Propagation of Electromagnetic Waves in a Waveguide Filled with Space-Time Multiperiodic Medium

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***Abstract*—In this article the interaction of transverse-electric (TE) and transverse-magnetic (TM) waves with space-time non-magnetic multiperiodically modulated filling of the waveguide is considered under the assumption of small modulation indexes. TE and TM fields in the waveguide are described by the longitudinal components of magnetic and electric vectors, respectively. The wave equations for the specified components are received from Maxwell equations, which are reduced to the Mathieu-Hill equations by changing variables. The solutions of these equations in the frequency domain of "weak" interaction of signal wave with modulation wave (the Wulff-Bragg condition is not satisfied) was found with an accuracy of terms proportional to in the first degree of modulation indexes. It is shown, that TE and TM fields in the waveguide in the approximation, indicated above, consist of the sum of three space-time harmonics with different amplitudes. An analytical expression is also found for the frequency around which a "strong" interaction between the signal wave and the waveguide modulation wave can occur.**

***Keywords*—waveguide, interaction, multiperiodic, space-time, harmonics, equation**

I. Introduction

The propagation of electromagnetic waves in periodically modulated media has been considered in many articles [1-5]. In the articles [6] and [7] were discussed the features of the propagation of electromagnetic waves in a waveguide with two-periodic modulated dielectric filling. In this article the propagation of the signal TE and TM waves in the waveguide of arbitrary cross section filled with a multiperiodically modulated in space and time filling is considered. As noted in the articles [8] and [9] multiperiodically modulated media can represent the certain interest at carrying out of researches in various fields of science and in designing various radio engineering devices.

II. Statement of the Problem and its Solution

Let the axis of the waveguide of an arbitrary cross section and the axis oz of some rectangular coordinate system coincide and the dielectric permittivity of the waveguide filling is multiperiodically modulated in space and time according to the law

$$\varepsilon = \varepsilon_0 \left[1 + \sum_{j=1}^{5} m_j \cos k_j \left(z - ut \right) \right],$$

where ε_0 is the dielectric permittivity of the waveguide filling in the absence of modulation, k_j are the wave numbers of modulation waves, $m_j \left(m_j << 1 \right)$ are the small modulation indexes, u is the velocity of modulation waves. Let us consider the propagation of a signal TE and TM waves with a frequency ω_0 in a similar waveguide when the direction of the wave propagation coincides with the positive direction of the oz axis. From the system of Maxwell equations

$$curl\vec{E} = -\frac{1}{c} \cdot \frac{\partial \vec{B}}{\partial t}, \; curl\vec{H} = \frac{1}{c} \cdot \frac{\partial \vec{D}}{\partial t},$$

$$div\vec{D} = 0, \; div\vec{B} = 0, \; \vec{D} = \varepsilon_0 \varepsilon \left(z \right) \vec{E}, \; \vec{B} = \mu_0 \vec{H},$$

where c is the velocity of light in vacuum, it is possible to obtain the wave equations for the longitudinal components of the magnetic vector and electric vector (H_z and E_z completely determine the TE and TM fields in the waveguide) in the form

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} - \frac{1}{c^2} \cdot \frac{\partial}{\partial t} \left(\varepsilon \frac{\partial H_z}{\partial z} \right) = 0. \quad (1)$$

$$\frac{\partial^2 \tilde{E}_z}{\partial x^2} + \frac{\partial^2 \tilde{E}_z}{\partial y^2} + \varepsilon \frac{\partial}{\partial z} \left(\frac{1}{\varepsilon} \cdot \frac{\partial \tilde{E}_z}{\partial z} \right) - \frac{\varepsilon}{c^2} \cdot \frac{\partial}{\partial t} \left(\frac{\partial \tilde{E}_z}{\partial t} \right) = 0, \quad (2)$$

where $\tilde{E}_z = \varepsilon \cdot E_z$.

Passing in equations (1) and (2) to variables ξ and η by the formulas

$$\xi = z - ut, \eta = \frac{z}{u} - \frac{1}{u} \int_0^{\xi} \frac{d\xi}{1 - \beta^2 \frac{\varepsilon(\xi)}{\varepsilon_0}},$$

where $\beta^2 = u^2 \varepsilon_0 / c^2$, we get

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial}{\partial \xi} \left[\left(1 - \beta^2 \frac{\varepsilon}{\varepsilon_0} \right) \frac{\partial H_z}{\partial \xi} \right] - \frac{\varepsilon}{\left(1 - \beta^2 \frac{\varepsilon}{\varepsilon_0} \right) c^2} \cdot \frac{\partial^2 H_z}{\partial \eta^2} = 0. \quad (3)$$

$$\frac{\partial^2 \tilde{E}_z}{\partial x^2}+\frac{\partial^2 \tilde{E}_z}{\partial y^2}+\varepsilon\frac{\partial}{\partial \xi}\left[\frac{1}{\varepsilon}\left(1-\beta^2\frac{\varepsilon}{\varepsilon_0}\right)\frac{\partial \tilde{E}_z}{\partial \xi}\right]- -\frac{\varepsilon}{\left(1-\beta^2\frac{\varepsilon}{\varepsilon_0}\right)c^2}\cdot\frac{\partial^2 \tilde{E}_z}{\partial \eta^2}=0. \quad (4)$$

If the solution of equations (3) and (4) is sought in the form as an expansion of the eigenfunctions $\hat{\psi}_n(x,y)$ and $\psi_n(x,y)$ of the second and first boundary value problems for the waveguide cross section, that is

$$H_z=\sum_{n=0}^{\infty}H_n(z,t)\cdot\hat{\psi}_n(x,y)=\sum_{n=0}^{\infty}e^{j\gamma\eta}\cdot H_n(\xi)\cdot\hat{\psi}_n(x,y), \quad (5)$$

$$\tilde{E}_z=\sum_{n=0}^{\infty}\tilde{E}_n(z,t)\cdot\psi_n(x,y)=\sum_{n=0}^{\infty}e^{j\gamma\eta}\cdot\tilde{E}_n(\xi)\cdot\psi_n(x,y), \quad (6)$$

and it means, that $\hat{\psi}_n(x,y)$ and $\psi_n(x,y)$ satisfy the equations

$$\frac{\partial^2\hat{\psi}_n(x,y)}{\partial x^2}+\frac{\partial^2\hat{\psi}_n(x,y)}{\partial y^2}+\hat{\lambda}_n^2\hat{\psi}_n(x,y)=0,\left.\frac{\partial\hat{\psi}_n(x,y)}{\partial\vec{n}}\right|_{\Sigma}=0, \quad (7)$$

$$\frac{\partial^2\psi_n(x,y)}{\partial x^2}+\frac{\partial^2\psi_n(x,y)}{\partial y^2}+\lambda_n^2\psi_n(x,y)=0,\left.\psi_n(x,y)\right|_{\Sigma}=0, \quad (8)$$

where $\hat{\lambda}_n$ and λ_n are the eigenvalues of the second and first boundary value problems for the waveguide cross section, Σ is the contour of the waveguide cross section, $\vec{n}$ is the normal to Σ, then for $H_n(\xi)$ and $\tilde{E}_n(\xi)$ we obtain the following ordinary second order differential equations

$$\frac{d}{d\xi}\left[\left(1-\beta^2\cdot\frac{\varepsilon}{\varepsilon_0}\right)\cdot\frac{dH_n(\xi)}{d\xi}\right]+\frac{\hat{\chi}_n^2}{1-\beta^2\cdot\frac{\varepsilon}{\varepsilon_0}}\cdot H_n(\xi)=0. \quad (9)$$

$$\varepsilon\frac{d}{d\xi}\left[\frac{1}{\varepsilon}\left(1-\beta^2\cdot\frac{\varepsilon}{\varepsilon_0}\right)\cdot\frac{d\tilde{E}_n(\xi)}{d\xi}\right]+\frac{\chi_n^2}{1-\beta^2\cdot\frac{\varepsilon}{\varepsilon_0}}\cdot\tilde{E}_n(\xi)=0, \quad (10)$$

Note, that in (9) and (10) $\hat{\chi}_n^2$ and χ_n^2 have the form

$$\hat{\chi}_n^2=\frac{\gamma^2}{c^2}\varepsilon-\hat{\lambda}_n^2\left(1-\beta^2\cdot\frac{\varepsilon}{\varepsilon_0}\right),\chi_n^2=\frac{\gamma^2}{c^2}\varepsilon-\lambda_n^2\left(1-\beta^2\cdot\frac{\varepsilon}{\varepsilon_0}\right).$$

In (9) and (10) we introduce a new variables by the formulas

$$\hat{s}=gb\int_0^{\xi}\frac{d\xi}{1-\beta^2\cdot\frac{\varepsilon}{\varepsilon_0}},\ s=\frac{gb}{\varepsilon_0}\int_0^{\xi}\frac{\varepsilon\, d\xi}{1-\beta^2\cdot\frac{\varepsilon}{\varepsilon_0}}.$$

where $b=1-\beta^2$, $g=\sum_{j=1}^{5}(r_j\cdot k_j)/2, r_j\in N$. It is easy to notice, that then the equations (9) and (10) take the form

$$\frac{d^2H_n(\hat{s})}{d\hat{s}^2}+\hat{f}_n(\hat{s})H_n(\hat{s})=0,\frac{d^2\tilde{E}_n(s)}{ds^2}+f_n(s)\tilde{E}_n(s)=0, \quad (11)$$

where

$$\hat{f}_n(\hat{s})=h\left[\hat{\chi}_{n0}^2+\left(\frac{\gamma^2\varepsilon_0}{c^2\beta^2}+\hat{\lambda}_n^2\right)\cdot\sum_{j=1}^{5}l_j\cos(2\alpha_j\hat{s})\right],$$

$$f_n(s)=h\left[\chi_{n0}^2+d_n\cdot\sum_{j=1}^{5}l_j\cos(2\alpha_j s)\right],$$

$$\hat{\chi}_{n0}^2=\frac{\gamma^2}{c^2b}\varepsilon_0-\hat{\lambda}_n^2,\ \chi_{n0}^2=\frac{\gamma^2}{c^2b}\varepsilon_0-\lambda_n^2,\ \alpha_j=\frac{k_j}{2g},$$

$$h=\frac{1}{bg^2},l_j=\frac{\beta^2}{b}m_j,\ d_n=\frac{\gamma^2\varepsilon_0}{bc^2\beta^2}+\lambda_n^2\cdot\frac{2-\beta^2}{\beta^2}.$$

If now the functions $\hat{f}_n(\hat{s})$ and $f_n(s)$ are decomposed into a Fourier series and is limited to the terms proportional to small parameters in the first degree taking into account the fact, that at small parameters m_j the quantities l_j are also small $(m_j\approx l_j)$, then the equations (11) take the form of Mathieu-Hill equation

$$\frac{d^2H_n(s)}{ds^2}+\left(\sum_{k=-1}^{1}\hat{\theta}_k^n\cdot e^{2iks}\right)\cdot H_n(s)=0, \quad (12)$$

$$\frac{d^2\tilde{E}_n(s)}{ds^2}+\left(\sum_{k=-1}^{1}\theta_k^n\cdot e^{2iks}\right)\cdot\tilde{E}_n(s)=0, \quad (13)$$

where

$$\hat{\theta}_0^n=h\left(\frac{\gamma^2}{c^2b}\varepsilon_0-\hat{\lambda}_n^2\right),\ \theta_0^n=h\left(\frac{\gamma^2}{c^2b}\varepsilon_0-\lambda_n^2\right), \quad (14)$$

$$\hat{\theta}_{\pm1}^n=\frac{h\hat{d}_n}{4\pi}\cdot\sum_{j=1}^{5}\frac{2\alpha_j\cdot\sin(2\pi\alpha_j)}{\alpha_j^2-1}\cdot l_j,\ \hat{d}_n=\frac{\gamma^2}{c^2\beta^2}\cdot\varepsilon_0+\hat{\lambda}_n^2,$$

$$\theta_{\pm1}^n=\frac{hd_n}{4\pi}\cdot\sum_{j=1}^{5}\frac{2\alpha_j\cdot\sin(2\pi\alpha_j)}{\alpha_j^2-1}\cdot l_j.$$

As is known, the solutions of equations (11) have the form [10]

$$H_n(\hat{s})=e^{i\hat{\mu}_n\hat{s}}\cdot\sum_{k=-1}^{1}\hat{c}_k^n\cdot e^{2ik\hat{s}}, \quad (15)$$

$$\tilde{E}_n(s)=e^{i\mu_n s}\cdot\sum_{k=-1}^{1}c_k^n\cdot e^{2iks}, \quad (16)$$

where $\hat{\mu}_n,\mu_n,\hat{c}_k^n$ and c_k^n yet unknown quantities. Substitution of (15) and (16) into (12) and (13) leads to the dispersion equations of the problem for determining the characteristic numbers $\hat{\mu}_n$ and μ_n, and to the systems of algebraic equations for determining the unknown coefficients $\widehat{c}_k^n$ and c_k^n. The dispersion equations with the accuracy of small parameters in the second degree have the form

$$\widehat{\mu}_n^2 \simeq \widehat{\theta}_0^n + \left(\widehat{\theta}_1^n\right)^2 \cdot \left[\frac{1}{\left(\widehat{\mu}_n - 2\right)^2 - \widehat{\theta}_0^n} + \frac{1}{\left(\widehat{\mu}_n + 2\right)^2 - \widehat{\theta}_0^n}\right], \quad (17)$$

$$\mu_n^2 \simeq \theta_0^n + \left(\theta_1^n\right)^2 \cdot \left[\frac{1}{\left(\mu_n - 2\right)^2 - \theta_0^n} + \frac{1}{\left(\mu_n + 2\right)^2 - \theta_0^n}\right]. \quad (18)$$

Solving the above-stated system of algebraic equations for coefficients $\widehat{c}_k^n$ and c_k^n with accuracy to small parameters in the first degree inclusive, we obtain

$$\widehat{c}_{\pm 1}^n \simeq \frac{\widehat{\theta}_1^n \cdot \widehat{c}_0^n}{\left(\widehat{\mu}_n \pm 2\right)^2 - \widehat{\theta}_0^n}, \quad c_{\pm 1}^n \simeq \frac{\theta_1^n \cdot c_0^n}{\left(\mu_n \pm 2\right)^2 - \theta_0^n},$$

where $\widehat{c}_0^n$ and c_0^n are determined from the normalization condition.

As is known (see, for example, [10]), when the conditions $\widehat{\mu}_n^2 \simeq \widehat{\theta}_0^n \neq 1$ and $\mu_n^2 \simeq \theta_0^n \neq 1$ are fulfilled, we fall into the frequency domain of weak interaction between the signal wave and the modulation wave. Passing to the variables z and t in (15) and (16), taking into account, that

$$\widehat{s} \simeq g \cdot \left[z - ut + \sum_{j=1}^{5} \frac{l_j}{k_j} \cdot \sin k_j \left(z - ut\right)\right],$$

$$s \simeq g \cdot \left[z - ut + \frac{1}{\beta^2} \cdot \sum_{j=1}^{5} \frac{l_j}{k_j} \cdot \sin k_j \left(z - ut\right)\right],$$

and $E_z = \tilde{E}_z / \varepsilon$, in specified above frequency domain from (5) and (6) for $H_z(x,y,z,t)$ and $E_z(x,y,z,t)$ we obtain the following expressions

$$H_z\left(x,y,z,t\right) = \sum_{n=0}^{\infty} \widehat{\psi}_n\left(x,y\right) \cdot \widehat{c}_0^n \cdot e^{i\left(\widehat{p}_0^n z - \omega_0 t\right)} \cdot$$

$$\cdot \sum_{k=-1}^{1} \left\{\frac{k}{2}\left[\sum_{j=1}^{5} \widehat{\Delta}_{k_j}^{nk} \cdot e^{ikk_j(z-ut)}\right] + \frac{\widehat{c}_k^n}{\widehat{c}_o^n} \cdot e^{2jkg(z-ut)}\right\}^{|k|},$$

$$E_z\left(x,y,z,t\right) = \sum_{n=0}^{\infty} \psi_n\left(x,y\right) \cdot c_0^n \cdot e^{i\left(p_0^n z - \omega_0 t\right)} \cdot$$

$$\cdot \sum_{k=-1}^{1} \left\{\frac{k}{2}\left[\sum_{j=1}^{5} \left(\Delta_{k_j}^{nk} - m_j\right) e^{ikk_j(z-ut)}\right] + \frac{c_k^n}{c_o^n} \cdot e^{2jkg(z-ut)}\right\}^{|k|},$$

where

$$\widehat{\Delta}_{k_j}^{nk} = \frac{1}{k_j} \cdot \left[g\left(\sqrt{\widehat{\theta}_0^n} + 2k\right) - \frac{\gamma}{ub}\right] \cdot l_j,$$

$$\widehat{p}_0^n = \frac{\gamma}{u} - \frac{\gamma}{ub} + g\sqrt{\widehat{\theta}_0^n}, \omega_0 = gu\sqrt{\widehat{\theta}_0^n} - \frac{\gamma}{b},$$

$$\Delta_{k_j}^{nk} = \frac{1}{k_j} \cdot \left[\frac{g}{\beta^2}\left(\sqrt{\widehat{\theta}_0^n} + 2k\right) - \frac{\gamma}{ub}\right] \cdot l_j,$$

$$p_0^n = \frac{\gamma}{u} - \frac{\gamma}{ub} + g\sqrt{\theta_0^n}, \omega_0 = gu\sqrt{\theta_0^n} - \frac{\gamma}{b}.$$

From Maxwell's equations in view of (7) and (8) for the transvers components of TE and TM fields in the waveguide can be obtained the following expressions

TE field

$$\vec{H}_\tau = \sum_{n=0}^{\infty} \widehat{\lambda}_n^{-2} \cdot \frac{\partial H_n\left(z,t\right)}{\partial z} \cdot \nabla \widehat{\psi}_n\left(x,y\right),$$

$$\vec{E}_\tau = \sum_{n=0}^{\infty} \widehat{\lambda}_n^{-2} \cdot \frac{\partial H_n\left(z,t\right)}{\partial t} \cdot \left[\vec{z}_0 \nabla \widehat{\psi}_n\left(x,y\right)\right],$$

TM field

$$\vec{H}_\tau = -\frac{1}{c} \cdot \sum_{n=0}^{\infty} \lambda_n^{-2} \cdot \frac{\partial\left(\varepsilon E_n\left(z,t\right)\right)}{\partial t} \cdot \left[\vec{z}_0 \nabla \psi_n\left(x,y\right)\right],$$

$$\vec{E}_\tau = \frac{1}{\varepsilon} \cdot \sum_{n=0}^{\infty} \lambda_n^{-2} \cdot \frac{\partial\left(\varepsilon E_n\left(z,t\right)\right)}{\partial z} \cdot \nabla \psi_n\left(x,y\right).$$

where $\nabla = \vec{i}\left(\partial/\partial x\right) + \vec{j}\left(\partial/\partial y\right)$, subscript τ indicates transverse components.

Note, that for $\widehat{\theta}_0^n \simeq 1$ and $\theta_0^n \simeq 1$ from dispersion equations (17) and (18) for the characteristic numbers $\widehat{\mu}_n$ and μ_n are obtained complex solutions [10]. This leads to the fact, that around a certain frequency there is a strong interaction between the signal wave and the modulation wave and the amplitude at minus first harmonic is of the same order as the amplitude at zero harmonic $\left(\widehat{c}_{-1}^n \simeq \widehat{c}_0^n, c_{-1}^n \simeq c_0^n\right)$, and the amplitude at plus first harmonic is proportional to the small parameters in the first degree $\left(\widehat{c}_1^n \simeq m_j, c_1^n \simeq m_j\right)$. The expressions for the frequency of strong interaction can be obtained from (14) taking into account, that $\widehat{\theta}_0^n \simeq 1$ and $\theta_0^n \simeq 1$. Calculations lead to the following

$$\omega_{0s}^{(TE)} = \frac{gu}{\beta} \cdot \left(\widehat{\eta}_n + \beta\right), \widehat{\eta}_n = \sqrt{1 + \frac{\widehat{\lambda}_n^2}{g^2 b}}, \quad (19)$$

$$\omega_{0s}^{(TM)} = \frac{gu}{\beta} \cdot \left(\eta_n + \beta\right), \eta_n = \sqrt{1 + \frac{\lambda_n^2}{g^2 b}}. \quad (20)$$

If in formulas (19) and (20) go to the limit at $u \rightarrow 0$, then we obtain the frequencies of strong interaction between the signal wave and the modulation wave in the case of multiperiodically modulated inhomogeneous waveguide filling in the form

$$\omega_{0s}^{(TE)}\Big|_{u=0} = \frac{gc}{\sqrt{\varepsilon_0}} \cdot \sqrt{1+\frac{\hat{\lambda}_n^2}{g^2}}, \quad \omega_{0s}^{(TM)}\Big|_{u=0} = \frac{gc}{\sqrt{\varepsilon_0}} \cdot \sqrt{1+\frac{\lambda_n^2}{g^2}}.$$

CONCLUSIONS

The results obtained in this article show, that the TE field in the waveguide with a multiperiodic space-time modulated filling is the sum of waves with different amplitudes and frequencies. It is shown, that there is a frequency around which the Wulff-Bragg condition of the first order is satisfied. We also note, that with help of method developed in this article it is possible to solve the problems of transition radiation of sources moving at a constant velocity in the waveguide filled with a space-time multiperiodically modulated medium.

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