)^{B(s_A,s_B)} \times \frac{\min(y_A^*,y_B^*)}{\max(y_A^*,y_B^*)}, \tag{31}$$

where $s_A = a_4 - a_1, s_B = b_4 - b_1$ and

$$B(s_A,s_B) = \begin{cases} 1, & \text{if } s_A + s_B > 0; \\ 0, & \text{if } s_A + s_B = 0. \end{cases} \tag{32}$$

Consider the sets (A_1,A_2) and (A_1,A_3) defined by the GFNs $A_1 = (0.000,0.225,0.225,0.450;0.225,0.225)$, $A_2 = (0.450,0.675,0.675,0.900;0.255,0.255)$ and $A_3 = (0.675,0.675,0.675,0.675;0.150,0.150)$ as shown in Fig. 5(a) and (b). Now by Chen and Chen $S(A_1,A_2) = 0.3045$ and $S(A_1,A_3) = 0.3045$, but the respective figures depicts that these are different sets of GFNs. Hence, there are some drawbacks in Chen and Chen's method.

3.5. Yong et al.'s similarity measures between fuzzy numbers

In 2004, Yong et al. (2004) proposed another similarity measure of GFNs with different heights by replacing Chen and Chen's COG by ROG. If $A = (a_1,a_2,a_3,a_4;\omega_A)$ is a GFN with height $0 \leqslant \omega_A \leqslant 1$, then ROG point (r_A^x,r_A^y) using the Eqs. (21) and (22) in Definition 2.12 are obtained as

$$r_A^x = \begin{cases} \frac{\sqrt{3}}{3}\omega_A, & \text{if } a_1 = a_2 = a_3 = a_4; \\ \sqrt{\frac{I_{x1} + I_{x2} + I_{x3}}{0.5 \times \omega_A(a_3 - a_2 + a_4 - a_1)}}, & \text{otherwise}, \end{cases} \tag{33}$$

$$r_A^y = \begin{cases} a_1, & \text{if } a_1 = a_2 = a_3 = a_4; \\ \sqrt{\frac{I_{y1} + I_{y2} + I_{y3}}{0.5 \times \omega_A(a_3 - a_2 + a_4 - a_1)}}, & \text{otherwise}, \end{cases} \tag{34}$$

where

$$I_{x1} = \frac{\omega_A^3(a_2 - a_1)}{12}, I_{x2} = \frac{\omega_A^3(a_3 - a_2)}{3}, I_{x1} = \frac{\omega_A^3(a_4 - a_3)}{12}, \tag{35}$$

$$I_{y1} = \frac{\omega_A(a_2 - a_1)^3}{12} + \frac{\omega_A a_1^2(a_2 - a_1)}{2} + \frac{2\omega_A a_1(a_2 - a_1)^2}{3}, \tag{36}$$

$$I_{y2} = \frac{\omega_A(a_3 - a_2)^3}{3} + \omega_A a_2^2(a_3 - a_2) + \omega_A a_2(a_3 - a_2)^2, \tag{37}$$

$$I_{y3} = \frac{\omega_A(a_4 - a_3)^3}{12} + \frac{\omega_A a_3^2(a_4 - a_3)}{2} + \frac{2\omega_A a_3(a_4 - a_3)^2}{3}. \tag{38}$$

Then, the degree of similarity measure $S(A,B)$ between GFNs $A = (a_1,a_2,a_3,a_4;\omega_A)$ and $B = (b_1,b_2,b_3,b_4;\omega_B)$ and with different heights ω_A and ω_B is defined as

$$S(A,B) = \left(1 - \frac{1}{4}\sum_{i=1}^{4} |a_i - b_i|\right)(1 - |r_A^y - r_B^y|)^{B(s_A,s_B)} \times \frac{\min(r_A^x,r_B^x)}{\max(r_A^x,r_B^x)}, \tag{39}$$

where $s_A = a_4 - a_1, s_B = b_4 - b_1$ and

$$B(s_A,s_B) = \begin{cases} 1, & \text{if } s_A + s_B > 0; \\ 0, & \text{if } s_A + s_B = 0. \end{cases} \tag{40}$$

Consider the sets (A,B) and (A,C) of GFNs where $A = (0.5,0.5,0.5;1.0,1.0), B = (0.0,0.0,0.0,0.0;1.0,1.0)$ and $C = (1.0,1.0,1.0,1.0;1.0,1.0)$ shown in Fig. 6(a) and (b). The sets (A,B) and (A,C) are non-identical crisp-valued fuzzy numbers, but Yong et al.'s degree of similarity is 0.5 for each. This signifies that the (A,B) and (A,C) are identical. However, the graphical representations depict that similarity should not be same. Hence, there are drawbacks in this method.

3.6. Wei and Chen's similarity measures between fuzzy numbers

The similarity measure by Wei and Chen (2009) between the GFNs

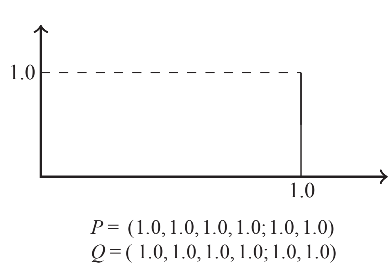

Fig. 4. Graphical representation of different sets of the GFNs P and Q.

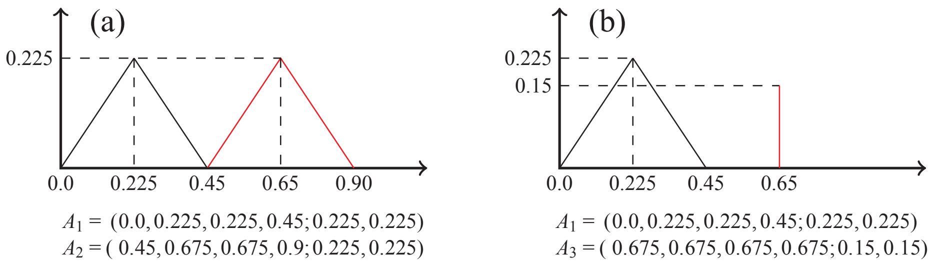

Fig. 5. Graphical representation of different sets of fuzzy numbers constructed from A_1,A_2, and A_3.

$A = (a_1,a_2,a_3,a_4;\omega_A)$ and $B = (b_1,b_2,b_3,b_4;\omega_B)$ is defined as

$$S(A,B) = \left(1 - \frac{1}{4}\sum_{i=1}^{4}|a_i - b_i|\right) \times \frac{\min(P(A),P(B)) + \min(\omega_A,\omega_B)}{\max(P(A),P(B)) + \max(\omega_A,\omega_B)} \quad (41)$$

where

$$P(A) = (a_3 - a_2) + (a_4 - a_1) + \sqrt{(a_1 - a_2)^2 + \omega_A^2} + \sqrt{(a_3 - a_4)^2 + \omega_A^2}, \quad (42)$$

$$P(B) = (b_3 - b_2) + (b_4 - b_1) + \sqrt{(b_1 - b_2)^2 + \omega_B^2} + \sqrt{(b_3 - b_4)^2 + \omega_B^2}. \quad (43)$$

are the perimeters of the GFNs A and B respectively.

Consider the sets of GFNs (P_1,P_2) and (P_1,P_3), shown in Fig. 7(a) and (b) where P_1,P_2 and P_3 are defined as $P_1 = (0.3,0.4,0.6,0.7;1.0,1.0)$,$P_2 = (0.3,0.4,0.4,0.5;1.0,1.0)$ and $P_3 = (0.4,0.5,0.5,0.6;1.0,1.0)$ respectively. According to Wei and Chen $S(P_1,P_2) = S(P_1,P_3) = 0.8003$, but these are different sets of GFNs. This shows that the method bears some drawbacks.

3.7. Xu et al.'s similarity measures between fuzzy numbers

Xu et al. (2010) proposed a new method based on the COG to calculate the similarity measure between GFNs. For the GFN $A = (a_1,a_2,a_3,a_4;w_A)$, the COG point (x_A^*,y_A^*) is defined as follows:

$$y_A^* = \begin{cases} \dfrac{\omega_A \times \left(\frac{a_3 - a_2}{a_4 - a_1} + 2\right)}{6}, & \text{if } a_4 \neq a_1; \\ \dfrac{\omega_A}{2}, & \text{if } a_4 = a_1 \end{cases} \quad (44)$$

$$x_A^* = \begin{cases} \dfrac{y_A^* \times (a_3 + a_2) + (a_4 + a_1)(\omega_A - y_A^*)}{2\omega_A}, & \text{if } \omega_A \neq 0; \\ \dfrac{a_4 + a_1}{2}, & \text{if } \omega_A = 0. \end{cases} \quad (45)$$

Then, the similarity measure between the GFNs $A = (a_1,a_2,a_3,a_4;\omega_A)$ and $B = (b_1,b_2,b_3,b_4;\omega_B)$ is defined as

$$S(A,B) = 1 - \frac{1}{8}\sum_{i=1}^{4}|a_i - b_i| - \frac{1}{2}d(A,B) \quad (46)$$

where

$$d(A,B) = \frac{(x_A^* - x_B^*)^2 + (y_A^* - y_B^*)^2}{\sqrt{1.25}}. \quad (47)$$

Consider the sets of GFNs (A_1,A_2) and (A_1,A_3) shown in Fig. 5(a) and (b). These sets of GFNs are different but similarity measure by Xu et al. are $S(A_1,A_2) = S(A_1,A_3) = 0.5737$. Hence, Xu et al.'s method bear some drawbacks.

3.8. Hejazi et al.'s similarity measures between fuzzy numbers

In 2011, Hejazi et al. (2011) proposed another similarity measure between the GFNs $A = (a_1,a_2,a_3,a_4;\omega_A)$ and $B = (b_1,b_2,b_3,b_4;\omega_B)$ which is defined as

$$S(A,B) = \left(1 - \frac{1}{4}\sum_{i=1}^{4}|a_i - b_i|\right) \times \frac{\min(P(A),P(B))}{\max(P(A),P(B))} \times \frac{\min(\mathrm{ar}(A),\mathrm{ar}(B)) + \min(\omega_A,\omega_B)}{\max(\mathrm{ar}(A),\mathrm{ar}(B)) + \max(\omega_A,\omega_B)} \quad (48)$$

where $P(A)$ and $P(B)$ are as defined in Eqs. (42) and (43) respectively and $\mathrm{ar}(A)$ and $\mathrm{ar}(B)$ are defined as

$$\mathrm{ar}(A) = \frac{\omega_A(a_4 + a_3 - a_2 - a_1)}{2} \quad (49)$$

and

$$\mathrm{ar}(B) = \frac{\omega_B(b_4 + b_3 - b_2 - b_1)}{2} \quad (50)$$

respectively.

Consider the sets of GFNs (P_1,P_2) and (P_1,P_3) shown in Fig. 7(a) and (b). The similarity degree are $S(P_1,P_2) = S(P_1,P_3) = 0.6448$. It can be seen that the different sets of GFNs have same similarity degree which reflect the drawbacks of the method.

3.9. Patra and Mondal's similarity measures between fuzzy numbers

Recently, Patra and Mondal (2015) formulated a new similarity measure between GFNs based on three parameters geometric distance, areas and heights. Let $A = (a_1,a_2,a_3,a_4;\omega_A)$ and $B = (b_1,b_2,b_3,b_4;\omega_B)$ be two GFNs, then the similarity measure is defined as

$$S(A,B) = \left(1 - \frac{1}{4}\sum_{i=1}^{4}|a_i - b_i|\right) \times \left(1 - \frac{1}{2}\{|\mathrm{ar}(A) - \mathrm{ar}(B)| + |\omega_A - \omega_B|\}\right) \quad (51)$$

where $A(A)$ and $A(B)$ are the areas of the GFNs A and B as defined in Eqs. (49) and (50) respectively.

Consider the sets of GFNs (B_1,B_2) and shown in Fig. 8(a) and (b). These sets are different sets of GFNs. However, Patra and Mondal's

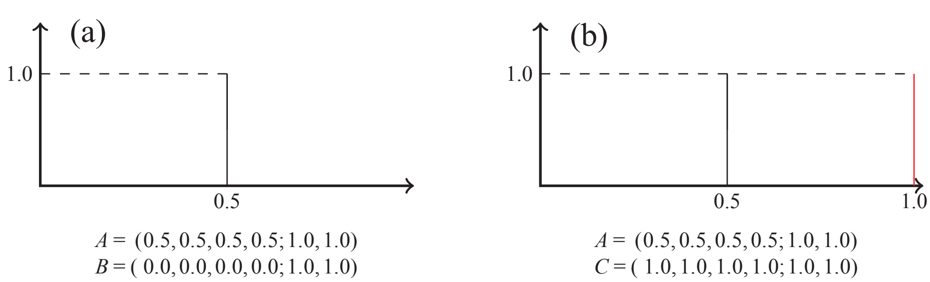

Fig. 6. Graphical representation of different sets (A,B) and (A,C) of GFNs.

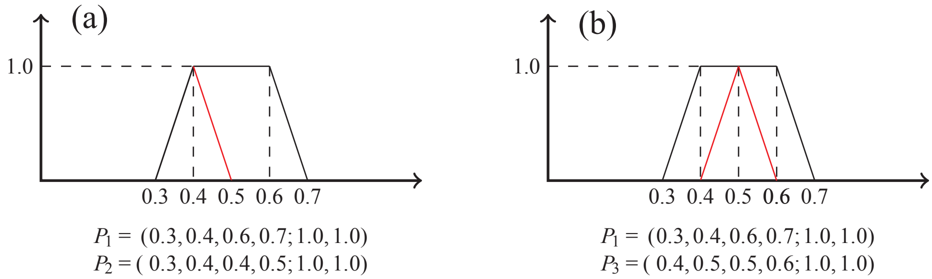

Fig. 7. Graphical representation of different sets (P_1,P_2) and (P_1,P_3) of GFNs.

similarity measures are $S(B_1,B_2) = S(B_1,B_3) = 0.88$. Thus there are some drawbacks in this method.

Consider another sets of GFNs (P_1,P_2) and (P_1,P_3) as shown in Fig. 7(a) and (b). Then, according to Patra and Mondal

$$S(P_1,P_2) = \left(1-\tfrac{1}{4}(0.0+0.0+0.2+0.2)\right)\Big(1 - \tfrac{1}{2}((0.3-0.1)+(1.0-1.0))\Big) = 0.81$$

$$S(P_1,P_3) = \left(1-\tfrac{1}{4}(0.1+0.1+0.1+0.1)\right)\Big(1 - \tfrac{1}{2}((0.3-0.1)+(1.0-1.0))\Big) = 0.81.$$

This shows that the $S(P_1,P_2) = S(P_1,P_3)$. But these two are different sets of GFNs. Hence, there are some drawbacks in this method.

3.10. Khorshidi and Nikfalazar's similarity measures between fuzzy numbers

Very recently in 2017, Khorshidi and Nikfalazar (2017) proposed a modified method on similarity measure using the existing concepts geometric distance, area, height and perimeter of GFNs. Consider the GFNs $A = (a_1,a_2,a_3,a_4;\omega_A)$ and $B = (b_1,b_2,b_3,b_4;\omega_B)$, then the similarity measure is defined as

$$S(A,B) = \left(1-\left\{\frac{1}{4}\sum_{i=1}^{4}|a_i-b_i|\right\}\times d(A,B)\right)\times\left(1-\frac{1}{3}\left\{|\mathrm{ar}(A)-\mathrm{ar}(B)|+|\omega_A-\omega_B|+\frac{|P(A)-P(B)|}{\max(P(A),P(B))}\right\}\right), \tag{52}$$

where $d(A,B)$ is given by the Eq. (47) of Xu et al. (2010), $\mathrm{ar}(A)$ and $\mathrm{ar}(B)$ are the areas of the GFNs A and B as defined in Eqs. (49) and (50) respectively, and $P(A)$ and $P(B)$ are the perimeters of GFNs A and B as defined in (42) and (43) respectively of Wei and Chen (2009).

Consider the sets (A,B) and (A,C) of GFNs where $A = (0.5,0.5,0.5,0.5;1.0,1.0)$, $B = (0.0,0.0,0.0,0.0;1.0,1.0)$ and $C = (1.0,1.0,1.0,1.0;1.0,1.0)$ shown in Fig. 6(a) and (b). The sets (A,B) and (A,C) are non-identical crisp-valued fuzzy numbers, but Yong et al.'s degree of similarity is 0.7764 for each. This signifies that the (A,B) and (A,C) are identical. However, the graphical representations depict that similarity should not be same. Hence, there are drawbacks in this method.

These counter examples are evident that the existing methods of similarity measure cannot properly give the correct result. Thus, it is utmost necessary to develop a new and complete method to determine the degree of similarity between GFNs. Hence, an effort has been made to develop a method so that such drawbacks and limitations are eliminated. In the next section the definition of the proposed similarity measure and some related properties are described.

4. Proposed similarity measure between GFNs with different left heights and right heights

For now, it is understood that existing methods of similarity measure between GFNs often encounter with limitations and drawbacks. Hence, a new similarity measure is being proposed. The proposed method is based on ambiguity, value, area, left height and right height of GFN.

Definition 4.1. If $A = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$ and $B = (b_1,b_2,b_3,b_4;\omega_1',\omega_2')$ are two non-empty GFNs with different left heights and right heights. Then, the degree of similarity between these two GFNs, denoted as $S(A,B)$, is defined as

$$\begin{aligned}S(A,B) = &\left(1-\frac{1}{4}\sum_{i=1}^{4}|a_i-b_i|\right)\\ &\times\left(1-\frac{1}{2}[|\mathrm{Amb}(A)-\mathrm{Amb}(B)|+|\mathrm{Val}(A)-\mathrm{Val}(B)|]\right)\\ &\times\left(1-\frac{1}{2}[|\omega_1-\omega_1'|)+|\omega_2-\omega_2'|]\right)\\ &\times\frac{\min(r_x^A,r_x^B)+\min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B)+\max(r_y^A,r_y^B)}\end{aligned} \tag{53}$$

where $\mathrm{Amb}(A)$,$\mathrm{Amb}(B)$,$\mathrm{Val}(A)$,$\mathrm{Val}(B)$ are the values and the ambiguity of GFNs A and B which are obtained using Definitions 2.8 and 2.9 respectively as

$$\mathrm{Amb}(A) = \tfrac{1}{6}[(a_3-a_1)\omega_1^2+(a_4-a_2)\omega_2^2+(a_3-a_2)\omega_1\omega_2],$$
$$\mathrm{Amb}(B) = \tfrac{1}{6}[(b_3-b_1)\omega_1'^2+(b_4-b_2)\omega_2'^2+(b_3-b_2)\omega_1'\omega_2'],$$
$$\mathrm{Val}(A) = \tfrac{1}{6}[(a_1-a_3)\omega_1^2+(a_2+a_4+4a_3)\omega_2^2+(a_2-a_3)\omega_1\omega_2],$$
$$\mathrm{Val}(A) = \tfrac{1}{6}[(b_1-b_3)\omega_1'^2+(b_2+b_4+4b_3)\omega_2'^2+(b_2-b_3)\omega_1'\omega_2'],$$

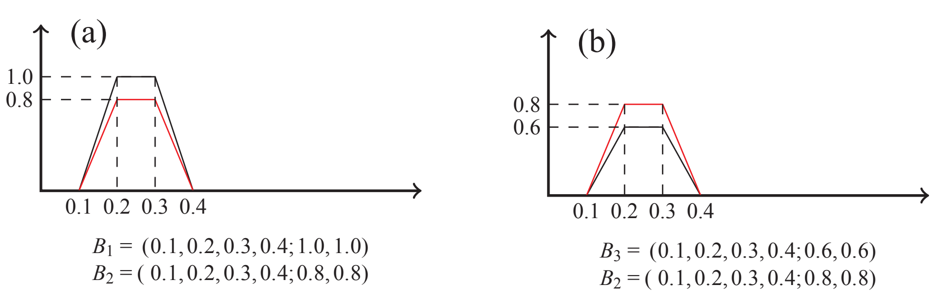

Fig. 8. Graphical representation of the GFNs B_1,B_2 and B_3.

and the r_x^A,r_x^B,r_y^A and r_y^B are the elements of the ROG points of the GFNs A and B as defined in Eqs. (68) and (69) respectively. The detail derivations of the ROG points for the GFN are available in the appendix.

The proposed similarity measure can overcome all the mentioned drawbacks and limitations of the above discussed methods. These will be discussed through numerical examples in a later section. Further, this method can measure the degree of similarity between any type of GFNs. Also the larger the value of $S(A,B)$ gives the more similarity between the GFNs. Here are some of the properties of the proposed definition of similarity measure of GFNs.

Property 4.1. *If A and B are two GFNs, then* $S(A,B) = 1$ *if and only if GFNs A and B are identical.*

Proof. Consider two GFNs $A = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$ and $B = (b_1,b_2,b_3,b_4;\omega_1',\omega_2')$ with different left height and right heights. Assume that A and B are identical, then $a_i = b_i$ for $i = 1,2,3,4,\mathrm{Amb}(A) = \mathrm{Amb}(B),\mathrm{Val}(A) = \mathrm{Val}(B),\omega_1 = \omega_1',\omega_2 = \omega_2',r_x^A = r_x^B$ and $r_y^A = r_y^B$. Hence, by the definition of the proposed similarity measure,

$$\begin{aligned}S(A,B) &= \left(1-\tfrac{1}{4}\sum_{i=1}^{4}|a_i-b_i|\right)\\&\quad\times\left(1-\tfrac{1}{2}[|\mathrm{Amb}(A)-\mathrm{Amb}(B)|+|\mathrm{Val}(A)-\mathrm{Val}(B)|]\right)\\&\quad\times\left(1-\tfrac{1}{2}[|\omega_1-\omega_1'|)+|\omega_2-\omega_2'|]\right)\times\frac{\min(r_x^A,r_x^B)+\min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B)+\max(r_y^A,r_y^B)},\\&=\left(1-\tfrac{1}{2}\times 0\right)\left(1-\tfrac{1}{2}[0+0]\right)\left(1-\tfrac{1}{2}(0+0)\right)\times 1,\\&=1.\end{aligned}$$

Conversely, let $S(A,B) = 1$, then

$$\left(1-\frac{1}{4}\sum_{i=1}^{4}|a_i-b_i|\right)\times\left(1-\frac{1}{2}[|\mathrm{Amb}(A)-\mathrm{Amb}(B)|+|\mathrm{Val}(A)-\mathrm{Val}(B)|]\right)\times\left(1-\frac{1}{2}[|\omega_1-\omega_1'|)+|\omega_2-\omega_2'|]\right)\times\frac{\min(r_x^A,r_x^B)+\min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B)+\max(r_y^A,r_y^B)}=1.$$

Now, as

$$\frac{\min(r_x^A,r_x^B)+\min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B)+\max(r_y^A,r_y^B)}\leqslant 1.$$

Therefore, this implies that $\sum_{i=1}^{4}|a_i-b_i| = 0$ and $|\omega_1-\omega_1'|+|\omega_2-\omega_2'| = 0$. Hence, $a_i = b_i$ for $i = 1,2,3,4,\omega_1 = \omega_1'$ and $\omega_2 = \omega_2'$. Thus A and B are identical. □

Property 4.2. *If* $A = (a,a,a,a;1.0,1.0)$ *and* $B = (b,b,b,b;1.0,1.0)$*, then*

$$S(A,B)=\begin{cases}(1-|a-b|)\left(1-\tfrac{1}{2}|a-b|\right)\frac{1+\sqrt{3}b}{1+\sqrt{3}a}, & \text{if } a>b;\\(1-|a-b|)\left(1-\tfrac{1}{2}|a-b|\right)\frac{1+\sqrt{3}a}{1+\sqrt{3}b}, & \text{if } a<b;\end{cases}\tag{54}$$

Proof. Since A and B are real numbers; hence, $\mathrm{Amb}(A) = 0.0,\mathrm{Amb}(B) = 0.0,\mathrm{Val}(A) = a$ and $\mathrm{Val}(B) = b$. Also, $r_x^A = \frac{1}{\sqrt{3}},r_y^A = a,r_x^B = \frac{1}{\sqrt{3}}$ and $r_y^B = b$. Hence, by the definition of the proposed similarity measure

$$\begin{aligned}S(A,B) &= \left(1-\tfrac{1}{4}\sum_{i=1}^{4}|a_i-b_i|\right)\\&\quad\times\left(1-\tfrac{1}{2}[|\mathrm{Amb}(A)-\mathrm{Amb}(B)|+|\mathrm{Val}(A)-\mathrm{Val}(B)|]\right)\\&\quad\times\left(1-\tfrac{1}{2}(|1-1|)+|1-1|\right)\frac{\min(r_x^A,r_x^B)+\min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B)+\max(r_y^A,r_y^B)},\\&=(1-|a-b|)\left(1-\tfrac{1}{2}|a-b|\right)\frac{\frac{1}{\sqrt{3}}+\min(a,b)}{\frac{1}{\sqrt{3}}+\max(a,b)},\\&=\begin{cases}(1-|a-b|)\left(1-\tfrac{1}{2}|a-b|\right)\frac{1+\sqrt{3}b}{1+\sqrt{3}a}, & \text{if } a>b;\\(1-|a-b|)\left(1-\tfrac{1}{2}|a-b|\right)\frac{1+\sqrt{3}a}{1+\sqrt{3}b}, & \text{if } a<b;\end{cases}\end{aligned}$$ □

Property 4.3. $S(A,B) = S(B,A)$.

Proof. Let $A = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$ and $B = (b_1,b_2,b_3,b_4;\omega_1',\omega_2')$ be two GFNs. Then, by the definition of the proposed similarity measure

$$\begin{aligned}S(A,B) &= \left(1-\tfrac{1}{4}\sum_{i=1}^{4}|a_i-b_i|\right)\\&\quad\times\left(1-\tfrac{1}{2}[|\mathrm{Amb}(A)-\mathrm{Amb}(B)|+|\mathrm{Val}(A)-\mathrm{Val}(B)|]\right)\\&\quad\times\left(1-\tfrac{1}{2}[|\omega_1-\omega_1'|)+|\omega_2-\omega_2'|]\right)\frac{\min(r_x^A,r_x^B)+\min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B)+\max(r_y^A,r_y^B)},\\&=\left(1-\tfrac{1}{4}\sum_{i=1}^{4}|b_i-a_i|\right)\\&\quad\times\left(1-\tfrac{1}{2}[|\mathrm{Amb}(B)-\mathrm{Amb}(A)|+|\mathrm{Val}(B)-\mathrm{Val}(A)|]\right)\\&\quad\times\left(1-\tfrac{1}{2}[|\omega_1'-\omega_1|)+|\omega_2'-\omega_2|]\right)\frac{\min(r_x^B,r_x^A)+\min(r_y^B,r_y^A)}{\max(r_x^B,r_x^A)+\max(r_y^B,r_y^A)},\\&=S(B,A).\end{aligned}$$ □

Property 4.4. *If* A,B *and* C *are GFNs such that* $A\subset B\subset C$*, then* $S(A,B) > S(A,C)$ *and* $S(B,C) > S(A,C)$.

Proof. Consider the GFNs $A = (a_1,a_2,a_3,a_4;\omega_1,\omega_2),B = (b_1,b_2,b_3,b_4;\omega_1',\omega_2')$ and $C = (c_1,c_2,c_3,c_4;\omega_1'',\omega_2'')$ such that $A\subset B\subset C$. Hence, the relations $c_1\leqslant b_1\leqslant a_1$, $c_2\leqslant b_2\leqslant a_2$, $a_3\leqslant b_3\leqslant c_3$, $a_4\leqslant b_4\leqslant c_4$, $\omega_1\leqslant\omega_1'\leqslant\omega_1''$, $\omega_2\leqslant\omega_2'\leqslant\omega_2''$, $r_x^A\leqslant r_x^B\leqslant r_x^C$ and $r_y^A\leqslant r_y^B\leqslant r_y^C$ are obtained and valid. The similarity measures $S(A,B)$ and $S(A,C)$ are defined as

$$\begin{aligned}S(A,B) &= \left(1-\frac{1}{4}\sum_{i=1}^{4}|a_i-b_i|\right)\\&\quad\times\left(1-\frac{1}{2}[|\mathrm{Amb}(A)-\mathrm{Amb}(B)|+|\mathrm{Val}(A)-\mathrm{Val}(B)|]\right)\\&\quad\times\left(1-\frac{1}{2}[|\omega_1-\omega_1'|)+|\omega_2-\omega_2'|]\right)\\&\quad\times\frac{\min(r_x^A,r_x^B)+\min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B)+\max(r_y^A,r_y^B)},\end{aligned}$$

$$\begin{aligned}S(A,C) &= \left(1-\frac{1}{4}\sum_{i=1}^{4}|a_i-c_i|\right)\\&\quad\times\left(1-\frac{1}{2}[|\mathrm{Amb}(A)-\mathrm{Amb}(C)|+|\mathrm{Val}(A)-\mathrm{Val}(C)|]\right)\\&\quad\times\left(1-\frac{1}{2}[|\omega_1-\omega_1''|)+|\omega_2-\omega_2''|]\right)\\&\quad\times\frac{\min(r_x^A,r_x^C)+\min(r_y^A,r_y^C)}{\max(r_x^A,r_x^C)+\max(r_y^A,r_y^C)}.\end{aligned}$$

Now the following relations are valid:

$$\sum_{i=1}^{4}|a_i-c_i| \geqslant \sum_{i=1}^{4}|a_i-b_i|,$$

$$|\mathrm{Amb}(A)-\mathrm{Amb}(C)| \geqslant |\mathrm{Amb}(A)-\mathrm{Amb}(B)| \text{and} |\mathrm{Val}(A)-\mathrm{Val}(C)| \geqslant |\mathrm{Val}(A)-\mathrm{Val}(B)|,$$

$$\frac{\min(r_x^A,r_x^C)+\min(r_y^A,r_y^C)}{\max(r_x^A,r_x^C)+\max(r_y^A,r_y^C)} \leqslant \frac{\min(r_x^A,r_x^B)+\min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B)+\max(r_y^A,r_y^B)}.$$

Thus it can be concluded that $S(A,B) > S(A,C)$. Similarly it can be proved that $S(B,C) > S(A,C)$. □

Property 4.5. *Let A,B and C are GFNs such that $S(A,B) = S(A,C)$, then $S(B,C) = 1$.*

Proof. Let $S(B,C) \neq 1$, then B and C are non-identical. Trivially, $S(A,B) \neq S(A,C)$. Hence, by the proof of contrapositive if $S(A,B) = S(A,C)$, then $S(B,C) = 1$. □

Property 4.6. *If $A = (a,a,a,a;\omega,\omega)$ and $B = (b,b,b,b;\omega',\omega')$, then*

$$S(A,B) = \begin{cases} (1-|a-b|)\left(1-\frac{1}{2}|a-b|\right)(1-(|\omega-\omega'|))\frac{\omega'+\sqrt{3}b}{\omega+\sqrt{3}a}, & \text{if } a>b,\omega>\omega'; \\ (1-|a-b|)\left(1-\frac{1}{2}|a-b|\right)(1-(|\omega-\omega'|))\frac{\omega+\sqrt{3}b}{\omega'+\sqrt{3}a}, & \text{if } a>b,\omega<\omega'; \\ (1-|a-b|)\left(1-\frac{1}{2}|a-b|\right)(1-(|\omega-\omega'|))\frac{\omega'+\sqrt{3}a}{\omega+\sqrt{3}b}, & \text{if } a<b,\omega>\omega'; \\ (1-|a-b|)\left(1-\frac{1}{2}|a-b|\right)(1-(|\omega-\omega'|))\frac{\omega+\sqrt{3}a}{\omega'+\sqrt{3}b}, & \text{if } a<b,\omega<\omega'; \end{cases}$$

Proof. The proof is very trivial. □

Property 4.7. *If $A = (0,0,0,0;\omega,\omega)$ and $B = (0,0,0,0;\omega',\omega')$, then*

$$S(A,B) = \begin{cases} (1-|\omega-\omega'|)\frac{\omega'}{\omega}, & \text{if } \omega>\omega'; \\ (1-|\omega-\omega'|)\frac{\omega}{\omega'}, & \text{if } \omega<\omega'; \end{cases}$$

Proof. The proof is very trivial. □

Property 4.8. *If $A = (a,b,c,d;\omega_1,\omega_2)$ and $B = (a,b,c,d;\omega_1',\omega_2')$, then*

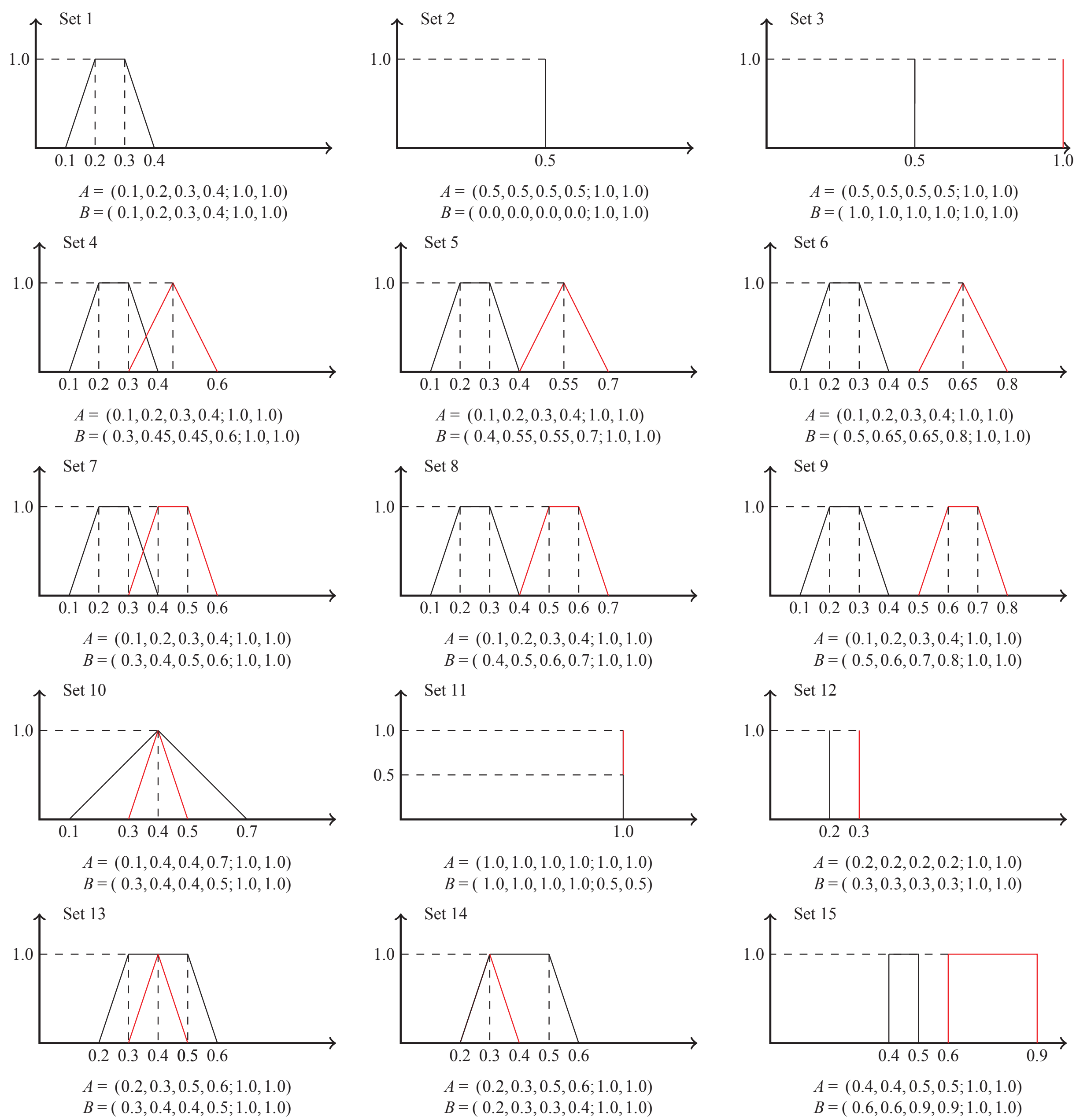

Fig. 9. Different sets of GFNs for comparative study in Section 5.

$$S(A,B) = \left(1 - \frac{1}{2}[|\mathrm{Amb}(A) - \mathrm{Amb}(B)| + |\mathrm{Val}(A) - \mathrm{Val}(B)|]\right) \times \left(1 - \frac{1}{2}[|\omega_1 - \omega_1'|) + |\omega_2 - \omega_2'|]\right)\frac{\min(r_x^A,r_x^B) + \min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B) + \max(r_y^A,r_y^B)}.$$

Proof. The proof is very trivial. □

Property 4.9. $0 \leqslant S(A,B) \leqslant 1$.

Proof. Let $A = (a_1,a_2,a_3,a_4;\omega_1,\omega_2)$ and $B = (b_1,b_2,b_3,b_4;\omega_1',\omega_2')$ be two GFNs such that $0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4 \leqslant 1$ and $0 \leqslant b_1 \leqslant b_2 \leqslant b_3 \leqslant b_4 \leqslant 1$. Then, by the definition of the proposed similarity measure

$$S(A,B) = \left(1 - \frac{1}{4}\sum_{i=1}^{4}|a_i - b_i|\right) \times \left(1 - \frac{1}{2}[|\mathrm{Amb}(A) - \mathrm{Amb}(B)| + |\mathrm{Val}(A) - \mathrm{Val}(B)|]\right) \times \left(1 - \frac{1}{2}[|\omega_1 - \omega_1'|) + |\omega_2 - \omega_2'|]\right)\frac{\min(r_x^A,r_x^B) + \min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B) + \max(r_y^A,r_y^B)}.$$

As $|\mathrm{Amb}(A) - \mathrm{Amb}(B)| \geqslant 0, |\mathrm{Val}(A) - \mathrm{Val}(B)| \geqslant 0$ and $0 \leqslant$

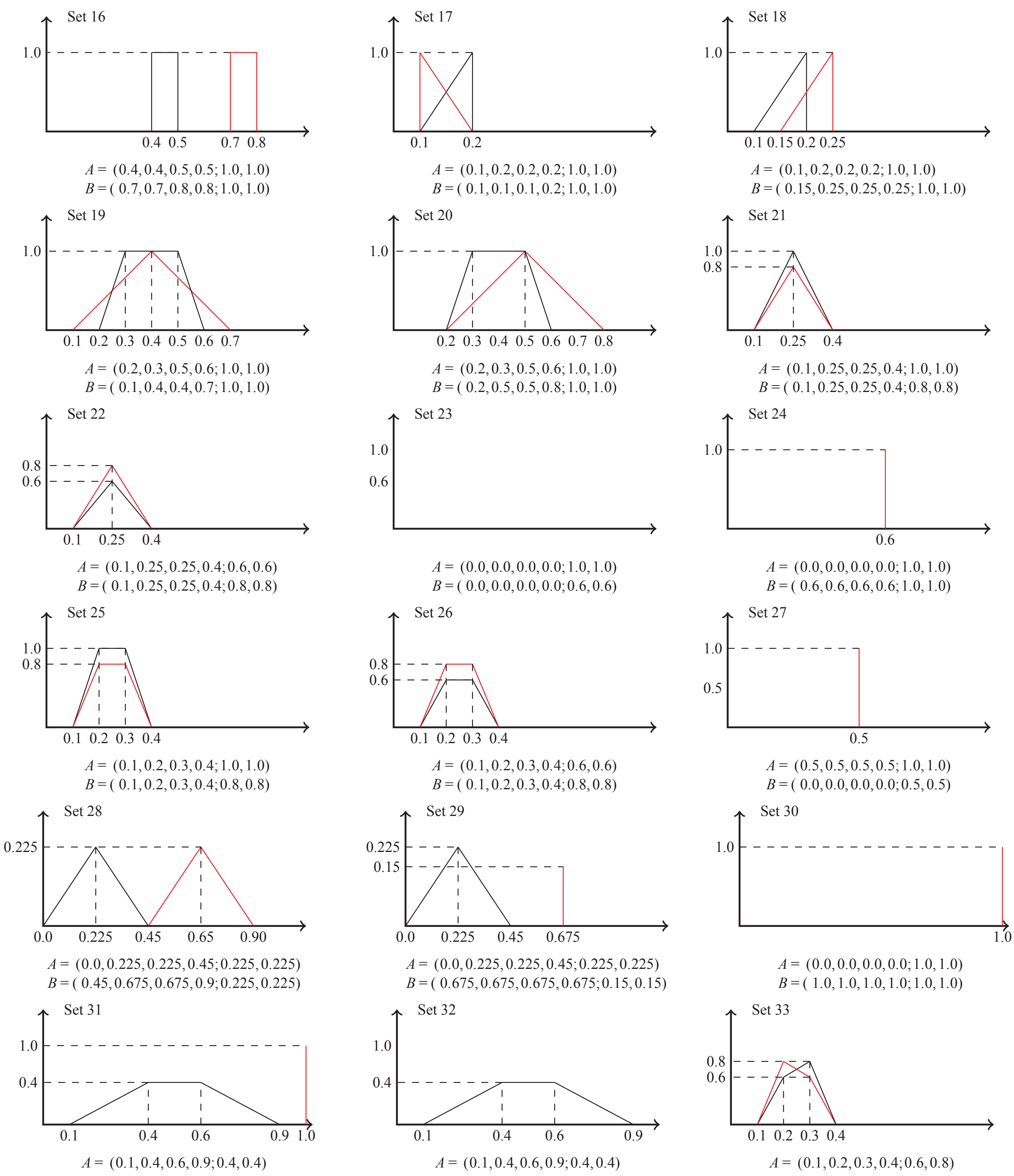

Fig. 10. Different sets of GFNs for comparative study in Section 5.

$\mathrm{Amb}(A),\mathrm{Amb}(B),\mathrm{Val}(A),\mathrm{Val}(B)\leqslant 1$. Therefore,

$$\left(1-\frac{1}{2}[|\mathrm{Amb}(A)-\mathrm{Amb}(B)|+|\mathrm{Val}(A)-\mathrm{Val}(B)|]\right)\leqslant 1.$$

Also,

$$\left(1-\frac{1}{4}\sum_{i=1}^{4}|a_i-b_i|\right)\leqslant 1,$$

$$\frac{\min(r_x^A,r_x^B)+\min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B)+\max(r_y^A,r_y^B)}\leqslant 1,$$

$$\left(1-\frac{1}{2}[|\omega_1-\omega_1'|)+|\omega_2-\omega_2'|]\right)\leqslant 1.$$

Therefore, $S(A,B)\leqslant 1$.

Again as $0\leqslant a_1\leqslant a_2\leqslant a_3\leqslant a_4\leqslant 1$ and $0\leqslant b_1\leqslant b_2\leqslant b_3\leqslant b_4\leqslant 1$, therefore $|a_i-b_i|\leqslant 1$. Hence, $\left(1-\frac{1}{4}\sum_{i=1}^{4}|a_i-b_i|\right)\geqslant 0$. As $0\leqslant \mathrm{Amb}(A),\mathrm{Amb}(B),\mathrm{Val}(A),\mathrm{Val}(B)\leqslant 1$; hence, $|\mathrm{Amb}(A)-\mathrm{Amb}(B)|\leqslant 1$and $|\mathrm{Val}(A)-\mathrm{Val}(B)|\leqslant 1$. Therefore,

$$\left(1-\frac{1}{2}[|\mathrm{Amb}(A)-\mathrm{Amb}(B)|+|\mathrm{Val}(A)-\mathrm{Val}(B)|]\right)\geqslant 0.$$

Also $\frac{\min(r_x^A,r_x^B)+\min(r_y^A,r_y^B)}{\max(r_x^A,r_x^B)+\max(r_y^A,r_y^B)}>0$. As $\omega_1,\omega_2,\omega_1',\omega_2'\leqslant 1$, therefore

$$\left(1-\frac{1}{2}[|\omega_1-\omega_1'|)+|\omega_2-\omega_2'|]\right)\geqslant 0.$$

Therefore, $S(A,B)\geqslant 0$. Hence, the property $0\leqslant S(A,B)\leqslant 1$. □

Property 4.10. *If* $A=(0,0,0,0;1,1)$ *and* $B=(1,1,1,1;1,1)$, *then* $S(A,B)=0$.

Proof. The proof is very trivial. □

5. Comparative analysis

Different sets of GFNs have been considered for the comparative study of the proposed method with other methods by Chen (1996), Hsieh and Chen (1999), Lee (2002), Chen and Chen (2001), Wei and Chen (2009), Xu et al. (2010), Hejazi et al. (2011) and Patra and Mondal (2015). The sets of fuzzy numbers are shown in Fig. 9 and 10. The similarity measure by various methods and the proposed method are displayed in Table 1. Some of the similarity measures are highlighted in Table 1 to show that the measures are these measures are incorrect.

(1) In Fig. 9 (Set 1), A and B are two identical GFNs. Hence, $S(A,B)=1$. The proposed method and all the other methods give an equal degree of similarity which is logical.
(2) Similarity measure, according to Chen's (1996) method for the sets 11, 21, 22, 23, 25, 26 and 33 is 1 although the fuzzy numbers in the sets are non-identical. Thus, limitations and drawbacks are observed in this method. However, the proposed method overcomes such drawbacks and give non-identical and non-unity similarity measures which are admissible from the graphical representations in Figs. 9 and 10. Similarity measure by Chen's method for the Sets 4 and 7 is same, which is illogical. In these sets one of the members in one set is identical to one of the members in other set and the other members are non-identical. Thus, the similarity measure of these sets need not be same, which is also clear from their graphical representations. However, the proposed method overcomes such limitations and gives different similarity measure for each of the sets, which is acceptable from the graphical representations in Fig. 9. A similar justification can be laid for the pairs of sets (2, 3), (5, 8), (6, 9), (10, 13), (13, 14), (15, 16), (17, 18), (19, 20), (14, 20), (25, 26), (28, 29) and (31, 32). However, the proposed method

Table 1
A comparison of similarity measure obtained from proposed method with existing methods.

	Chen (1996)	Hsieh and Chen (1999)	Lee (2002)	Chen and Chen (2001)	Yong et al. (2004)	Wei and Chen (2009)	Xu et al. (2010)	Hejazi et al. (2011)	Patra and Mondal (2015)	Khorshidi and Nikfalazar (2017)	The proposed method
Set 1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Set 2	0.5000	0.6666	0.0000	0.5000	0.5000	0.5000	0.5264	0.5000	0.5000	0.7764	0.2009
Set 3	0.5000	0.6666	0.0000	0.5000	0.5000	0.5000	0.5264	0.5000	0.5000	0.7764	0.2561
Set 4	0.8000	0.8333	0.6000	0.5485	0.5252	0.7794	0.8071	0.7387	0.7800	0.9352	0.4936
Set 5	0.7000	0.7692	0.5000	0.4200	0.4028	0.6820	0.7135	0.6464	0.6825	0.8917	0.3691
Set 6	0.6000	0.7142	0.4285	0.3085	0.2965	0.5845	0.6193	0.5540	0.5850	0.8309	0.2717
Set 7	0.8000	0.8333	0.6000	0.6400	0.6429	0.8000	0.8105	0.8000	0.8000	0.9642	0.5719
Set 8	0.7000	0.7692	0.5000	0.4900	0.4931	0.7000	0.7158	0.7000	0.7000	0.9195	0.4281
Set 9	0.6000	0.7142	0.4285	0.3600	0.3630	0.6000	0.6211	0.6000	0.6000	0.8569	0.3156
Set 10	0.9000	1.0000	0.8333	0.9000	0.8854	0.7833	0.9500	0.6260	0.8100	0.8740	0.8529
Set 11	1.0000	1.0000	**	0.5000	0.5000	0.5000	0.8881	0.2500	0.7500	0.6666	0.2553
Set 12	0.9000	0.9090	0.0000	0.9000	0.9000	0.9000	0.9052	0.9000	0.9000	0.9911	0.7575
Set 13	0.9000	1.0000	0.7500	0.7200	0.6914	0.8002	0.9127	0.6448	0.8100	0.8757	0.7392
Set 14	0.9000	0.9090	0.7500	0.6480	0.6222	0.8002	0.8917	0.6448	0.8100	0.8719	0.6144
Set 15	0.7000	0.7692	0.4000	0.4900	0.4872	0.6222	0.7158	0.5011	0.6300	0.8110	0.4322
Set 16	0.7000	0.7692	0.2500	0.4900	0.4903	0.7000	0.7158	0.7000	0.7000	0.9195	0.4607
Set 17	0.9500	0.9375	0.5000	0.9183	0.9187	0.9500	0.9600	0.9500	0.9500	0.9985	0.8659
Set 18	0.9500	0.9523	0.6666	0.9025	0.9029	0.9500	0.9526	0.9500	0.9500	0.9978	0.8529
Set 19	0.9000	1.0000	0.8333	0.7200	0.6915	0.8809	0.9127	0.8738	0.9000	0.9829	0.7663
Set 20	0.9000	0.9090	0.8333	0.6480	0.6243	0.8809	0.8917	0.8738	0.9000	0.9788	0.6599
Set 21	1.0000	1.0000	1.0000	0.8000	0.8000	0.8211	0.9701	0.6641	0.8850	0.8667	0.6639
Set 22	1.0000	1.0000	1.0000	0.7500	0.7500	0.7833	0.9701	0.5979	0.8850	0.8557	0.6592
Set 23	1.0000	1.0000	**	0.6000	0.6000	0.6000	0.9105	0.3600	0.8000	0.7333	0.3600
Set 24	0.4000	0.6250	0.0000	0.4000	0.4000	0.4000	0.4316	0.4000	0.4000	0.6780	0.1373
Set 25	1.0000	1.0000	1.0000	0.8000	0.8000	0.8247	0.9652	0.6680	0.8800	0.8650	0.6528
Set 26	1.0000	1.0000	1.0000	0.7500	0.7500	0.7881	0.9652	0.7500	0.8800	0.8544	0.6468
Set 27	0.5000	0.6666	0.0000	0.2500	0.2500	0.2500	0.5000	0.1250	0.3750	0.5000	0.0502
Set 28	0.5500	0.6896	0.5000	0.3025	0.3089	0.5500	0.5737	0.5500	0.5500	0.8189	0.2355
Set 29	0.5500	0.6896	0.3333	0.3025	0.2945	0.1887	0.5737	0.0826	0.5154	0.5870	0.2179
Set 30	0.0000	0.5000	0.0000	0.0000	0.0000	0.0000	0.0527	0.0000	0.0000	0.1056	0.0000
Set 31	0.5000	0.6666	0.4444	0.0750	0.0883	0.4000	0.4771	0.1666	0.3000	0.5332	0.0479
Set 32	0.5000	0.6666	0.4444	0.0750	0.0791	0.4000	0.4771	0.1666	0.3000	0.5332	0.0331
Set 33	1.0000	1.0000	1.0000	–	–	–	–	–	–	–	0.7456

gives different similarity measure for these sets. Hence, there are drawbacks in Chen's method. Nevertheless, no such drawbacks are observed in the proposed method. Hence, the proposed method is far superior to Chen's method.

(3) Hsieh and Chen (1999) gives identical similarity measure for the non-identical pairs of sets (2, 3), (4, 7), (5, 8), (6, 9), (14, 20), (15, 16), (28, 29) and (31, 32). However, the proposed method gives non-identical, admissible and logical similarity measure. Also, the similarity measure for the sets 3, 10, 11, 13, 19, 21, 22, 23, 25, 26 and 33 is 1, which implies that members in these sets are identical. However, the graphical representations do not admit such similarity. Nevertheless, the proposed method gives non-unity similarity measure. Thus, the superiority of the proposed method to Hsieh and Chen's method is evident. The graphical representations of the sets discussed may be found in Figs. 9 and 10.

(4) Lee's (2002) similarity measure either fails or gives non-similar measure when the fuzzy numbers are crisp-valued fuzzy numbers (Sets 11, 12, 23, 24). However, under such circumstances the proposed method gives admissible similarity measures. Moreover, Lee's method also fails when the fuzzy numbers are of types non-normal GFN and normal GFN (Sets 21, 22, 25 and 26). As like Chen's and Hsieh and Chen's methods, Lee's method also gives an identical similarity measure for the pairs of sets (2, 3), (4, 7), (5, 8), (6, 9), (13, 14), (19, 20) and (31, 32) where one member in each pair are identical and other members are non-identical. Similarity measure by the proposed method coincides with the graphical representations. This signifies the superiority of the proposed method over Lee's method. The graphical representations of the sets discussed may be found in Figs. 9 and 10.

(5) Chen and Chen's (2001) similarity measure also retains the drawbacks of Chen's and Hsieh's, Chen's and Lee's method where one member is identical and the other are non-identical (sets (2, 3), (15, 16), (28, 29) and (31, 32)). As always, no such drawbacks and limitations are observed in the proposed method. Thus, the superiority of the proposed method over Chen and Chen's method is established. The graphical representations of the sets discussed may be found in Figs. 9 and 10.

(6) Yong et al. (2004) proposed a very promising similarity measure. However, it fails to give a correct similarity for crisp-valued fuzzy numbers (sets 2 and 3 in Fig. 9) as discussed in subSection 3.5. However, the proposed method overcome such drawback and gives a justified similarity measure for these sets of fuzzy numbers. Hence, the out-performance of the proposed method is evident.

(7) Limitations of similar types like the one in Chen and Chen's has been seen in the method by Wei and Chen (2009) (namely, the pairs of sets (2, 3), (13, 14), (17, 18), (19, 20) and (31, 32)). Also, such types of limitations are observed in Xu et al. (2010) (namely, the pairs of sets (2, 3), (15, 16), (21, 22), (25, 26), (28, 29) and (31, 32)) and Hejazi et al. (2011) (namely, the pairs of sets (2, 3), (13, 14), (17, 18), (19, 20) and (31, 32)). The most recent method by Patra and Mondal (2015) also bears such types of anomalies (namely, the pairs of sets (2, 3), (10, 13), (13, 14), (17, 18), (19, 20), (21, 22), (25, 26) and (31, 32)). However, no such drawbacks are seen in the proposed method which signifies the superiority of the proposed method. The graphical representations of the sets discussed may be found in Figs. 9 and10.

(8) Moreover, a very recent study by Khorshidi and Nikfalazar (2017) also fails to overcome the Yong et al.'s drawback. Also, the drawback is seen in the sets 31 and 32 as an incorrect similarity measure is given for the non-identical fuzzy numbers in those sets. As like the other methods for the sets 2 and 3 Khorshidi and Nikfalazar also fails to give a correct similarity. However, the proposed method overcomes the limitations and drawbacks of the recent study as well.

(9) Property 4.9 makes it clear that for any two arbitrary fuzzy numbers A and $B, 0 \leqslant S(A,B) \leqslant 1$. Thus, $S(A,B)=0$ signifies complete dissimilarity between A and B. Gradually, as $S(A,B)$ increases similarity increases. And, $S(A,B) = 1$ leads to complete similarity between A and B. Eventually, it can be concluded by intuition as well as the proposed method that if A and B are crisp-valued fuzzy numbers such that $A = (0,0,0,0;1,1)$ and $B = (1,1,1,1;1,1)$ (Set 30, Fig. 10), then $S(A,B) = 0$. However, the methods by Hsieh and Chen (1999), Xu et al. (2010) and Khorshidi and Nikfalazar (2017), $S(A,B) \neq 0$ is completely unreasonable.

These numerical examples are evident that the existing methods of similarity measure bear many drawbacks and limitations. However, the proposed method can overcome all the limitations and shortcomings and outperform in all situations. Further, the proposed method can properly deal with GFN with different left heights and right heights (Set 33). Thus, it is claimed that the proposed method is much better than existing methods.

6. Application of the proposed method in risk analysis

In this section, a real-life problem of risk analysis in poultry farming has been discussed by using the proposed fuzzy similarity measure. Schmucke (1984) first discussed the risk analysis problem under fuzzy environment using the parameters probability of failure and severity of loss. A thorough overview of computing with words and risk assessment is forwarded by Liu, Martínez, Wang, Rodríguez, and Novozhilov (2010). Generally, the probabilistic values of these parameters are not precise due to its nature. Thus, these parameters are more precisely expressed as linguistic terms such as high, low, medium, etc. Further, these parameters are generally expressed as fuzzy numbers. A lot of literature are available that describes risk analysis problem using these parameters. Some of the studies in risk analysis problem using linguistic terms as fuzzy numbers are Zhang (1986), Chen (1996), Chen and Chen (2008, 2009), Wei and Chen (2009), Chen et al. (2012), Zhu and Xu (2012) and Patra and Mondal (2015).

6.1. Fuzzy risk analysis

Assume a production system C consisting of n sub-components $A_i, i = 1,2,\cdots,n$. Each sub-components is assessed by two parameters

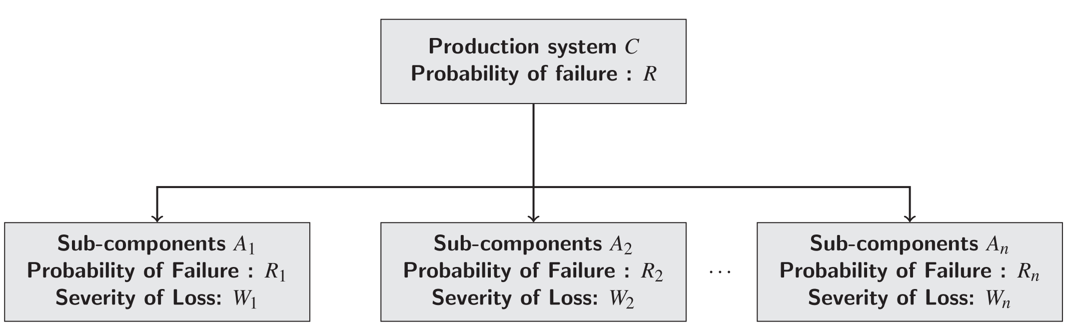

Fig. 11. Graphical representation of fuzzy risk analysis

probability of failure $R_i, i = 1,2,\cdots,n$ and severity of loss $W_i, i = 1,2,\cdots,n$ which are linguistic terms. The structure of risk analysis under fuzzy environment is shown in Fig. 11 (Schmucke, 1984). The algorithm of fuzzy risk analysis is expressed as the following steps.

Step (1) Consider probability of failure R_i and severity of loss W_i for each sub-component $A_i, i = 1,2,\cdots,n$, in linguistic terms such as low, medium, high, etc where n is the number of sub-components in the production system.

Step (2) Use the fuzzy weighted mean method and the GFN arithmetic operation to get the total risk R of the production system C integrating R_i and W_i of each sub-component A_i as follows

$$R = \frac{\sum_{i=1}^{n} W_i \otimes R_i}{\sum_{i=1}^{n} W_i} \tag{55}$$

$$= (r_1,r_2,r_3,r_4;\omega_1,\omega_2). \tag{56}$$

Step (3) Standardize the GFN of total risk R into R^* where

$$R^* = \left(\frac{r_1}{k},\frac{r_2}{k},\frac{r_3}{k},\frac{r_4}{k};\omega_1,\omega_2\right) \tag{57}$$

$$= (r_1^*,r_2^*,r_3^*,r_4^*;\omega_1,\omega_2) \tag{58}$$

$$k = \max_i(\lceil |r_i| \rceil,1)\, i = 1,2,3,4. \tag{59}$$

where $|r_i|$ denotes the absolute value of r_i, $\lceil |r_i| \rceil$ denotes taking the upper bound of $|r_i|, 1 \leqslant i \leqslant 4$.

Step (4) Similarity measure of total risk R^* with all given linguistic terms are measured using the proposed method.

Step (5) The largest similarity between the total risk R^* and the linguistic terms is considered as a risk value of the system in linguistic term.

6.2. A case study

Poultry farming plays a major role in contributing towards addressing key national development goals and improving the standard of living of people through poverty alleviation and creating employment opportunities. India has made tremendous progress in poultry production during the last decades. Assam is a state in North-Eastern region of India, where not much industrialization has happened so far. Hence, unemployment is a major issue in recent times, mostly in the rural Assam. A case study has been done on the fuzzy risk analysis on poultry farming in rural Assam.

The key to successful poultry production depends on following sub-components $A_i, i = 1,2,\cdots,8$ where A_1: availability of land, A_2: availability of expert labor, A_3: financial support, A_4: availability of clean water, A_5: transportation, A_6: availability of electricity, A_7: food supply and A_8: good poultry baby (Chutia, 2017). As the probabilistic values of the sub-components $A_1,A_2,\cdots,A_8$ are not precise therefore such types of values are more used in linguistic terms. Generally these linguistic terms are expressed as fuzzy numbers. The linguistic terms to the sub-components are assigned based on intuition and on basis of discussion with some experienced poultry farmers in rural Assam. The linguistic terms to the sub-components are depicted in Table 3 and are discussed below which are assigned from Table 2.

A_1: Probability of failure R_1 due to insufficient land is 'Very low' as there exist lots of unused land in rural Assam. Thus, the severity of loss W_1 is 'Absolutely low'.

Table 2
A 9-member linguistic term set (Schmucke, 1984).

Linguistic term	GFN
Absolutely-low	(0.00, 0.00, 0.00, 0.00; 1.0, 1.0)
Very-low	(0.00, 0.00, 0.02, 0.07; 1.0, 1.0)
Low	(0.04, 0.10, 0.18, 0.23; 1.0, 1.0)
Fairly-low	(0.17, 0.22, 0.36, 0.42; 1.0, 1.0)
Medium	(0.32, 0.41, 0.58, 0.65; 1.0, 1.0)
Fairly-high	(0.58, 0.63, 0.80, 0.86; 1.0, 1.0)
High	(0.72, 0.78, 0.92, 0.97; 1.0, 1.0)
Very-high	(0.93, 0.98, 1.00, 1.00; 1.0, 1.0)
Absolutely-high	(1.00, 1.00, 1.00, 1.00; 1.0, 1.0)

Table 3
Linguistic values of R_i and W_i for eight sub-components $A_1,A_2,\cdots,A_8$.

Sub-component A_i	Linguistic value of R_i	Linguistic value of W_i
A_1	Very low	Absolutely low
A_2	Very low	Fairly high
A_3	High	Fairly low
A_4	Very low	Fairly low
A_5	Low	Fairly low
A_6	Fairly low	Very low
A_7	Fairly low	Low
A_8	Low	Fairly high

A_2: Probability of failure R_2 is 'Very low' as expert labor is shortage in rural area. However, one can hire well-trained labor, but well-trained labor has more demand; hence, to minimize the labor wages the farmer has to hire inexpert labor in which case it might lead to greater risk. So, probability in severity of loss W_2 is a 'Fairly high'.

A_3: Probability of failure R_3 due to capital is 'High' as the farmer might not have enough capital in hand. However, nowadays Government take a lot of steps to provide financial support to the farmers who wanted to start such mini-projects for self-employment. So, the severity of loss W_3 due to insufficient capital is 'Fairly low'.

A_4: Probability of failure R_4 due to insufficient water is 'Very low' as rural area has enough availability of water from different sources. However, the river and pond water might be infected. So, severity of loss W_4 is 'Fairly low'.

A_5: Probability of failure R_5 due to transpiration is 'Low' as there are various means of transportation. However, due to bad road condition transportation cost might be higher; hence, severity of loss W_5 will be 'Fairly low'.

A_6: In case of rural Assam electricity supply is very irregular. However, during power-cut one can use solar and generators. Therefore, there is a 'Fairly low' probability of failure R_6 due to irregularity of electricity. Hence, the severity of loss W_6 will be 'Very low'.

A_7: Probability of failure R_7 due to insufficient food is 'Fairly low' as poultry feed is easily available in both rural and urban areas. Hence, severity of loss W_7 due to insufficient poultry feed is 'Low'.

A_8: Mostly, good quality of poultry baby is available in the market; hence, probability of failure R_8 due to bad quality poultry baby is 'Low'. However, in case of inexpert farmer it is difficult to recognize the best quality of poultry baby. As most of the farmers are ignorant of good quality of poultry baby; hence, severity of loss W_8 is 'Fairly high'.

Now the question comes, what is the risk for a rural farmer in terms of linguistic variables that exists in the systems to produce the maximum amount of good quality of poultry under these circumstances. And this question is generally answered by the similarity measure of fuzzy numbers. The proposed method of similarity measure plays an important role in determining the risk in terms of linguistic variable.

Table 4
A comparison of similarity measure R and linguistic terms F_i,i = 1,2,⋯,9 by different methods.

Linguistics terms F_i	Chen (1996)	Hsieh and Chen (1999)	Lee (2002)	Chen and Chen (2001)	Yong et al. (2004)	Wei and Chen (2009)	Xu et al. (2010)	Hejazi et al. (2011)	Patra and Mondal (2015)	Khorshidi and Nikfalazar (2017)	The proposed method
Absolutely low	0.7509	0.8100	0.5357	0.4549	0.4875	0.6037	0.7533	0.4064	0.6184	0.7448	0.3672
Very low	0.7735	0.8222	0.5777	0.5553	0.5494	0.6407	0.7824	0.4574	0.6543	0.7805	0.4160
Low	0.8868	0.9122	0.7720	0.7722	0.7615	0.7778	0.8964	0.6191	0.7901	0.8608	0.6624
Fairly low	0.8982	0.9459	0.7915	0.8276	0.8537	0.8168	0.9310	0.6951	0.8273	0.9027	0.8046
Medium	0.7590	0.7954	0.5994	0.5669	0.5829	0.7133	0.7755	0.6435	0.7199	0.8912	0.5023
Fairly high	0.5315	0.6747	0.4227	0.2689	0.2742	0.4919	0.5593	0.4322	0.4975	0.7448	0.2464
High	0.4015	0.6196	0.3505	0.1572	0.1670	0.3651	0.4367	0.3107	0.3698	0.6197	0.1527
Very high	0.2715	0.5723	0.2344	0.0716	0.0776	0.2249	0.3146	0.1605	0.2297	0.4360	0.0780
Absolutely high	0.2490	0.5664	0.2107	0.0522	0.0622	0.2002	0.2897	0.1348	0.2051	0.3943	0.0680

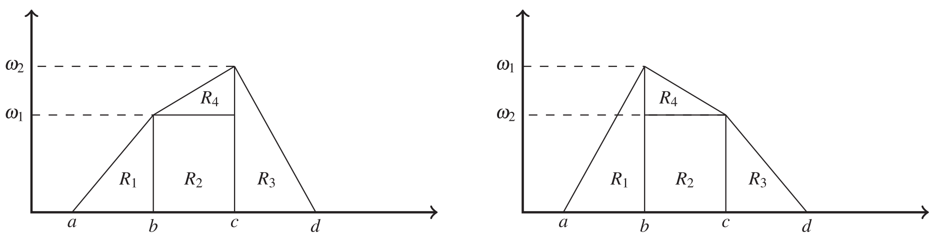

Fig. 12. Graphical representation of GFN A with different left heights and right heights.

The risk of probability of failure using Eq. (56) and taking the parameters from Table 3 is given by

$$R = \frac{\sum_{i=1}^{8} W_i \otimes R_i}{\sum_{i=1}^{8} W_i} = (0.04850.09670.31450.5365;1.0,1.0).$$

The degree of similarity between the total risk R and the linguistic terms in Table 2 using the proposed similarity measure are depicted in Table 4. The largest similarity value degree is 0.8046, which is between the total risk of failure R and the linguistic term 'Fairly low'. Hence, the probability of failure under such circumstance is fairly low. Hence, it can be concluded that under such circumstances, one can take up the poultry farming as a self-employment project. For the purpose of validation, the result of the proposed method is compared with the other existing methods. It is evident that risk of failure by the proposed method is the same as that of the existing method as shown in Table 4 (marked in gray). The case study does not involve fuzzy numbers depicting drawbacks and limitations of the existing methods. Hence, the results of the existing methods are justifiable. Therefore, comparison of the results of the existing method with the proposed method is also justifiable.

7. Conclusions

Owing to the limitations and drawbacks of the existing method of similarity measure. A new similarity measure based on ambiguity, value, area, heights and geometric distance of GFN has been proposed. The proposed method is much more generalized than the method described as it can deal with any type of GFNs. This measure has been discussed on GFN with different left heights and right heights. This measure is not just confined to GFNs with different left heights and right heights, but also can deal with arbitrary fuzzy numbers.

Different sets of GFNs are considered to see the out-performance of the method through comparison with other method. The out-performance of the current method is evident from the discussed numerical examples. It has been observed that most of the method gives an equal similarity for the sets where one of the members in one set is identical to one of the members in other set and the other members are non-identical (namely, (2, 3), (5, 8), (6, 9), (10, 13), (13, 14), (15, 16), (17, 18), (19, 20), (14, 20), (25, 26), (28, 29) and (31, 32) in Figs. 9 and 10). Nevertheless, the proposed method gives justified similarity for these sets. Further, the proposed method can handle GFN with different left height and right height. It has been evident that all the drawbacks and limitations of the existing methods of similarity measure is being overcome by the proposed method.

Further, the proposed method of similarity measure has been applied to the risk analysis problem on poultry farming. As the final risk so obtained is a fuzzy number which is sometimes called as generalized interval. Hence, a linguistic variable has to be given to the obtained risk, which is possible by the proposed similarity measure. The parameters used the risk analysis problem are probability of failure and severity of loss. Due to the nature, these parameters are expressed in terms of linguistic terms which are basically fuzzy numbers. Under the current study the probability of failure obtained using the proposed method is 'Fairly low'. Hence, rural farmers can take up such project.

Generally risk analysis problems may be studied under the Z-number (Bakar & Gegov, 2015; Zadeh, 2011) as it discusses about restriction and reliability. Hence, this concept can be further implemented in similarity measure and risk analysis problem.

Acknowledgements

The authors would like to express his sincere thanks and gratitude to the anonymous referees for their valuable comments and suggestions which improved the paper a lot in quality and clarity point of view.

Appendix A

Consider a GFN $A = (a,b,c,d;\omega_1,\omega_2)$ given by the membership function

$$\mu_A(x) = \begin{cases} \frac{\omega_1(x-a)}{b-a}, & \text{if } a \leqslant x \leqslant b; \\ \frac{\omega_1(c-b)+(\omega_2-\omega_1)(x-b)}{c-b}, & \text{if } b \leqslant x \leqslant c; \\ \frac{\omega_2(d-x)}{d-c}, & \text{if } c \leqslant x \leqslant d; \\ 0, & \text{otherwise.} \end{cases} \tag{60}$$

The graphical representations of the GFNs A is shown in Fig. 12 depending on the heights ω_1 and ω_2.

The ROG points of the GFN A is denote as (r_x^A, r_y^A) whose value can be obtained by using the Eqs. (21) and (22) in Definition 2.12. To evaluate the moment of inertia the GFN A is divided into regions R_1,R_2,R_3 and R_4. Hence, the moment of inertia of the areas R_1,R_2,R_3 and R_4 about the x and y axis can be calculated, according to Eqs. (19) and (20) in Definition 2.11, as

$$(I_x)_{R_1} = \int_{R_1} y^2 dR_1 = \int_0^{\omega_1} y^2 \left\{ b - a - \frac{y(b-a)}{\omega_1} \right\} dy = \frac{(b-a)\omega_1^3}{12}. \tag{61}$$

$$(I_y)_{R_1} = \int_{R_1} x^2 dR_1 = \int_a^b x^2 \frac{\omega_1(x-a)}{b-a} dx = \frac{\omega_1}{12}(3b^3 - a^3 - a^2 b - ab^2). \tag{62}$$

$$(I_x)_{R_2} = \begin{cases} \frac{(c-b)\omega_1^3}{3}, & \text{if } \omega_1 \leqslant \omega_2; \\ \frac{(c-b)\omega_2^3}{3}, & \text{if } \omega_1 \geqslant \omega_2. \end{cases} \tag{63}$$

$$(I_x)_{R_3} = \begin{cases} \frac{(c-b)}{12}(\omega_2^3 - 3\omega_1^3 + \omega_1^2\omega_2 + \omega_1\omega_2^2), & \text{if } \omega_1 \leqslant \omega_2; \\ \frac{(c-b)}{12}(\omega_1^3 - 3\omega_2^3 + \omega_1^2\omega_2 + \omega_1\omega_2^2), & \text{if } \omega_1 \geqslant \omega_2. \end{cases} \tag{64}$$

$$(I_x)_{R_4} = \frac{(d-c)\omega_2^3}{12}. \tag{65}$$

$$(I_y)_{(R_2+R_4)} = \frac{\omega_1}{3}(c^3 - b^3) + \frac{(\omega_2 - \omega_1)}{12}(3c^3 - b^3 - c^2 b - cb^2). \tag{66}$$

$$(I_y)_{R_3} = \frac{\omega_2}{12}(d^3 - 3c^3 + c^2 d + cd^2). \tag{67}$$

Hence, the ROG point of the GFN A can be obtained using the above equations as

$$r_x^A = \sqrt{\frac{(I_x)_{R_1} + (I_x)_{R_2} + (I_x)_{R_3} + (I_x)_{R_4}}{\mathrm{ar}(A)}}, \tag{68}$$

$$r_y^A = \sqrt{\frac{(I_y)_{R_1} + (I_y)_{(R_2+R_4)} + (I_y)_{R_3}}{\mathrm{ar}(A)}}. \tag{69}$$

where $\mathrm{ar}(A)$ is the area of the GFN A as defined in Definition 2.5.

If the GFN is such that $a = b = c = d$ and $\omega_1 = \omega_2 = \omega$, then the ROG point is given by $r_x^A = \frac{\omega}{\sqrt{3}}$ and $r_y^A = a$. For the detail derivation one may refer to Yong et al. (2004).

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