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Innovative Applications of O.R.

Minimal repair of failed components in coherent systems

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ABSTRACT

The minimal repair replacement is a reasonable assumption in many practical systems. Under this assumption a failed component is replaced by another one whose reliability is the same as that of the component just before the failure, i.e., a used component with the same age. In this paper we study the minimal repair in coherent systems. We consider both the cases of independent and dependent components. Three replacement policies are studied. In the first one, the first failed component in the system is minimally repaired while, in the second one, we repair the component which causes the system failure. A new technique based on the relevation transform is used to compute the reliability of the systems obtained under these replacement policies. In the third case, we consider the replacement policy which assigns the minimal repair to a fixed component in the system. We compare these three options under different stochastic criteria and for different system structures. In particular, we provide the optimal strategies for all the coherent systems with 1–4 independent and identically distributed components.

1. Introduction

Nowadays, people demand more and more reliable systems. Several techniques have been developed to model and improve the reliability of a system. The basic concepts used in Reliability Theory were introduced in the classic book by Barlow and Proschan (1975). Recent developments can be seen, for example, in Aven and Jensen (2013) and Natvig (2011). A good way to improve the reliability of a system is to consider some redundancy or maintenance actions. These actions can be performed in different ways as, for example, by planning some replacement strategies, minimal repairs, imperfect repairs, redundancies, etc. On the one hand, it is addressed in literature the concept of *active* or *hot redundancy*, where some additional components are included in the system by using parallel structures, see Valdés and Zequeira (2006), Zhao, Chan, Li, and Ng (2013), Zhao, Chan, and Ng (2012), and Zhao, Zhang, and Chen (2017), or Belzunce, Martínez-Puertas, and Ruiz (2013) and Zhang, Amini-Seresht, and Ding (2017) for systems having independent and dependent components, respectively. On the other hand, it is addressed the concept of *standby* or *cold redundancy*, where a component is replaced or repaired when it fails. Among the standby policies, many papers study the case of *perfect repairs* when the broken unit is replaced by a new and identical unit, see, e.g., Misra, Misra, and Dhariyal (2011), Singh and Misra (1994) and You and Li (2014). Nevertheless, there exist many options of replacement for a failed component. A nice summary of these cases is described in Aven (2014). In this paper we focus on *minimal repairs* as a particular case of cold redundancy. Under this assumption a failed component is repaired to be just as it was before its failure. This is equivalent to replace this unit by another one whose reliability is the same as that of the component just before the failure, that is, it is replaced by a used component with the same distribution and the same age. This concept allows us to describe many repairs in real cases where it is not unrealistic to think that repairs basically bring the system to the same condition it was just before the failure.

The basic minimal repair model was introduced in Barlow and Hunter (1960). To formalize this idea, the basic model assumes that the repair time is negligible and the number of failures that occur in the interval $(0, t]$ follows a nonhomogeneous Poisson process (NHPP) with an intensity function $\lambda(t)$. Since then, many works have been published attempting to extend the minimal repair concept. For example, Brown and Proschan (1983) examined the case of imperfect repair which uses a perfect repair with probability p and a minimal repair with probability $1-p$. This model was generalized by Block, Borges, and Savits (1985) by considering that the probability of perfect repair depends on the system's state and by Shaked and Shanthikumar (1986) for the multivariate case. Phelps (1983) obtained an optimal policy for

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the replacement problem with minimal repair, under the assumption of an increasing failure rate. Stadje and Zuckerman (1991) studied a maintenance model in which the degree of repair is a decision variable determined by a controller and it varies between minimal and perfect repairs. Beichelt (1993) proposed a new common framework, based on a general failure model, to include different replacement policies under minimal repair. Finkelstein (2004) generalized the notion of minimal repair to the case when the lifetime distribution function is a continuous or a discrete mixture of distributions, that is, a heterogeneous population. Aven and Castro (2008) and Zequeira and Berenguer (2006) analysed an optimal strategy of maintenance from two types of failures in the system and the associated repair costs. More recently, Balakrishnan, Kamps, and Kateri (2009) introduced minimal repair processes under a simple step-stress test in the context of life-testing reliability experiments. Some authors split minimal repairs into two cases: physical minimal repairs and statistical minimal repairs. The former is used when a component of the system is repaired and the latter, also called black box minimal repair, when the system is repaired, see Aven (2014) and Aven and Jensen (2000, 2013) for further details and illustrative examples. As it is nicely discussed in Aven (2014), the need to be precise with the level of information leads to some author to frame their works in the theory of point processes by taking into account the history of the system. Some valuables contributions in this sense are given by Arjas and Norros (1989), Aven (1983, 1987, 1996), Aven and Jensen (2000, 2013), Bergman (1985), Gåsemyr and Natvig (2017) and Natvig (1979, 1990).

In the literature there exist alternative approaches not based in processes as well. For example, some stochastic comparisons of repairable coherent systems with independent components were obtained in Belzunce, Martínez-Riquelme, and Ruiz (2018), Chahkandi, Ruggeri, and Suárez-Llorens (2016) and El-Neweihi and Sethuraman (1993) and some preservation results and aging properties of repairable systems under minimal repair were established in Chahkandi, Ahmadi, and Baratpour (2014). Recently, a new representation for the reliability function of a coherent system with possibly dependent components was obtained by using copulas, see, e.g., Miziuła and Navarro (2017) and Navarro, Pellerey, and Di Crescenzo (2015) or expression (2.4) below. This expression is very useful since the distortion (or aggregation) function $\bar{Q}$ contains all the information about the structure of the system and the dependency between the components (its survival copula). This representation was used in Arriaza, Navarro, and Suárez-Llorens (2018) to compare different replacement policies under minimal repairs when we have a limited maximum number k of repairs and they are assigned to fixed components in the system.

In this paper we use expression (2.4) as an alternative approach to model and compare the lifetimes of the repaired systems. Our approach could be considered as a good alternative to the classical approach based on processes and, in our opinion, satisfies some advantages. Firstly, the representation of the system reliability function in terms of distortion functions leads to simplify the complex algebraic expressions derived from the computation of the system's reliability. Secondly, our results can be applied to systems with independent or dependent components. Furthermore, this approach can be used to deal with systems having heterogeneous components. Finally, the main results allow us to get distribution-free comparisons (i.e. comparisons that do not depend on the distributions of the components) of the repaired systems. We study different repair policies based on minimal repairs of the failed components in the system. We will focus on comparing three different repair policies. The first policy, denoted by case I, consists in a minimal repair of the component that fails first. The second one, denoted by case II, consists in a minimal repair of the component that causes the system failure. The last one, denoted by case III, consists in assigning a minimal repair to a fixed component in the system. This last case is the one studied in Arriaza et al. (2018) when $k = 1$. In all these cases we will consider only one repair and we will compare the resulting systems under different stochastic criteria. Moreover, we show that the same technique can be applied to study k replacements and other replacement policies. In particular, we prove that the replacement policy of case II is better than that of case I under the assumption of independent and identically distributed (IID) components. However, some examples prove that they are not ordered with case III. We also apply this procedure to determine the best replacement policy in terms of the usual stochastic order for all the systems with 1–4 IID components.

The rest of the paper is organized as follows. In Section 2 we introduce the notation and the tools needed in the paper including the basic properties on the relevation transform and on coherent systems. The main results are given in Section 3, where we give a procedure to determine the reliability functions of the systems obtained with the replacement policies of the cases mentioned above. The expressions obtained are based on distortion functions. These representations are used in Section 4 to compare the different replacement policies. There we also provide some general results for systems with IID components. The conclusions are placed in Section 5.

Throughout the paper, we say that a function g is increasing (resp. decreasing) if $g(x) \le g(y)$ $(\ge)$ for all $x \le y$. If G: $[0, 1]^n \to [0, 1]$, then $\partial_i G$ represents the partial derivative of G with respect to the ith variable.

2. Notation and preliminary results

2.1. Relevation transform

Let X and Y be two nonnegative independent random variables with absolutely continuous reliability (survival) functions $\bar{F}$ and $\bar{G}$. Then the reliability function of $X + Y$ (convolution) is

$$\bar{F} * \bar{G}(t) = \Pr(X + Y > t) = \int_t^\infty f(x)dx + \int_0^t \int_{t-x}^\infty g(y)f(x)dydx$$
$$= \bar{F}(t) + \int_0^t \bar{G}(t-x)f(x)dx,$$

where f and g are the respective probability density functions. Under a *perfect repair* in a cold standby procedure, the unit X is replaced when failed by an independent unit Y having the same distribution as X (when new). Then the resulting reliability is

$$\bar{F} * \bar{F}(t) = \bar{F}(t) + \int_0^t \bar{F}(t-x)f(x)dx.$$

If X and Y are dependent, we obtain the expression included in the following definition.

Definition 2.1. If X and Y are two nonnegative dependent random variables with reliability functions $\bar{F}$ and $\bar{G}$, then the relevation transform (or conditional convolution) $\bar{F}\#\bar{G}$ is the reliability of $X + Y$ given by

$$\bar{F}\#\bar{G}(t) = \bar{F}(t) + \int_0^t \bar{G}_x(t-x)f(x)dx, \tag{2.1}$$

where f is the probability density function of X and $\bar{G}_x$ is the reliability function of $(Y|X = x)$.

Under a *classic relevation transform*, the unit X is replaced when it fails at a time x by a unit having reliability $\bar{G}$ but with the same age as X, that is, by $Y_x = (Y - x|Y > x)$ with reliability

$$\bar{G}_x(y) = \Pr(Y - x > y|Y > x) = \frac{\bar{G}(x+y)}{\bar{G}(x)}$$

for $y \geq 0$. Hence,

$$\bar{F}\#\bar{G}(t) = \Pr(X + Y_X > t) = \bar{F}(t) + \int_0^t \frac{\bar{G}(t)}{\bar{G}(x)} f(x)dx. \quad (2.2)$$

Under a *minimal repair*, the failed unit X is replaced by a unit having the same reliability as X and with the same age (that is, it is repaired to be as it was just before its failure). Then, from (2.2), the resulting reliability is

$$\bar{F}\#\bar{F}(t) = \bar{F}(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} f(x)dx = \bar{F}(t) - \bar{F}(t)\ln \bar{F}(t).$$

After k replacements, the resulting reliability is

$$\bar{F}\#^k\bar{F}(t) = \bar{F}(t)\sum_{i=0}^{k}\frac{1}{i!}[-\ln \bar{F}(t)]^i,$$

where $\bar{F}\#^0\bar{F} = \bar{F}$, $\bar{F}\#^1\bar{F} = \bar{F}\#\bar{F}$, $\bar{F}\#^2\bar{F} = (\bar{F}\#\bar{F})\#\bar{F}$ and so on. Note that $(\bar{F}\#\bar{F})\#\bar{F} \neq \bar{F}\#(\bar{F}\#\bar{F})$. We shall write it as $\bar{F}\#^k\bar{F}(t) = \bar{q}_k(\bar{F}(t))$ with

$$\bar{q}_k(u) = u\sum_{i=0}^{k}\frac{1}{i!}(-\ln u)^i. \quad (2.3)$$

The distributions that can be written in this way are called *distorted distributions* (see, e.g., Navarro, del Águila, Sordo, & Suárez-Llorens, 2013; Navarro & Rychlik, 2010 and the references therein). Thus, we say that a distribution function F_q is a distortion of another distribution F if $F_q(t) = q(F(t))$ for a *distortion function* q: $[0, 1] \to [0, 1]$ increasing, continuous and such that $q(0) = 0$ and $q(1) = 1$. A similar representation holds for the respective reliability functions, that is, $\bar{F}_q(t) = \bar{q}(\bar{F}(t))$, where $\bar{q}(u) = 1 - q(1 - u)$ for $u \in [0, 1]$. It is also a distortion function, that is, it is an increasing continuous function in [0,1] such that $\bar{q}(0) = 0$ and $\bar{q}(1) = 1$ (see, e.g., (2.3)). It is called the *dual distortion function* associated to q.

2.2. Coherent systems

Let T be the lifetime of a coherent system with component lifetimes $X_1, \dots, X_n$. In the general case, the components can be dependent and this possible dependency will be represented by the joint reliability of the components lifetimes which can be written as

$$\bar{\boldsymbol{F}}(x_1, \dots, x_n) = \Pr(X_1 > x_1, \dots, X_n > x_n) = K(\bar{F}_1(x_1), \dots, \bar{F}_n(x_n)),$$

where K is the survival copula and $\bar{F}_i$ is the reliability function of the ith component for $i = 1, \dots, n$ (see, e.g., Durante and Sempi (2015, p. 33)). Note that the case of independent components is included here and that it is represented by the product copula $K = \Pi$, where $\Pi(u_1, \dots, u_n) = u_1 \dots u_n$ for $u_1, \dots, u_n \in [0, 1]$. From now on we assume that $\bar{\boldsymbol{F}}$ is absolutely continuous with joint probability density function

$$\boldsymbol{f}(x_1, \dots, x_n) = k(\bar{F}_1(x_1), \dots, \bar{F}_n(x_n))f_1(x_1)\dots f_n(x_n),$$

where f_i is the probability density function of X_i and

$$k(u_1, \dots, u_n) = \partial_1 \dots \partial_n K(u_1, \dots, u_n) = \frac{\partial^n}{\partial u_1 \dots \partial u_n}K(u_1, \dots, u_n)$$

is the probability density function associated to K.

Then it is well known (see, e.g., Miziuła & Navarro, 2017; Navarro et al., 2015) that the system reliability can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t)), \quad (2.4)$$

where $\bar{Q}$ is a *distortion function*, that is, a continuous increasing function $\bar{Q} : [0, 1]^n \to [0, 1]$ such that $\bar{Q}(0, \dots, 0) = 0$ and $\bar{Q}(1, \dots, 1) = 1$ which depends on the system structure and on K (the dependence structure). In particular, if the components are identically distributed (ID), then (2.4) reduces to $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$ (see, e.g., Navarro & Rychlik, 2010) where $\bar{F}$ is the common reliability function of the components and $\bar{q}(u) = \bar{Q}(u, \dots, u)$ (i.e., T has a distorted distribution from the common distribution of the components). If the components are just independent, then $\bar{Q}$ is a multinomial expression (see Barlow & Proschan, 1975, p. 21). Finally, if they are independent and identically distributed (IID), then $\bar{q}(u) = \sum_{i=1}^n a_i u^i$, where $(a_1, \dots, a_n)$ is called the *minimal signature* of the system (see, e.g., Navarro & Rubio, 2010).

For example, if $n = 2$, the reliability function of the parallel system $X_{2:2} = \max(X_1, X_2)$ is

$$\begin{aligned}\bar{F}_{2:2}(t) &= \Pr(\{X_1 > t\} \cup \{X_2 > t\}) = \bar{F}_1(t) + \bar{F}_2(t) \\ &\quad - \Pr(X_1 > t, X_2 > t) = \bar{Q}_{2:2}(\bar{F}_1(t), \bar{F}_2(t)),\end{aligned}$$

where $\bar{Q}_{2:2}(u, v) = u + v - K(u, v)$ and, in the IID case, $\bar{q}_{2:2}(u) = \bar{Q}_{2:2}(u, u) = 2u - u^2$.

2.3. Reliability of systems using the relevation transform

Let us see how the relevation transform can also be used to compute the system's reliability. This new technique will be used in the following sections to compute the reliability of systems with minimal repairs on failed components. As in the preceding section we consider the simple case of a two-component parallel system.

Example 2.2. Let us consider $X_{2:2} = \max(X_1, X_2)$. If the component lifetimes X_1, X_2 are IID with a common reliability $\bar{F}$, then

$$\bar{F}_{2:2}(t) = \bar{F}_{1:2}\#\bar{F}(t) = \bar{F}_{1:2}(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} f_{1:2}(x)dx$$

and, as $\bar{F}_{1:2}(t) = \bar{F}^2(t)$ and $f_{1:2}(t) = 2\bar{F}(t)f(t)$, we have

$$\begin{aligned}\bar{F}_{2:2}(t) &= \bar{F}^2(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} 2\bar{F}(x)f(x)dx \\ &= \bar{F}^2(t) + 2\bar{F}(t)F(t) = 2\bar{F}(t) - \bar{F}^2(t).\end{aligned}$$

Let us assume now that both components can be dependent with a survival copula K. Then

$$\begin{aligned}\bar{F}_{2:2}(t) &= \Pr(X_1 < X_2)\Pr(X_{2:2} > t | X_1 < X_2) \\ &\quad + \Pr(X_2 < X_1)\Pr(X_{2:2} > t | X_2 < X_1) \\ &= \Pr(X_1 < X_2)\bar{F}_1^{(X_1<X_2)}\#\bar{G}_1(t) + \Pr(X_2 < X_1)\bar{F}_2^{(X_2<X_1)}\#\bar{G}_2(t),\end{aligned}$$

where $\bar{F}_1^{(X_1<X_2)}(t) = \Pr(X_1 > t | X_1 < X_2)$, $\bar{F}_2^{(X_2<X_1)}(t) = \Pr(X_2 > t | X_2 < X_1)$, $\bar{G}_{1,x}(y) = \Pr(X_2 - x > y | X_1 = x, X_2 > x)$ and $\bar{G}_{2,x}(y) = \Pr(X_1 - x > y | X_2 = x, X_1 > x)$. Note that

$$\begin{aligned}p_1 &= \Pr(X_1 < X_2) = \int_0^\infty \int_x^\infty f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))dydx \\ &= \int_0^\infty f_1(x)\partial_1 K(\bar{F}_1(x), \bar{F}_2(x))dx\end{aligned}$$

when $\lim_{u\to 0^+} \partial_1 K(\bar{F}_1(x), u) = 0$ (see Navarro & Sordo, 2018). Analogously,

$$p_2 = \Pr(X_2 < X_1) = 1 - p_1 = \int_0^\infty f_2(x)\partial_2 K(\bar{F}_1(x), \bar{F}_2(x))dx$$

when $\lim_{u\to 0^+} \partial_2 K(u, \bar{F}_2(y)) = 0$. The joint density of $(X_1, X_2 | X_1 < X_2)$ is $\boldsymbol{h}(x, y) = \boldsymbol{f}(x, y)/p_1$ for all $x \leq y$ (0 otherwise). Then the marginal density of $(X_1 | X_1 < X_2)$ is

$$\begin{aligned}h_1(x) &= \frac{1}{p_1}\int_x^\infty \boldsymbol{f}(x, y)dy \\ &= \frac{1}{p_1}\int_x^\infty f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))dy \\ &= \frac{1}{p_1}f_1(x)\partial_1 K(\bar{F}_1(x), \bar{F}_2(x)).\end{aligned}$$

Hence, the conditional density of $(X_2|X_1=x, X_2>x)$ is

$$h_{2|1}(y|x)=\frac{\boldsymbol{h}(x,y)}{h_1(x)}=\frac{f_2(y)\partial_{1,2}K(\bar F_1(x),\bar F_2(y))}{\partial_1 K(\bar F_1(x),\bar F_2(x))}$$

and then the reliability function $\bar G_{1,x}$ is given by

$$\bar G_{1,x}(y)=\int_{x+y}^{\infty}h_{2|1}(z|x)dz=\int_{x+y}^{\infty}\frac{f_2(z)\partial_{1,2}K(\bar F_1(x),\bar F_2(z))}{\partial_1 K(\bar F_1(x),\bar F_2(x))}dz$$
$$=\frac{\partial_1 K(\bar F_1(x),\bar F_2(x+y))}{\partial_1 K(\bar F_1(x),\bar F_2(x))}. \tag{2.5}$$

In a similar way (by the symmetry), we get

$$\bar G_{2,x}(y)=\frac{\partial_2 K(\bar F_1(x+y),\bar F_2(x))}{\partial_2 K(\bar F_1(x),\bar F_2(x))}. \tag{2.6}$$

Therefore, from (2.1), we obtain

$$\begin{aligned}\bar F_1^{(X_1<X_2)}\#\bar G_1(t)&=\bar F_1^{(X_1<X_2)}(t)+\int_0^t \bar G_{1,x}(t-x)h_1(x)dx\\&=\bar F_1^{(X_1<X_2)}(t)+\frac{1}{p_1}\int_0^t f_1(x)\partial_1 K(\bar F_1(x),\bar F_2(t))dx\\&=\bar F_1^{(X_1<X_2)}(t)+\frac{1}{p_1}\left[\bar F_2(t)-K(\bar F_1(t),\bar F_2(t))\right].\end{aligned}$$

Analogously, we have

$$\bar F_2^{(X_2<X_1)}\#\bar G_2(t)=\bar F_2^{(X_2<X_1)}(t)+\frac{1}{p_2}[\bar F_1(t)-K(\bar F_1(t),\bar F_2(t))].$$

Then

$$\begin{aligned}\bar F_{2:2}(t)&=p_1\bar F_1^{(X_1<X_2)}\#\bar G_1(t)+p_2\bar F_2^{(X_2<X_1)}\#\bar G_2(t)\\&=p_1\bar F_1^{(X_1<X_2)}(t)+p_2\bar F_2^{(X_2<X_1)}(t)+\bar F_1(t)\\&\quad+\bar F_2(t)-2K(\bar F_1(t),\bar F_2(t))\\&=p_1\Pr(X_{1:2}>t|X_1<X_2)+p_2\Pr(X_{1:2}>t|X_2<X_1)\\&\quad+\bar F_1(t)+\bar F_2(t)-2K(\bar F_1(t),\bar F_2(t))\\&=\Pr(X_{1:2}>t)+\bar F_1(t)+\bar F_2(t)-2K(\bar F_1(t),\bar F_2(t))\\&=\bar F_1(t)+\bar F_2(t)-K(\bar F_1(t),\bar F_2(t)).\end{aligned}$$

These expressions can be simplified if $\boldsymbol{F}$ is exchangeable (EXC), that is, K is permutation invariant and the components are ID. In this case we have $\bar F_{2:2}=\bar F_{1:2}\#\bar G$, where

$$\bar G_x(y)=\Pr(X_2-x>y|X_1=x,X_2>x)=\frac{\Pr(X_2>x+y|X_1=x)}{\Pr(X_2>x|X_1=x)}.$$

Then, from (2.5), we get $\bar G_x(y)=\dfrac{\partial_1 K(\bar F(x),\bar F(x+y))}{\partial_1 K(\bar F(x),\bar F(x))}$. Hence, from (2.1), we have

$$\begin{aligned}\bar F_{1:2}\#\bar G(t)&=\bar F_{1:2}(t)+\int_0^t \bar G_x(t-x)f_{1:2}(x)dx\\&=\bar F_{1:2}(t)+\int_0^t\frac{\partial_1 K(\bar F(x),\bar F(t))}{\partial_1 K(\bar F(x),\bar F(x))}f_{1:2}(x)dx,\end{aligned}$$

where $\bar F_{1:2}(x)=K(\bar F(x),\bar F(x))$ and $f_{1:2}(x)=2f(x)\partial_1 K(\bar F(x),\bar F(x))$. Therefore,

$$\begin{aligned}\bar F_{1:2}\#\bar G(t)&=\bar F_{1:2}(t)+2\int_0^t\partial_1 K(\bar F(x),\bar F(t))f(x)dx\\&=K(\bar F(t),\bar F(t))-2K(\bar F(t),\bar F(t))+2K(1,\bar F(t))\\&=2\bar F(t)-K(\bar F(t),\bar F(t)).\end{aligned}$$

Another approach for the general case is

$$\bar F_{2:2}=\bar F_{1:2}\#\bar G, \tag{2.7}$$

where

$$\begin{aligned}\bar G_x(y)&=p_1(x)\Pr(X_2-x>y|X_1=x,X_2>x)\\&\quad+p_2(x)\Pr(X_1-x>y|X_2=x,X_1>x)\\&=p_1(x)\frac{\Pr(X_2>x+y|X_1=x)}{\Pr(X_2>x|X_1=x)}+p_2(x)\frac{\Pr(X_1>x+y|X_2=x)}{\Pr(X_1>x|X_2=x)},\end{aligned}$$

$p_1(x)=\Pr(X_1<X_2|X_{1:2}=x)$ and $p_2(x)=\Pr(X_2<X_1|X_{1:2}=x)$.

Similar expressions can be obtained for other order statistics (k-out-of-n systems), that is, for $X_{i:n}$, $i=1,\dots,n$. For example, in the IID case, the reliability of $X_{2:3}$ can be written as $\bar F_{2:3}=\bar F_{1:3}\#\bar F_{1:2}$ or that of $X_{3:3}$ as $\bar F_{3:3}=(\bar F_{1:3}\#\bar F_{1:2})\#\bar F$. Analogous (but more complicated) expressions hold for general coherent systems.

3. Main results

With the notation introduced in the preceding section, let us assume that we have a coherent system with lifetime T based on n components with lifetimes $X_1,\dots,X_n$. If we apply a single minimal repair to the system then the main options are:

Case I: To repair the component which fails first.

Case II: To repair the component which leads to the system failure.

Case III: To repair a fixed component (e.g., to repair the ith component).

Other options will be considered later. If we can choose among these options (this is not always the case in practice), we need to determine which one is the best one under some stochastic criteria. To do this, we need to obtain the reliability of the resulting systems after these replacement policies.

From now on, we will denote by T_I and T_{II} the lifetimes associated to the resulting system under the policy I and II, respectively. In the third option, if we repair the ith component, the resulting system lifetime will be represented by $T_{III}^{(i)}$. If the dependence structure does not change after the replacement, then the reliability of $T_{III}^{(i)}$ is

$$\bar F_{T_{III}^{(i)}}(t)=\bar Q(\bar F_1(t),\dots,\bar F_{i-1}(t),\bar q_1(\bar F_i(t)),\bar F_{i+1}(t),\dots,\bar F_n(t)),$$

where $\bar q_1$ is given in (2.3). If the components are ID, then $\bar F_{T_{III}^{(i)}}(t)=\bar q_{III}^{(i)}(\bar F(t))$, where

$$\bar q_{III}^{(i)}(u)=\bar Q(u,\dots,u,\bar q_1(u),u,\dots,u) \tag{3.1}$$

and $\bar q_1$ is placed at the ith position. Comparison results for these kinds of replacements were given in Arriaza et al. (2018). Let us study the other two cases.

3.1. Case I

In this case we repair the component which fails first. Its lifetime is $X=X_{1:n}$. Then the broken component is minimally repaired and the resulting system has the same structure as T but we know that all the components are working and have age X. Hence its reliability is

$$\bar F_{T_I}(t)=\bar F_{1:n}\#\bar G(t), \tag{3.2}$$

where

$$\begin{aligned}\bar G_x(y)&=\Pr(T-x>y|X_1>x,\dots,X_n>x)\\&=\frac{\Pr(T>x+y,X_1>x,\dots,X_n>x)}{\Pr(X_1>x,\dots,X_n>x)}\end{aligned}$$

when $X=x$. In Proposition 3 of Navarro (2018) is proved that this reliability can be written as $\bar G_x(t)=\bar Q_x(\bar F_{1,x}(t),\dots,\bar F_{n,x}(t))$, where $\bar F_{i,x}(t)=\Pr(X_i-x>t|X_i>x)=\bar F_i(t+x)/\bar F_i(x)$ for $i=1,\dots,n$ and $\bar Q_x$

is a distortion function (see Section 2). Hence, from (2.1), we have,

$$\begin{aligned}\bar{F}_{T_I}(t) &= \bar{F}_{1:n}(t) + \int_0^t \bar{G}_x(t-x) f_{1:n}(x)dx \\ &= \bar{F}_{1:n}(t) + \int_0^t \bar{Q}_x(\bar{F}_{1,x}(t-x), \dots, \bar{F}_{n,x}(t-x)) f_{1:n}(x)dx. \quad (3.3)\end{aligned}$$

Let us see an example.

Example 3.1. If $T = X_{2:2}$ (a parallel system with two components), then

$$\begin{aligned}&\bar{G}_x(y) \\ &= \Pr(T - x > y | X_1 > x, X_2 > x) \\ &= \frac{\Pr(T > x+y, X_1 > x, X_2 > x)}{\Pr(X_1 > x, X_2 > x)} \\ &= \frac{\Pr(X_1 > x+y, X_2 > x) + \Pr(X_2 > x+y, X_1 > x) - \Pr(X_1 > x+y, X_2 > x+y)}{\Pr(X_1 > x, X_2 > x)} \\ &= \frac{K(\bar{F}_1(x+y), \bar{F}_2(x)) + K(\bar{F}_1(x), \bar{F}_2(x+y)) - K(\bar{F}_1(x+y), \bar{F}_2(x+y))}{K(\bar{F}_1(x), \bar{F}_2(x))} \\ &= \bar{Q}_x(\bar{F}_{1,x}(y), \bar{F}_{2,x}(y))\end{aligned}$$

for $y \geq 0$, with $\bar{F}_{1,x}(y) = \bar{F}_1(x+y)/\bar{F}_1(x)$, $\bar{F}_{2,x}(y) = \bar{F}_2(x+y)/\bar{F}_2(x)$ and

$$\begin{aligned}&\bar{Q}_x(u_1, u_2) \\ &= \frac{K(u_1\bar{F}_1(x), \bar{F}_2(x)) + K(\bar{F}_1(x), u_2\bar{F}_2(x)) - K(u_1\bar{F}_1(x), u_2\bar{F}_2(x))}{K(\bar{F}_1(x), \bar{F}_2(x))}\end{aligned}$$

whenever $K(\bar{F}_1(x), \bar{F}_2(x)) > 0$. Hence, from (2.1) and (3.2),

$$\begin{aligned}\bar{F}_{T_I}(t) &= \bar{F}_{1:2}(t) + \int_0^t \bar{Q}_x(\bar{F}_{1,x}(t-x), \bar{F}_{2,x}(t-x)) f_{1:2}(x)dx \\ &= \bar{F}_{1:2}(t) + \bar{F}_{1:2}(t)\ln(\bar{F}_{1:2}(t)) \\ &\quad + \int_0^t \frac{K(\bar{F}_1(t), \bar{F}_2(x)) + K(\bar{F}_1(x), \bar{F}_2(t))}{\bar{F}_{1:2}(x)} f_{1:2}(x)dx \quad (3.4)\end{aligned}$$

holds. In particular, if the components are IID, then

$$\begin{aligned}\bar{F}_{T_I}(t) &= \bar{F}^2(t) + 2\bar{F}^2(t)\ln(\bar{F}(t)) \\ &\quad + \int_0^t \frac{\bar{F}(t)\bar{F}(x) + \bar{F}(x)\bar{F}(t)}{\bar{F}^2(x)} 2f(x)\bar{F}(x)dx \\ &= \bar{F}^2(t) + 2\bar{F}^2(t)\ln(\bar{F}(t)) + 4\bar{F}(t)F(t).\end{aligned}$$

Therefore, $\bar{F}_{T_I}(t) = \bar{q}_I(\bar{F}(t))$ with $\bar{q}_I(u) = 4u - 3u^2 + 2u^2\ln(u)$. A straightforward calculation shows that $\bar{q}_{III}^{(i)}(u) = 2u - u^2 - u\ln u + u^2\ln u$ and $\bar{q}_I \leq \bar{q}_{III}^{(i)}$ for $i = 1, 2$. So, $T_I \leq_{ST} T_{III}^{(i)}$ holds for all F, that is, in this system, it is better to replace a fixed component than to replace the first failure.

If the components are just ID, from (3.4), we get

$$\begin{aligned}\bar{F}_{T_I}(t) &= \bar{F}_{1:2}(t) + \bar{F}_{1:2}(t)\ln(\bar{F}_{1:2}(t)) \\ &\quad + \int_0^t \frac{K(\bar{F}(t), \bar{F}(x)) + K(\bar{F}(x), \bar{F}(t))}{\bar{F}_{1:2}(x)} f_{1:2}(x)dx \\ &= \bar{F}_{1:2}(t) + \bar{F}_{1:2}(t)\ln(\bar{F}_{1:2}(t)) \\ &\quad + \int_0^t \frac{K(\bar{F}(t), \bar{F}(x)) + K(\bar{F}(x), \bar{F}(t))}{K(\bar{F}(x), \bar{F}(x))} \\ &\quad \times [\partial_1 K(\bar{F}(x), \bar{F}(x)) + \partial_2 K(\bar{F}(x), \bar{F}(x))] f(x)dx,\end{aligned}$$

where $\bar{F}_{1:2}(t) = K(\bar{F}(t), \bar{F}(t))$. Now, if we do the change $v = \bar{F}(x)$, then

$$\begin{aligned}\bar{F}_{T_I}(t) &= \delta_K(\bar{F}(t)) + \delta_K(\bar{F}(t))\ln(\delta_K(\bar{F}(t))) \\ &\quad + \int_{\bar{F}(t)}^1 \frac{K(\bar{F}(t), v) + K(v, \bar{F}(t))}{\delta_K(v,v)} \delta_K'(v)dv,\end{aligned}$$

where $\delta_K(v) = K(v,v)$ is the diagonal section of the copula K and $\delta_K'(v) = \partial_1 K(v,v) + \partial_2 K(v,v)$ for $v \in (0,1)$. Therefore $\bar{F}_{T_I}(t) = \bar{q}_I(\bar{F}(t))$ with

$$\bar{q}_I(u) = \delta_K(u) + \delta_K(u)\ln(\delta_K(u)) + \int_u^1 \frac{K(u,v) + K(v,u)}{\delta_K(v)} \delta_K'(v)dv.$$

A similar representation is obtained in the following theorem for an arbitrary coherent system.

Theorem 3.2. *Let T be the lifetime of a coherent system with ID components having a common reliability $\bar{F}$. Then the reliability function of T_I can be written as*

$$\bar{F}_{T_I}(t) = \bar{q}_I(\bar{F}(t)) \quad (3.5)$$

for all $t \geq 0$ and a distortion function $\bar{q}_I$ which does not depend on $\bar{F}$.

Proof. In the ID case, the general representation obtained in (3.3), can be written as

$$\begin{aligned}\bar{F}_{T_I}(t) &= \bar{F}_{1:n}(t) + \int_0^t \bar{G}_x(t-x) f_{1:n}(x)dx = \bar{F}_{1:n}(t) \\ &\quad + \int_0^t \bar{q}_x(\bar{F}_x(t-x)) f_{1:n}(x)dx, \quad (3.6)\end{aligned}$$

where $\bar{q}_x(u) = \bar{Q}_x(u, \dots, u)$ and $\bar{F}_x(t) = \Pr(X_i - x > t | X_i > x) = \bar{F}(t+x)/\bar{F}(x)$ for $i = 1, \dots, n$. Even more, in this case, $\bar{G}_x$ can be written as $\bar{G}_x(y) = \bar{q}(\bar{F}(x+y); \bar{F}(x))$, see Navarro (2018). Hence

$$\bar{F}_{T_I}(t) = \bar{F}_{1:n}(t) + \int_0^t \bar{q}(\bar{F}(t); \bar{F}(x)) f_{1:n}(x)dx$$

where $\bar{F}_{1:n}(t) = \delta_K(\bar{F}(t))$, $\delta_K(u) = K(u, \dots, u)$ and $f_{1:n}(t) = f(t)\delta_K'(\bar{F}(t))$. Then

$$\bar{F}_{T_I}(t) = \delta_K(\bar{F}(t)) + \int_0^t \bar{q}(\bar{F}(t); \bar{F}(x)) \delta_K'(\bar{F}(x)) f(x)dx.$$

Finally, if we do the change $u = \bar{F}(x)$, then

$$\bar{F}_{T_I}(t) = \delta_K(\bar{F}(t)) + \int_{\bar{F}(t)}^1 \bar{q}(\bar{F}(t); u)\delta_K'(u)du \quad (3.7)$$

and therefore (3.5) holds. □

The dual distortion function $\bar{q}_I$ in (3.5) depends on the structure of the system and on the underlying survival copula K. In the next sections we will show how to compute it. However, we must say that, sometimes, it is not easy to get an explicit expression for it (since we have to solve the integral in (3.7)). In the IID case, the preceding theorem can be simplified as follows.

Theorem 3.3. *Let T be the lifetime of a coherent system with IID components having a common reliability $\bar{F}$. Then the reliability function of T_I can be written as $\bar{F}_{T_I}(t) = \bar{q}_I(\bar{F}(t))$ where*

$$\bar{q}_I(u) = n\sum_{i=1}^{n-1} \frac{a_i}{n-i} u^i + \left(1 - n\sum_{i=1}^{n-1} \frac{a_i}{n-i}\right) u^n - n a_n u^n \ln u \quad (3.8)$$

and $(a_1, \dots, a_n)$ is the minimal signature of the system.

Proof. If the components are independent, then $\bar{G}_x(t) = \bar{Q}(\bar{F}_{1,x}(t), \dots, \bar{F}_{n,x}(t))$ holds from Proposition 5 in Navarro (2018), that is, $\bar{Q}_x = \bar{Q}$, where $\bar{Q}$ is the distortion function in (2.4). Then, if they are IID, we have $\bar{G}_x(t) = \bar{q}(\bar{F}_x(t))$, where $\bar{F}_x(t) = \bar{F}(t+x)/\bar{F}(x)$ and $\bar{q}(u) = \sum_{i=1}^n a_i u^i$ (see Section 2). Hence, from (3.6), we have

$$\begin{aligned}\bar{F}_{T_I}(t) &= \bar{F}_{1:n}(t) + \int_0^t \bar{q}(\bar{F}_x(t-x)) f_{1:n}(x)dx \\ &= \bar{F}^n(t) + \int_0^t \bar{q}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right) n\bar{F}^{n-1}(x) f(x)dx\end{aligned}$$

$$= \bar{F}^n(t) + n\sum_{i=1}^{n} a_i \bar{F}^i(t)\int_0^t \bar{F}^{n-i-1}(x)f(x)dx$$

$$= \bar{F}^n(t)+n\sum_{i=1}^{n-1}\frac{a_i}{n-i}\bar{F}^i(t)(1-\bar{F}^{n-i}(t)) + na_n\bar{F}^n(t)(-\ln\bar{F}(t))$$

which concludes the proof. □

The minimal signatures of all the coherent systems with 1–5 IID components were obtained in Navarro and Rubio (2010). Hence, from the preceding theorem, we have explicit expressions for $\bar{q}_I$ for all these systems.

3.2. Case II

Let us assume now that we repair the component which is critical for the system. We may expect that this option leads to a better performance since the most relevant components for the system have higher probabilities of being repaired. Note that, in case I, we just repair the first failure and so, for example, if the components are exchangeable, then all the components have the same probability of being repaired. However, we must note that case II is not always available in practice for all systems.

In this case it is not easy to obtain the reliability $\bar{F}_{T_{II}}$ of the resulting system lifetime T_{II}. Let us see a simple example. If the system is a series system, then cases I and II coincide since the first failure is always critical for the system. So let us consider again a parallel system.

Example 3.4. If $T = X_{2:2}$ and the components are IID, then, from (2.1), we have

$$\bar{F}_{T_{II}}(t) = \bar{F}_T\#\bar{F}(t) = \bar{F}_T(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} f_T(x)dx,$$

where $\bar{F}_T(t) = 2\bar{F}(t) - \bar{F}^2(t)$ and $f_T(t) = 2(1-\bar{F}(t))f(t)$. Hence

$$\bar{F}_{T_{II}}(t) = 2\bar{F}(t) - \bar{F}^2(t) + 2\bar{F}(t)\int_0^t \frac{1-\bar{F}(x)}{\bar{F}(x)} f(x)dx$$
$$= \bar{F}^2(t) - 2\bar{F}(t)\ln\bar{F}(t) = \bar{q}_{II}(\bar{F}(t))$$

with $\bar{q}_{II}(u) = u^2 - 2u\ln u$. So T_{II} also has a distorted distribution from F. Hence it is easy to compare the three replacement policies for this system just by comparing the three distortion functions. Thus a straightforward calculation leads to $\bar{q} \le \bar{q}_I \le \bar{q}_{III}^{(i)} \le \bar{q}_{II}$ and so $T \le_{ST} T_I \le_{ST} T_{III}^{(i)} \le_{ST} T_{II}$ for all $\bar{F}$ and $i = 1, 2$, that is, the best option in this system is to repair the component which is critical for the system. The second best option is to replace a fixed component and, of course, the three options are better than the original system T. They are also better than a parallel system with three components (active redundancy) with $\bar{q}_{3:3}(u) = 3u - 3u^2 + u^3$.

Let us assume now that the component lifetimes are just exchangeable. Then, proceeding as in Section 2, we have $\bar{F}_{T_{II}}(t) = \bar{F}_T\#\bar{G}(t)$, where

$$\bar{G}_x(y) = \Pr(X_2 - x > y | X_1 \le x, X_2 > x) = \frac{\Pr(X_1 \le x, X_2 > x+y)}{\Pr(X_1 \le x, X_2 > x)}$$
$$= \frac{\Pr(X_2 > x+y) - \Pr(X_1 > x, X_2 > x+y)}{\Pr(X_2 > x) - \Pr(X_1 > x, X_2 > x)}$$
$$= \frac{\bar{F}(x+y) - K(\bar{F}(x), \bar{F}(x+y))}{\bar{F}(x) - K(\bar{F}(x), \bar{F}(x))}$$

for $x, y \ge 0$. Hence, from (2.1), we have

$$\bar{F}_{T_{II}}(t) = \bar{F}_T(t) + \int_0^t \bar{G}_x(t-x) f_T(x)dx$$
$$= \bar{F}_T(t) + \int_0^t \frac{\bar{F}(t) - K(\bar{F}(x), \bar{F}(t))}{\bar{F}(x) - K(\bar{F}(x), \bar{F}(x))} f_T(x)dx,$$

where $\bar{F}_T(t) = 2\bar{F}(t) - K(\bar{F}(t), \bar{F}(t))$ and $f_T(t) = 2(1 - \partial_1 K(\bar{F}(t), \bar{F}(t)))f(t)$. Therefore

$$\bar{F}_{T_{II}}(t) = 2\bar{F}(t) - K(\bar{F}(t), \bar{F}(t)) + 2\int_0^t \frac{\bar{F}(t) - K(\bar{F}(x), \bar{F}(t))}{\bar{F}(x) - K(\bar{F}(x), \bar{F}(x))}$$
$$\times (1 - \partial_1 K(\bar{F}(x), \bar{F}(x)))f(x)dx$$
$$= 2\bar{F}(t) - K(\bar{F}(t), \bar{F}(t)) + 2\int_{\bar{F}(t)}^1 \frac{\bar{F}(t) - K(v, \bar{F}(t))}{v - K(v,v)}$$
$$\times (1 - \partial_1 K(v,v))dv = \bar{q}_{II}(\bar{F}(t))$$

with

$$\bar{q}_{II}(u) = 2u - K(u,u) + 2\int_u^1 \frac{u - K(v,u)}{v - K(v,v)}(1 - \partial_1 K(v,v))dv. \quad (3.9)$$

Note that we need K (and to solve this integral) to get an explicit expression for $\bar{q}_{II}$. Of course, if $K(u, v) = uv$, then we obtain the expression obtained above for the IID case.

Finally, in the general case, proceeding as in (2.7), we get $\bar{F}_{T_{II}}(t) = \bar{F}_T\#\bar{G}(t)$, where

$$\bar{G}_x(y) = p_1(x)\Pr(X_2 - x > y | X_1 \le x, X_2 > x)$$
$$+ p_2(x)\Pr(X_1 - x > y | X_2 \le x, X_1 > x)$$
$$= p_1(x)\frac{\Pr(X_1 \le x, X_2 > x+y)}{\Pr(X_1 \le x, X_2 > x)} + p_2(x)\frac{\Pr(X_2 \le x, X_1 > x+y)}{\Pr(X_2 \le x, X_1 > x)}$$
$$= p_1(x)\frac{\Pr(X_2 > x+y) - \Pr(X_1 > x, X_2 > x+y)}{\Pr(X_2 > x) - \Pr(X_1 > x, X_2 > x)}$$
$$+ p_2(x)\frac{\Pr(X_1 > x+y) - \Pr(X_1 > x+y, X_2 > x)}{\Pr(X_1 > x) - \Pr(X_1 > x, X_2 > x)}$$
$$= p_1(x)\frac{\bar{F}_2(x+y) - K(\bar{F}_1(x), \bar{F}_2(x+y))}{\bar{F}_2(x) - K(\bar{F}_1(x), \bar{F}_2(x))}$$
$$+ p_2(x)\frac{\bar{F}_1(x+y) - K(\bar{F}_1(x+y), \bar{F}_2(x))}{\bar{F}_1(x) - K(\bar{F}_1(x), \bar{F}_2(x))}, \quad (3.10)$$

$p_1(x) := \Pr(X_1 < X_2 | T = x)$ and $p_2(x) := \Pr(X_2 < X_1 | T = x) = 1 - p_1(x)$ for $x, y \ge 0$. To compute $p_1(x)$, we need the joint reliability of $(X_1, X_{2:2})$ given by

$$\bar{\boldsymbol{H}}(x,y) = \Pr(X_1 > x, X_{2:2} > y)$$
$$= \Pr(X_1 > x, X_1 > y) + \Pr(X_1 > x, X_2 > y)$$
$$- \Pr(X_1 > x, X_1 > y, X_2 > y)$$
$$= \bar{F}_1(y) + K(\bar{F}_1(x), \bar{F}_2(y)) - K(\bar{F}_1(y), \bar{F}_2(y))$$

for all $x \le y$. Hence, its joint density is $\boldsymbol{h}(x,y) = f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))$ for all $x \le y$ (0 otherwise) and the conditional density function of $(X_1 | X_{2:2} = y)$ is

$$\boldsymbol{h}_{1|2}(x|y) = \frac{f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))}{f_T(y)}$$

for $0 \le x \le y$. Therefore

$$p_1(y) = \Pr(X_1 < X_2 | T = y) = \Pr(X_1 < X_{2:2} | T = y) = \int_0^y \boldsymbol{h}_{1|2}(x|y)dx$$
$$= \int_0^y \frac{f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))}{f_T(y)}dx$$
$$= \frac{f_2(y) - f_2(y)\partial_2 K(\bar{F}_1(y), \bar{F}_2(y))}{f_T(y)} \quad (3.11)$$

when $\lim_{u\to 1^-}\partial_2 K(u, \bar{F}_2(y)) = 1$ (see Navarro & Sordo, 2018). Analogously, we get

$$p_2(y) = \Pr(X_2 < X_1 | T = y) = \frac{f_1(y) - f_1(y)\partial_1 K(\bar{F}_1(y), \bar{F}_2(y))}{f_T(y)}. \quad (3.12)$$

Hence, from (2.1), (3.10), (3.11) and (3.12), we have

$$\begin{aligned}\bar{F}_{T_{II}}(t) &= \bar{F}_T(t) + \int_0^t \bar{G}_x(t-x) f_T(x)dx \\ &= \bar{F}_1(t) + \bar{F}_2(t) - K(\bar{F}_1(t), \bar{F}_2(t)) \\ &\quad + \int_0^t [1-\partial_2 K(\bar{F}_1(x), \bar{F}_2(x))] \frac{\bar{F}_2(t)-K(\bar{F}_1(x), \bar{F}_2(t))}{\bar{F}_2(x)-K(\bar{F}_1(x), \bar{F}_2(x))} f_2(x)dx \\ &\quad + \int_0^t [1-\partial_1 K(\bar{F}_1(x), \bar{F}_2(x))] \frac{\bar{F}_1(t)-K(\bar{F}_1(t), \bar{F}_2(x))}{\bar{F}_1(x)-K(\bar{F}_1(x), \bar{F}_2(x))} f_1(x)dx.\end{aligned}$$

In the exchangeable case, we have $\Pr(X_1 < X_2 | T = y) = \Pr(X_2 < X_1 | T = y) = 1/2$ and (3.9).

The preceding example shows that it is not easy to get an expression for the reliability in the general case. So, we are going to try to solve the case of exchangeable components. In this case, we know that the system's reliability can be written as

$$\bar{F}_T(t) = \sum_{i=1}^n s_i \bar{F}_{i:n}(t), \tag{3.13}$$

where $\boldsymbol{s} = (s_1, \dots, s_n)$ is the *signature* of the system and $s_i = \Pr(T = X_{i:n})$ for $i = 1, \dots, n$. We can use this representation to obtain the following result.

Theorem 3.5. *Let T be the lifetime of a coherent system with components having an absolutely continuous exchangeable joint reliability. Then the reliability function of T_{II} can be written as*

$$\bar{F}_{T_{II}}(t) = \bar{q}_{II}(\bar{F}(t)) \tag{3.14}$$

for all $t \geq 0$ and for a distortion function $\bar{q}_{II}$ which does not depend on $\bar{F}$.

Proof. Let us consider the events $E_\sigma = \{X_{\sigma(1)} < \cdots < X_{\sigma(n)}\}$ for σ in the set P_n of all the permutations of order n. If the components are exchangeable, then $\Pr(E_\sigma) = 1/n!$. Let us divide the set P_n in the disjoint subsets $A_1, \dots, A_k$ where A_j contains all the permutations which lead to $T = X_{i_j:n}$ and to a fixed repaired system T_j. Let $H_j = \cup_{\sigma \in A_j} E_\sigma$. Then $p_j := \Pr(H_j) = |A_j|/n!$, where $|A_j|$ is the cardinal of the set A_j for $j = 1, \dots, k$. Hence

$$\bar{F}_{T_{II}}(t) = \Pr(T_{II} > t) = \sum_{j=1}^k p_j \Pr(T_{II} > t | H_j). \tag{3.15}$$

Note that under H_j, we know which component failure causes the system failure. Moreover $(T|H_j) =_{ST} (X_{i_j:n}|H_j)$. Also note that $X_{i_j:n} =_{ST} (X_{i_j:n}|H_j)$ due to the assumption about exchangeable components. Proceeding as in Section 2, we get $\Pr(T_{II} > t | H_j) = \bar{F}_{i_j:n} \# \bar{G}_j(t)$, where

$$\bar{G}_{j,x}(y) = \Pr(T_j - x > y | X_{i_j:n} = x, H_j) \tag{3.16}$$

and T_j is the system obtained after a minimal repair of the component broken in the i_jth position and at a given time x under H_j. Note that the structure of this system is completely determined by H_j. This event also determines which components are working and which have failed at time x. Hence, from (2.1),

$$\bar{F}_{T_{II}}(t) = \sum_{j=1}^k p_j \left[\bar{F}_{i_j:n}(t) + \int_0^t \bar{G}_{j,x}(t-x) f_{i_j:n}(x)dx \right] \tag{3.17}$$

holds. Note that the semi-coherent system T_j has $n - i_j + 1$ working components (some of them can be irrelevant for the system). These components are exchangeable and the corresponding joint reliability function $\bar{\boldsymbol{H}}(y_1, \dots, y_{n-i_j+1})$ is given by

$$\Pr(X_{i_j} - x > y_1, \dots, X_n - x > y_{n-i_j+1} | X_1 \leq x, \dots, X_{i_j-1} \leq x, X_{i_j} > x, \dots, X_n > x).$$

Proceeding as in case I, this joint reliability can be written as

$$\bar{\boldsymbol{H}}(y_1, \dots, y_{n-i_j+1}) = \bar{Q}_x(\bar{F}_x(y_1), \dots, \bar{F}_x(y_{n-i_j+1})) \tag{3.18}$$

for a distortion function $\bar{Q}_x$ which depends on $\bar{F}(x)$, where $\bar{F}_x(y) = \bar{F}(x+y)/\bar{F}(x)$. Let $\bar{H}^j_{1:n}, \dots, \bar{H}^j_{n-i_j+1:n}$ be the reliability functions of the order statistics obtained from these exchangeable components and let $(s^j_1, \dots, s^j_{n-i_j+1})$ be the signature (of order $n - i_j + 1$) of T_j. Then

$$\bar{F}_{T_{II}}(t) = \sum_{j=1}^k p_j \left[\bar{F}_{i_j:n}(t) + \sum_{\ell=1}^{n-i_j+1} s^j_\ell \int_0^t \bar{H}^j_{\ell:n}(t-x) f_{i_j:n}(x)dx \right].$$

It is well known (see, e.g., Navarro et al., 2013) that $\bar{F}_{i_j:n}(t)$ can be written as $\bar{F}_{i_j:n}(t) = \bar{q}_{i_j:n}(\bar{F}(t))$ where $\bar{q}_{i_j:n}$ depends on K. Analogously, from (3.18), we know that $\bar{H}^j_{\ell:n}$ can be written as $\bar{H}^j_{\ell:n}(y) = \bar{q}^j_{i:n}(\bar{F}(x+y); \bar{F}(x))$ where $\bar{q}^j_{i:n}$ depends on K. Therefore

$$\bar{F}_{T_{II}}(t) = \sum_{j=1}^k p_j \left[\bar{q}_{i_j:n}(\bar{F}(t)) + \sum_{\ell=1}^{n-i_j+1} s^j_\ell \int_0^t \bar{q}^j_{i:n}(\bar{F}(t); \bar{F}(x)) \bar{q}'_{i_j:n}(\bar{F}(x)) f(x)dx \right] \tag{3.19}$$

and by doing the change $v = \bar{F}(x)$ we get (3.14). □

The coefficients in the signature used in (3.13) can also be computed as $s_k = |B_k|/n!$, where B_k is the subset of P_n with the permutations which lead to $T = X_{k:n}$, that is, $B_k = \cup_{j:i_j=k} A_j$. Hence (3.19) can also be written as

$$\bar{F}_{T_{II}}(t) = \bar{F}_T(t) + \sum_{j=1}^k p_j \sum_{\ell=1}^{n-i_j+1} s^j_\ell \int_0^t \bar{q}^j_{i:n}(\bar{F}(t); \bar{F}(x)) \bar{q}'_{i_j:n}(\bar{F}(x)) f(x)dx, \tag{3.20}$$

where $\bar{F}_T(t) = \bar{q}_T(\bar{F}(t))$. These general expressions can be simplified in the IID case as follows.

Theorem 3.6. *Let T be the lifetime of a coherent system with IID components having a common absolutely continuous reliability $\bar{F}$. Then the reliability function of T_{II} can be expressed as $\bar{F}_{T_{II}}(t) = \bar{q}_{II}(\bar{F}(t))$ for all $t \geq 0$, where*

$$\bar{q}_{II}(u) = \sum_{i=1}^n c_i u^i + \sum_{i=1}^n d_i u^i \ln u \tag{3.21}$$

for some coefficients c_i, d_i, $i = 1, \dots, n$ which only depend on the structure of the system.

Proof. Let $a^j = (a^j_1, \dots, a^j_{n-i_j+1})$ be the minimal signature the system T_j considered in the proof of the preceding theorem for $j = 1, \dots, k$. In the IID case, this semi-coherent system has $n - i_j + 1$ IID components with the common reliability $\bar{F}_x(y) = \bar{F}(x+y)/\bar{F}(x)$. Hence the reliability in (3.16) is

$$\bar{G}_{j,x}(y) = \sum_{\ell=1}^{n-i_j+1} a^j_\ell \left(\frac{\bar{F}(x+y)}{\bar{F}(x)} \right)^\ell.$$

Therefore, from (3.17) and (3.20), we have

$$\bar{F}_{T_{II}}(t) = \bar{F}_T(t) + \sum_{j=1}^k p_j \sum_{\ell=1}^{n-i_j+1} a^j_\ell \int_0^t \frac{\bar{F}^\ell(t)}{\bar{F}^\ell(x)} \bar{q}'_{i_j:n}(\bar{F}(x)) f(x)dx$$

where $\bar{F}_{i:n}(t) = \bar{q}_{i:n}(\bar{F}(t))$ for a polynomial

$$\bar{q}_{i:n}(u) = \sum_{r=n-i+1}^n (-1)^{r-n+i-1} \binom{n}{r} \binom{r-1}{n-i} u^r$$

(see, e.g., David & Nagaraja, 2003, p. 46). So

$$f_{i:n}(t) = f(t)\bar{q}'_{i:n}(\bar{F}(t)) = f(t)\sum_{r=n-i+1}^{n}(-1)^{r-n+i-1}r\binom{n}{r}\binom{r-1}{n-i}\bar{F}^{r-1}(t).$$

Therefore, if $(a_1,\dots,a_n)$ is the minimal signature of T, then

$$\begin{aligned}\bar{F}_{T_{II}}(t) &= \bar{F}_T(t)+\sum_{j=1}^{k}p_j\sum_{\ell=1}^{n-i_j+1}a_\ell^j\bar{F}^\ell(t)\\ &\quad\times\sum_{r=n-i_j+1}^{n}r(-1)^{r-n+i_j-1}\binom{n}{r}\binom{r-1}{n-i_j}\int_0^t\bar{F}^{r-\ell-1}(x)f(x)dx\\ &= \sum_{j=1}^{n}a_j\bar{F}^j(t)+\sum_{j=1}^{k}p_j\sum_{\ell=1}^{n-i_j+1}a_\ell^j\bar{F}^\ell(t)\\ &\quad\times\sum_{r=n-i_j+1}^{n}r(-1)^{r-n+i_j-1}\binom{n}{r}\binom{r-1}{n-i_j}\phi_{r-\ell}(t),\end{aligned}$$

where $\phi_s(t)=(1-\bar{F}^s(t))/s$ if $s>0$ and $\phi_s(t)=-\ln\bar{F}(t)$ if $s=0$. This concludes the proof. □

Example 4.4 shows how to apply the preceding theorem. In Navarro, Arriaza, and Suárez-Llorens (2017) we provide an R-script to compute the coefficients c_i and d_i for a given coherent system with IID components.

In general it is not easy to compute the reliability function associated to the case II of a coherent system with dependent components. However, the reliability function of k-out-of-n systems can be obtained by assuming exchangeable components. Thus, if $T=X_{i:n}$ for a fixed $i\in\{2,\dots,n\}$ and the components are exchangeable, then $\bar{F}_{T_{II}}(t)=\bar{F}_{i:n}\#\bar{G}(t)$, where

$$\begin{aligned}\bar{G}_x(y) &= \Pr(X_i>x+y,\dots,X_n>x+y|X_1\le x,\dots,\\ &\qquad X_{i-1}\le x, X_i>x,\dots,X_n>x)\\ &= \frac{\Pr(X_1\le x,\dots,X_{i-1}\le x,X_i>x+y,\dots,X_n>x+y)}{\Pr(X_1\le x,\dots,X_{i-1}\le x,X_i>x,\dots,X_n>x)}\\ &= \frac{H_i(\bar{F}(x),\bar{F}(x+y))}{H_i(\bar{F}(x),\bar{F}(x))},\end{aligned}$$

with a function H_i such that $\Pr(X_1\le x,\dots,X_{i-1}\le x,X_i>t,\dots,X_n>t)=H_i(\bar{F}(x),\bar{F}(t))$ for all $0\le x\le t$. Note that H_i only depends on K. Therefore, from (2.1), we have

$$\Pr(T_{II}>t)=\bar{F}_{i:n}(t)+\int_0^t\frac{H_i(\bar{F}(x),\bar{F}(t))}{H_i(\bar{F}(x),\bar{F}(x))}f_{i:n}(x)dx. \tag{3.22}$$

If the components are IID, then the following result provide an explicit expression for (3.22).

Proposition 3.7. *Given an i-out-of-n system with IID components and lifetime $T=X_{i:n}$ for a fixed $i\in\{2,\dots,n\}$, then $\bar{F}_{T_{II}}(t)=\bar{q}_{II}(\bar{F}(t))$, where*

$$\begin{aligned}\bar{q}_{II}(u) &= \binom{n}{n-i+1}u^{n-i+1}+u^{n-i+1}\\ &\quad\times\sum_{k=n-i+2}^{n}(-1)^{k-n+i-1}\frac{k}{k-n+i-1}\binom{n}{k}\binom{k-1}{n-i}\\ &\quad+\sum_{k=n-i+2}^{n}(-1)^{k-n+i}\frac{n-i+1}{k-n+i-1}\\ &\quad\times\binom{n}{k}\binom{k-1}{n-i}u^k-i\binom{n}{i}u^{n-i+1}\ln u.\end{aligned}$$

Proof. If the components are IID, then

$$\begin{aligned}\bar{G}_x(y) &= \Pr(X_i>x+y,\dots,X_n>x+y|X_1\le x,\dots,\\ &\qquad X_{i-1}\le x,X_i>x,\dots,X_n>x)\\ &= \Pr(X_i>x+y|X_i>x)\dots\Pr(X_n>x+y|X_n>x)\\ &= \frac{\bar{F}^{n-i+1}(x+y)}{\bar{F}^{n-i+1}(x)}.\end{aligned}$$

Moreover, as $\bar{F}_{i:n}(t)=\sum_{k=n-i+1}^{n}(-1)^{k-n+i-1}\binom{n}{k}\binom{k-1}{n-i}\bar{F}^k(t)$ (see, e.g., David & Nagaraja, 2003, p. 46), we have

$$\begin{aligned}\Pr(T_{II}>t) &= \bar{F}_{i:n}(t)+\int_0^t\frac{\bar{F}^{n-i+1}(t)}{\bar{F}^{n-i+1}(x)}f_{i:n}(x)dx\\ &= \bar{F}_{i:n}(t)+\bar{F}^{n-i+1}(t)\sum_{k=n-i+1}^{n}(-1)^{k-n+i-1}k\binom{n}{k}\binom{k-1}{n-i}\\ &\quad\times\int_0^t\bar{F}^{k-n+i-2}(x)f(x)dx\\ &= \bar{F}_{i:n}(t)+\sum_{k=n-i+2}^{n}(-1)^{k-n+i-1}k\binom{n}{k}\binom{k-1}{n-i}\\ &\quad\times\frac{\bar{F}^{n-i+1}(t)-\bar{F}^k(t)}{k-n+i-1}-i\binom{n}{i}\bar{F}^{n-i+1}(t)\ln\bar{F}(t)\\ &= \binom{n}{n-i+1}\bar{F}^{n-i+1}(t)-\sum_{k=n-i+2}^{n}(-1)^{k-n+i-1}\\ &\quad\times\frac{n-i+1}{k-n+i-1}\binom{n}{k}\binom{k-1}{n-i}\bar{F}^k(t)\\ &\quad+\bar{F}^{n-i+1}(t)\sum_{k=n-i+2}^{n}(-1)^{k-n+i-1}\frac{k}{k-n+i-1}\\ &\quad\times\binom{n}{k}\binom{k-1}{n-i}-i\binom{n}{i}\bar{F}^{n-i+1}(t)\ln\bar{F}(t)\end{aligned}$$

which concludes the proof. □

3.3. Other cases

The purpose of this section is to show that we can study other cases following the procedures used above in cases I and II. For example, if we know that the system does not fail with the first component failure, we can consider to repair the system at the second component failure with a minimal repair of the broken component at this point. Then, if the components are exchangeable, the reliability function of the repaired system is $\bar{F}_{(2)}(t)=\bar{F}_{2:n}\#\bar{G}(t)$, where

$$\bar{G}_x(y)=\frac{1}{n}\sum_{i=1}^{n}\Pr(T_i-x>y|X_i\le x, X_j>t\text{ for all }j\neq i)$$

and T_i is the lifetime of the semi-coherent system obtained from T when we know that the ith component is broken. A similar expression can be obtained if the system is repaired at the jth failure for $j=3,4,\dots$.

In all the options studied above, we just repair one component. We can of course consider k replacements. For example, if $k=2$ and, in case III, we repair components i and j (for fixed $i<j$), then the reliability of the repaired system is

$$\begin{aligned}\bar{F}_{T_{III}^{(i,j)}}(t) &= \bar{Q}(\bar{F}_1(t),\dots,\bar{F}_{i-1}(t),\bar{q}_1(\bar{F}_i(t)),\bar{F}_{i+1}(t),\dots,\\ &\qquad\bar{F}_{j-1}(t),\bar{q}_1(\bar{F}_j(t)),\bar{F}_{j+1}(t),\dots,\bar{F}_n(t)),\end{aligned}$$

where $\bar{q}_1$ is given in (2.3). If the components are ID, then this representation can be reduced to $\bar{F}_{T_{III}^{(i,j)}}(t)=\bar{q}_{III}^{(i,j)}(\bar{F}(t))$, where $\bar{q}_{III}^{(i,j)}(u)=\bar{Q}(u,\dots,u,\bar{q}_1(u),u,\dots,u,\bar{q}_1(u),u,\dots,u)$ and $\bar{q}_1(u)$ is placed at the ith and jth positions. Analogously, if we repair the ith component twice, then

$$\bar{F}_{T_{III}^{(i,i)}}(t)=\bar{Q}(\bar{F}_1(t),\dots,\bar{F}_{i-1}(t),\bar{q}_2(\bar{F}_i(t)),\bar{F}_{i+1}(t),\dots,\bar{F}_n(t)),$$

where $\bar{q}_2$ is given in (2.3). If the components are ID, we get $\bar{F}_{T_{III}^{(i,i)}}(t) = \bar{q}_{III}^{(i,i)}(\bar{F}(t))$, where

$$\bar{q}_{III}^{(i,i)}(u) = \bar{Q}(u, \dots, u, \bar{q}_2(u), u, \dots, u)$$

and $\bar{q}_2$ is placed at the ith position. Other options with fixed repairs were studied in Arriaza et al. (2018).

We could consider other options with $k = 2$ minimal repairs. For example, we can repair the two first broken components. In this case, if $X_1, \dots, X_n$ are IID, the resulting reliability is

$$\bar{F}_I^{(2)}(t) = (\bar{F}_{1:n}\#\bar{G}_{1:n})\#\bar{G}(t),$$

where $\bar{F}_{1:n}(t) = \bar{F}^n(t)$ is the reliability function of $X_{1:n} = \min(X_1, \dots, X_n)$,

$$(\bar{G}_{1:n})_x(y) = \bar{F}_x^n(y) = \frac{\bar{F}^n(x+y)}{\bar{F}^n(x)}$$

is the reliability function of $Y_{1:n} = \min(Y_1, \dots, Y_n)$ (a series system with n IID components and a common reliability $\bar{F}_x(y) = \bar{F}(x+y)/\bar{F}(x)$) and $\bar{G}_y(z) = \bar{q}_T(\bar{F}_y(z))$ is the reliability of a system with the same structure as T, having n IID components with reliability $\bar{F}_y$ when $Y_{1:n} = y$. The reliability $\bar{H} = \bar{F}_{1:n}\#\bar{G}_{1:n}$ can be computed from (2.1) as

$$\bar{H}(t) = \bar{F}^n(t) + \int_0^t \frac{\bar{F}^n(t)}{\bar{F}^n(x)} n\bar{F}^{n-1}(x)f(x)dx = \bar{F}^n(t) - n\bar{F}^n(t)\ln\bar{F}(t).$$

Its density is $h(t) = -n^2\bar{F}^{n-1}(t)f(t)\ln\bar{F}(t)$. Then, by using (2.1) again, the system's reliability is

$$\begin{aligned}\bar{F}_I^{(2)}(t) &= \bar{H}(t) + \int_0^t \bar{G}_y(t-y)h(y)dy \\ &= \bar{H}(t) - n^2\int_0^t \bar{q}_T\left(\frac{\bar{F}(t)}{\bar{F}(y)}\right)\bar{F}^{n-1}(y)f(y)\ln\bar{F}(y)dy \\ &= \bar{H}(t) - n^2\sum_{i=1}^n a_i\bar{F}^i(t)\int_0^t \bar{F}^{n-i-1}(y)\ln\bar{F}(y)f(y)dy,\end{aligned}$$

where $(a_1, \dots, a_n)$ is the minimal signature of the system T. Then

$$\begin{aligned}\bar{F}_I^{(2)}(t) &= \bar{H}(t) - n^2a_n\bar{F}^n(t)\int_0^t \bar{F}^{-1}(y)\ln\bar{F}(y)f(y)dy - n^2 \\ &\quad\times\sum_{i=1}^{n-1} a_i\bar{F}^i(t)\int_0^t \bar{F}^{n-i-1}(y)\ln\bar{F}(y)f(y)dy \\ &= \bar{H}(t) + n^2\frac{a_n}{2}\bar{F}^n(t)\ln^2\bar{F}(t) - n^2 \\ &\quad\times\sum_{i=1}^{n-1} a_i\bar{F}^i(t)\int_0^t \bar{F}^{n-i-1}(y)\ln\bar{F}(y)f(y)dy.\end{aligned}$$

Finally, by doing the change $x = -\ln\bar{F}(y)$, in $I_i(t) = \int_0^t \bar{F}^{n-i-1}(y)\ln\bar{F}(y)f(y)dy$, we get

$$I_i(t) = \int_0^{-\ln\bar{F}(t)} xe^{-(n-i)x}dx = \frac{\bar{F}^{n-i}(t)\ln\bar{F}(t)}{n-i} + \frac{1-\bar{F}^{n-i}(t)}{(n-i)^2}.$$

Therefore

$$\begin{aligned}\bar{F}_I^{(2)}(t) &= \bar{q}_1(F^n(t)) + \frac{n^2a_n}{2}\bar{F}^n(t)\ln^2\bar{F}(t) + n^2 \\ &\quad\times\sum_{i=1}^{n-1} a_i\frac{\bar{F}^n(t)\ln\bar{F}(t)}{n-i} + n^2\sum_{i=1}^{n-1} a_i\frac{\bar{F}^i(t)-\bar{F}^n(t)}{(n-i)^2}.\end{aligned}$$

Note that the reliability can be written as $\bar{F}_I^{(2)}(t) = \bar{q}_I^{(2)}(\bar{F}(t))$ for a distortion function $\bar{q}_I^{(2)}$. For example, for $T = X_{1:n}$, we obtain $\bar{q}_I^{(2)}(u) = u^n - nu^n\ln u + (n^2/2)u^n(\ln u)^2$. For this system, if we repair the first k broken components, then we get $\bar{q}_I^{(k)}(u) = \sum_{i=0}^k n^iu^n(-\ln u)^i/i!$.

Other similar replacement policies can be studied in a similar way. However, in the following section we restrict ourselves to the cases with $k = 1$ to develop fair comparisons, that is comparisons of replacement policies with the same number of repairs (i.e. with the same cost).

4. Comparison results

The representations obtained in the preceding section can be used jointly with the ordering results for distorted distributions given in Navarro et al. (2013) and Navarro and Gomis (2016) to compare the different replacement policies. For sake of completeness we include some of these ordering results in the following theorem. We shall consider the following (well known) stochastic orders.

The main order is the *usual stochastic order*, denoted by $X \le_{ST} Y$, that compares the respective reliability functions $\bar{F}_X(t) \le \bar{F}_Y(t)$ for any time t. This ordering implies that $E(X) \le E(Y)$ (if these expectations exist). An alternative (stronger) order is the *hazard rate order*, denoted by $X \le_{HR} Y$, that compares the respective residual lifetimes $(X - t|X > t) \le_{ST} (Y - t|Y > t)$ for any time t. While the ST order compares new units, the HR order compares (in the ST order) used units with the same age t. Analogously, the *mean residual life order*, denoted by $X \le_{MRL} Y$, compares the respective mean (expected) residual lifetimes $E(X - t|X > t) \le E(Y - t|Y > t)$ for any time t. The HR order implies the MRL order. An order similar to the HR order is the *reversed hazard rate order*, denoted by $X \le_{RHR} Y$, that compares the inactivity times $(t - X|X < t) \ge_{ST} (t - Y|Y < t)$ for any time t. Finally, the *likelihood ratio order*, denoted by $X \le_{LR} Y$, holds if the ratio of their densities f_Y/f_X is increasing in the union of their supports. This order implies all the preceding orders. For basic properties and applications of these orders we refer the reader to Barlow and Proschan (1975) and Shaked and Shanthikumar (2007).

Theorem 4.1. *Let X_1 and X_2 be two random variables with distribution functions $F_{q_1} = q_1(F)$ and $F_{q_2} = q_2(F)$ obtained as distorted distributions from the same distribution function F and from the distortion functions q_1 and q_2, respectively. Let $\bar{q}_1$ and $\bar{q}_2$ be the respective dual distortion functions. Then:*

(i) $X_1 \le_{ST} X_2$ for all F $\iff \bar{q}_1(u) \le \bar{q}_2(u)$ [or $q_1(u) \ge q_2(u)$] for all $u \in (0, 1)$.

(ii) $X_1 \le_{HR} X_2$ for all F $\iff \bar{q}_2(u)/\bar{q}_1(u)$ is decreasing in $(0, 1)$.

(iii) $X_1 \le_{RHR} X_2$ for all F $\iff q_2(u)/q_1(u)$ is increasing in $(0, 1)$.

(iv) $X_1 \le_{LR} X_2$ for all F $\iff \bar{q}_2'(u)/\bar{q}_1'(u)$ is decreasing in $(0, 1)$.

(v) $X_1 \le_{MRL} X_2$ for all F $\Longleftarrow \bar{q}_2(u)/\bar{q}_1(u)$ is bathtub in $(0, 1)$ *and* $E(X_1) \le E(X_2)$.

We apply these ordering results in the following theorems and examples comparing the different replacement policies. In the first main result we prove that, for any system with IID components, the replacement policy of case II is always ST-better than that of case I.

Theorem 4.2. *Let T be the lifetime of a coherent system with IID components having a common absolutely continuous reliability $\bar{F}$. Let T_I and T_{II} be the system lifetimes obtained with the replacement policies of cases I and II, respectively. Then $T_I \le_{ST} T_{II}$ for all $\bar{F}$.*

Proof. If we assume that the component lifetimes $X_1, \dots, X_n$ are IID, then the system's reliability can be written as $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$ for a polynomial $\bar{q}(u)$. From Theorems 3.3 and 3.6, we also know that the reliability functions of T_I and T_{II} can be written as $\Pr(T_I > t) = \bar{q}_I(\bar{F}(t))$ and $\Pr(T_{II} > t) = \bar{q}_{II}(\bar{F}(t))$. So we just need to prove that $\bar{q}_I(u) \le \bar{q}_{II}(u)$ for all $u \in [0, 1]$.

From the proof of Theorem 3.3, we know that $T_I = X_{1:n} + Y^I$, where $X_{1:n} = \min(X_1, \dots, X_n)$,

$$\Pr(Y^I - x > y|X_{1:n} = x) = \Pr(T^* > y)$$

and T^* is the lifetime of a system with the same structure as T and having IID components with the common reliability function $\bar{F}_x(y) = \bar{F}(x+y)/\bar{F}(x)$ for $y \geq 0$. Hence

$$\Pr(Y^I - x > y | X_{1:n} = x) = \Pr(T^* > y) = \bar{q}(\bar{F}_x(y)).$$

On the other hand, from the proof of Theorem 3.6, we know that $T_{II} = T + Y^{II}$, where

$$\Pr(Y^{II} - x > y | T = x) = \Pr(T^{**} > y)$$

and T^{**} is a mixture of different semi-coherent systems with n (or less) IID components with the common reliability function $\bar{F}_x$.

Now let assume that the IID components are exponential with mean 1, that is, $\bar{F}(t) = e^{-t}$ for $t \geq 0$. This model has the lack of memory property and so $\bar{F}_x(y) = \bar{F}(y)$ for all $y \geq 0$. Hence

$$\Pr(Y^I - x > y | X_{1:n} = x) = \bar{q}(\bar{F}(y)) = \Pr(T > y)$$

for all $x, y \geq 0$, that is, $(Y^I - x | X_{1:n} = x) =_{ST} T$. So $X_{1:n}$ and Y^I are independent. Analogously, T^{**} is a mixture of different semi-coherent systems with n (or less) components and having IID components with the common reliability function $\bar{F}$. Hence T and Y^{II} are independent. Moreover, as all these semi-coherent systems are ST-better than $X_{1:n}$ (because they have n or less components), then $X_{1:n} \leq_{ST} T^{**}$. Finally, from Theorem 1.A.3, b, in Shaked and Shanthikumar (2007, p. 6), we get

$$T_I =_{ST} X_{1:n} + T^* \leq_{ST} T + T^{**} =_{ST} T_{II}$$

for $\bar{F}(t) = e^{-t}$, where $T^* =_{ST} T$. Hence $\bar{q}_I(e^{-t}) \leq \bar{q}_{II}(e^{-t})$ for all $t \geq 0$. So $\bar{q}_I(u) \leq \bar{q}_{II}(u)$ for all $u \in [0, 1]$ and the proof is completed. □

In the second theorem we prove that this property can be extended to the hazard rate order for the systems which preserve the IFR (increasing failure rate) aging property. A similar result can be stated for the likelihood ratio order from Theorem 1.C.9 in Shaked and Shanthikumar (2007, p. 46) and the preservation results for the ILR class of logconcave densities given in Proposition 2.2 of Navarro, del Águila, Sordo, and Suárez-Llorens (2014).

Theorem 4.3. *Let T be the lifetime of a coherent system with IID components having a common absolutely continuous reliability $\bar{F}$. Let T_I and T_{II} be the system lifetimes obtained with the replacement policies of cases I and II, respectively. Let $\bar{q}$ be the dual distortion function of T. If $\alpha(u) = u\bar{q}'(u)/\bar{q}(u)$ is decreasing in (0,1), then $T_I \leq_{HR} T_{II}$ for all $\bar{F}$.*

Proof. As in the preceding theorem, we have $\Pr(T_I > t) = \bar{q}_I(\bar{F}(t))$ and $\Pr(T_{II} > t) = \bar{q}_{II}(\bar{F}(t))$. So, from Theorem 4.1, (*ii*), we need to prove that $\bar{q}_{II}/\bar{q}_I$ is decreasing in (0,1). With the notation used in the proof of the preceding theorem, if we assume that $\bar{F}(t) = e^{-t}$ for $t \geq 0$ (exponential components), we have $T_I =_{ST} X_{1:n} + T^*$ and $T_{II} =_{ST} T + T^{**}$, where $T^* =_{ST} T$ and T^{**} is a mixture of semi-coherent systems of order n. Then its reliability can be written as

$$\Pr(T^{**} > t) = s_1^{**}\bar{F}_{1:n}(t) + \cdots + s_n^{**}\bar{F}_{n:n}(t)$$

for all $t \geq 0$. The vector $(s_1^{**}, \ldots, s_n^{**})$ is called the signature (of order n) of T^{**} (see, e.g., Navarro, Samaniego, Balakrishnan, & Bhattacharya, 2008). The signature of $X_{1:n}$ is $(1, 0, \ldots, 0)$. Hence, as $(1, 0, \ldots, 0) \leq_{HR} (s_1^{**}, \ldots, s_n^{**})$, from Theorem 4.4 in Navarro et al. (2008), we get $X_{1:n} \leq_{HR} T^{**}$ for $\bar{F}(t) = e^{-t}$. Moreover, we know that T^* is independent of $X_{1:n}$ and T^{**} is independent of T. Then we can apply Lemma 1.B.3 in Shaked and Shanthikumar (2007, p. 18) obtaining

$$T_I =_{ST} X_{1:n} + T^* \leq_{HR} T + T^{**} =_{ST} T_{II}$$

for $\bar{F}(t) = e^{-t}$ whenever T is IFR. Now we note that, from the results given in Navarro et al. (2014, p. 447), if the function α defined above is decreasing, then the system preserves the IFR property. So, as the exponential distribution is IFR, then T is also IFR and $T_I \leq_{HR} T_{II}$ holds for $\bar{F}(t) = e^{-t}$, that is,

$$\frac{\Pr(T_{II} > t)}{\Pr(T_I > t)} = \frac{\bar{q}_{II}(\bar{F}(t))}{\bar{q}_I(\bar{F}(t))} = \frac{\bar{q}_{II}(e^{-t})}{\bar{q}_I(e^{-t})}$$

is increasing for $t \geq 0$. Therefore, $\bar{q}_{II}(u)/\bar{q}_I(u)$ is decreasing in (0,1) and the proof is completed. □

Table 1
Repairing options for the system in Example 4.4.

j	A_j	H_j	$\lvert A_j\rvert$	T	i_j	T_j
1	$(1, i_2, i_3)$	$X_1 < X_{i_2} < X_{i_3}$	2	$T = X_{i_2}$	2	$\min(X_2, X_3)$
2	$(i_1, 1, i_3)$	$X_{i_1} < X_1 < X_{i_3}$	2	$T = X_1$	2	X_1
3	$(i_1, i_2, 1)$	$X_{i_1} < X_{i_2} < X_1$	2	$T = X_1$	3	X_1

The following example shows that, sometimes, to repair a fixed component (case III) is better than to repair the critical component of the system (case II).

Example 4.4. Let us consider a coherent system with three IID components and lifetime $T = \max(X_1, \min(X_2, X_3))$. Then the distortion functions of the system are $\bar{Q}(u_1, u_2, u_3) = u_1 + u_2u_3 - u_1u_2u_3$ and $\bar{q}(u) = \bar{Q}(u, u, u) = u + u^2 - u^3$. Furthermore, the dual distortion functions associated to the lifetimes obtained after the minimal repair of the components 1, 2 and 3 are given by

$$\bar{q}_{III}^{(1)}(u) = \bar{Q}(\bar{q}_1(u), u, u) = u + u^2 - u^3 - (u - u^3)\ln u$$

and

$$\bar{q}_{III}^{(2)}(u) = \bar{q}_{III}^{(3)}(u) = \bar{Q}(u, \bar{q}_1(u), u) = u + u^2 - u^3 - (u^2 - u^3)\ln u.$$

On the other hand, the distortion function for case I can be obtained from (3.8) as

$$\bar{q}_I(u) = \frac{3}{2}u + 3u^2 - \frac{7}{2}u^3 + 3u^3 \ln u.$$

Finally, we compute $\bar{q}_{II}$ from (3.21). The signature of the system is (0, 2/3, 1/3). It can be computed from the permutations given in Table 1. This table also contains the numbers i_j of component failures which cause the system failure and the expressions of the repaired system lifetimes T_j for each $j = 1, 2, 3$. Hence, from (3.15), we get

$$\Pr(T_{II} > t) = \frac{1}{3}\sum_{j=1}^{3} \Pr(T_{II} > t | H_j)$$

for the events H_j given in Table 1. The first probability can be computed as

$$\Pr(T_{II} > t | H_1) = \bar{F}_{i_1:3} \# \bar{G}_1(t) = \bar{F}_{2:3} \# \bar{G}_1(t),$$

where if $X_{2:3} = x$, then

$$\begin{aligned}\bar{G}_{1,x}(y) &= \Pr(T_1 - x > y | X_{2:3} = x, H_1) \\ &= \Pr(\min(X_2, X_3) - x > y | X_1 < x < X_2 < X_3) = \frac{\bar{F}^2(x+y)}{\bar{F}^2(x)}\end{aligned}$$

since the components are IID. Therefore, from (2.1), we have

$$\Pr(T_{II} > t | H_1) = \bar{F}_{2:3}(t) + \int_0^t \frac{\bar{F}^2(t)}{\bar{F}^2(x)} f_{2:3}(x)dx,$$

where $\bar{F}_{2:3}(t) = 3\bar{F}^2(t) - 2\bar{F}^3(t)$ and $f_{2:3}(t) = 6(\bar{F}(t) - \bar{F}^2(t))f(t)$. Hence

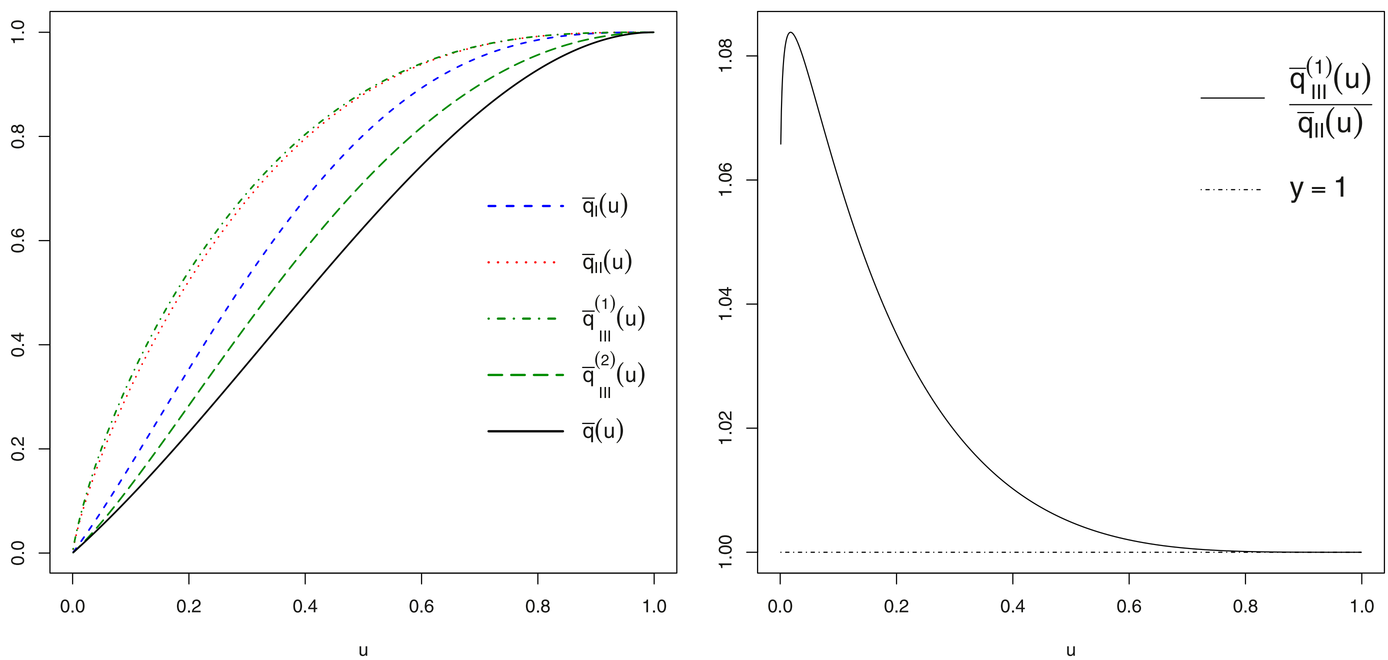

Fig. 1. Plots of the dual distortion functions for the cases: I, II, III ((1) and (2)) and for the system given in Example 4.4 (left). Ratio $\bar{q}^{(1)}_{III}/\bar{q}_{II}$ in the interval (0,1) (right).

$$
\begin{aligned}
\Pr(T_{II} > t|H_1) &= \bar{F}_{2:3}(t) + 6\bar{F}^2(t)\int_0^t \frac{\bar{F}(x) - \bar{F}^2(x)}{\bar{F}^2(x)} f(x)dx \\
&= \bar{F}_{2:3}(t) + 6\bar{F}^2(t)\int_0^t \left(\frac{1}{\bar{F}(x)} - 1\right) f(x)dx \\
&= \bar{F}_{2:3}(t) + 6\bar{F}^2(t)\big(-\log \bar{F}(t) - F(t)\big) \\
&= -3\bar{F}^2(t) + 4\bar{F}^3(t) - 6\bar{F}^2(t)\log \bar{F}(t).
\end{aligned}
$$

A straightforward (analogous) calculation for H_2 and H_3 leads us to

$$\Pr(T_{II} > t|H_2) = 3\bar{F}(t) - 3\bar{F}^2(t) + \bar{F}^3(t)$$

and

$$\Pr(T_{II} > t|H_3) = -\frac{3}{2}\bar{F}(t) + 3\bar{F}^2(t) - \frac{1}{2}\bar{F}^3(t) - 3\bar{F}(t)\log \bar{F}(t).$$

Hence

$$
\begin{aligned}
\Pr(T_{II} > t) &= \frac{1}{3}\Pr(T_{II} > t|H_1) + \frac{1}{3}\Pr(T_{II} > t|H_2) + \frac{1}{3}\Pr(T_{II} > t|H_3) \\
&= \frac{1}{2}\bar{F}(t) - \bar{F}^2(t) + \frac{3}{2}\bar{F}^3(t) - \bar{F}(t)\log \bar{F}(t) \\
&\quad - 2\bar{F}^2(t)\log \bar{F}(t) \\
&= \bar{q}_{II}(\bar{F}(t)),
\end{aligned}
$$

where $\bar{q}_{II}(u) = u/2 - u^2 + (3/2)u^3 - u\log u - 2u^2 \log u$ for $u \in (0, 1)$.

In Fig. 1 (left) we compare the distortion functions of the three cases. From these plots we conclude that $T \le_{ST} T^{(2)}_{III} \le_{ST} T_I \le_{ST} T_{II} \le_{ST} T^{(1)}_{III}$. In order to clarify the last inequality, we plot the ratio $\bar{q}^{(1)}_{III}/\bar{q}_{II}$ in the interval (0,1) (see Fig. 1, right). This quotient is always above the line $y = 1$. However it is not decreasing and therefore T_{II} and $T^{(1)}_{III}$ are not HR-ordered. Hence, we can state that against the expected, the replacement policy of case II is not always the best strategy in the case of IID components.

The following example shows that Theorem 4.2 is not true when the components are dependent.

Example 4.5. Let us consider a parallel system with 2 exchangeable components having a common absolutely continuous reliability function $\bar{F}$. Let us assume that both components are dependent and have the following Clayton-Oakes survival copula

$$K(u, v) = \frac{uv}{u + v - uv}.$$

Taking into account that both components are ID and have survival copula K, we get

$$\bar{F}_{1:2}(t) = K(\bar{F}(t), \bar{F}(t)) = \frac{\bar{F}(t)}{2 - \bar{F}(t)} \quad \text{and} \quad f_{1:2}(t) = \frac{2f(t)}{(2 - \bar{F}(t))^2},$$

where f represents the common density function of both components. Hence, the reliability function associated to T_I can be obtained from (3.4) as follows

$$
\begin{aligned}
\bar{F}_{T_I}(t) &= \bar{F}_{1:2}(t) + \bar{F}_{1:2}(t)\ln(\bar{F}_{1:2}(t)) + 2\int_0^t \frac{K(\bar{F}(t), \bar{F}(x))}{\bar{F}_{1:2}(x)} f_{1:2}(x)dx \\
&= \frac{\bar{F}(t)}{2 - \bar{F}(t)}\big(1 - -3\ln \bar{F}(t) - \ln(2 - \bar{F}(t))\big) = \bar{q}_I(\bar{F}(t)),
\end{aligned}
$$

where $\bar{q}_I(u) = (u - 3u\ln u - u\ln(2-u))/(2-u)$ represents the dual distortion associated to T_I.

On the other hand, we can obtain immediately the expression for the dual distortion associated to T_{II} just by replacing $K(u, v)$ in (3.9) as follows

$$
\begin{aligned}
\bar{q}_{II}(u) &= 2u - K(u, u) + 2\int_u^1 \frac{u - K(v, u)}{v - K(v, v)}(1 - \partial_1 K(v, v))dv \\
&= \frac{u(3 - 2u)}{2 - u} + \frac{u(3 - u)}{1 - u}\ln(2 - u) + \frac{u^2(5 - 3u)}{(2 - u)(1 - u)}\ln u.
\end{aligned}
$$

Finally, we obtain the dual distortion functions for the case III. Firstly, we note that both distortions must be the same because we are considering exchangeable components. Moreover, $\bar{Q}(u, v) = u + v - K(u, v)$. Hence, the dual distortion function of $T^{(1)}_{III}$ can be obtained as follows

$$
\begin{aligned}
\bar{F}_{T^{(1)}_{III}}(t) &= \bar{Q}(\bar{q}_1(\bar{F}(t)), \bar{F}(t)) = \bar{q}_1(\bar{F}(t)) + \bar{F}(t) \\
&\quad - \frac{\bar{q}_1(\bar{F}(t))\bar{F}(t)}{\bar{q}_1(\bar{F}(t)) + \bar{F}(t) - \bar{q}_1(\bar{F}(t))\bar{F}(t)} = \bar{q}^{(1)}_{III}(\bar{F}(t)),
\end{aligned}
$$

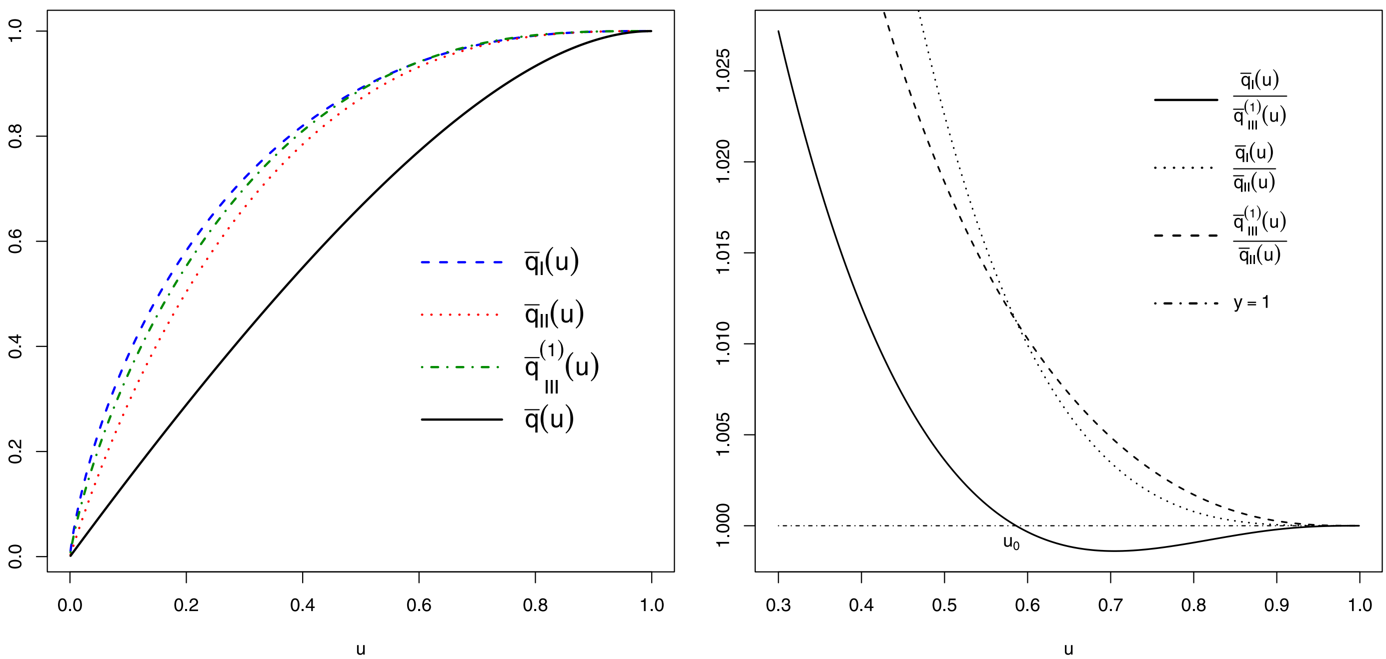

Fig. 2. Plots of the dual distortion functions for the system in Example 4.5 for cases I, II, III and without repairs (left) and plots of the ratios $\bar{q}_{III}^{(1)}/\bar{q}_{II}$, $\bar{q}_I/\bar{q}_{II}$ and $\bar{q}_I/\bar{q}_{III}^{(1)}$ in the interval (0.3,1) (right).

Table 2
Coefficients c_i and d_i associated to the dual distortion function $\bar{q}_{II}$ (see Theorem 3.6) for all the coherent systems with 1–4 IID components and the best replacement policy in the stochastic order. Cases I, II and III (i) are denoted by C_I, C_{II} and $C_{III}^{(i)}$, respectively.

N	$T=\phi(X_1,X_2,X_3,X_4)$	**c**	**d**	Best ST-policy
1	$X_{1:1}=X_1$	(1)	(-1)	$C_I\equiv C_{II}\equiv C_{III}^{(1)}$
2	$X_{1:2}=\min(X_1,X_2)$	(0,1)	(0,-2)	$C_I\equiv C_{II}$
3	$X_{2:2}=\max(X_1,X_2)$	(0,1)	(-2,0)	C_{II}
4	$X_{1:3}=\min(X_1,X_2,X_3)$	(0,0,1)	(0,0,-3)	$C_I\equiv C_{II}$
5	$\min(X_1,\max(X_2,X_3))$	(0,0,1)	(0,-4,1)	C_{II}
6	$X_{2:3}$ (2-out-of-3:F)	(0,-3,4)	(0,-6,0)	C_{II}
7	$\max(X_1,\min(X_2,X_3))$	(1/2,-1,3/2)	(-1,-2,0)	$C_{III}^{(1)}$
8	$X_{3:3}=\max(X_1,X_2,X_3)$	(-3/2,3,-1/2)	(-3,0,0)	C_{II}
9	$X_{1:4}=\min(X_1,X_2,X_3,X_4)$	(0,0,0,1)	(0,0,0,-4)	$C_I\equiv C_{II}$
10	$\max(\min(X_1,X_2,X_3),\min(X_2,X_3,X_4))$	(0,0,0,1)	(0,0,-6,2)	C_{II}
11	$\min(X_{2:3},X_4)$	(0,0,-3,4)	(0,0,-9,2)	C_{II}
12	$\min(X_1,\max(X_2,X_3),\max(X_2,X_4))$	(0, 1/2, -1, 3/2)	(0, -2, -3, 1)	C_{II}
13	$\min(X_1,\max(X_2,X_3,X_4))$	(0,-3/2,3,-1/2)	(0,-6,3,-1)	C_{II}
14	$X_{2:4}$ (2-out-of-4:F)	(0,0,-8,9)	(0,0,-12,0)	C_{II}
15	$\max(\min(X_1,X_2),\min(X_1,X_3,X_4),\min(X_2,X_3,X_4))$	(0,0,-4,5)	(0,-2,-6,0)	C_{II}
16	$\max(\min(X_1,X_2),\min(X_3,X_4))$	(0,0,0,1)	(0,-4,0,0)	C_{II}
17	$\max(\min(X_1,X_2),\min(X_1,X_3),\min(X_2,X_3,X_4))$	(0,-1,0,2)	(0,-4,-2,0)	C_{II}
18	$\max(\min(X_1,X_2),\min(X_2,X_3),\min(X_3,X_4))$	(0,-2,4,-1)	(0,-6,2,0)	C_{II}
19	$\max(\min(X_1,\max(X_2,X_3,X_4)),\min(X_2,X_3,X_4))$	(0,-3,4,0)	(0,-6,0,0)	C_{II}
20	$\min(\max(X_1,X_2),\max(X_1,X_3),\max(X_2,X_3,X_4))$	(0,-5,8,-2)	(0,-8,2,0)	C_{II}
21	$\min(\max(X_1,X_2),\max(X_3,X_4))$	(0,-4,8,-3)	(0,-8,4,0)	C_{II}
22	$\min(\max(X_1,X_2),\max(X_1,X_3,X_4),\max(X_2,X_3,X_4))$	(0,-8,12,-3)	(0,-10,2,0)	C_{II}
23	$X_{3:4}$ (3-out-of-4:F)	(0,-12,16,-3)	(0,-12,0,0)	C_{II}
24	$\max(X_1,\min(X_2,X_3,X_4))$	(2/3,0,-2,7/3)	(-1,0,-3,0)	$C_{III}^{(1)}$
25	$\max(X_1,\min(X_2,X_3),\min(X_2,X_4))$	(1/3,-3,5,-4/3)	(-1,-4,1,0)	C_{II}, $C_{III}^{(1)}$
26	$\max(X_{2:3},X_4)$	(5/6,-5,13/2,-4/3)	(-1,-4,0,0)	C_{II}, $C_{III}^{(4)}$
27	$\max(X_1,X_2,\min(X_3,X_4))$	(1/3,0,1,-1/3)	(-2,0,0,0)	C_{II}
28	$X_{4:4}=\max(X_1,X_2,X_3,X_4)$	(-10/3,6,-2,1/3)	(-4,0,0,0)	C_{II}

where

$$\bar{q}_{III}^{(1)}(u) = 2u - u\ln u - \frac{u - u\ln u}{1 + (1-u)(1-\ln u)}.$$

We compare $\bar{q}_I$, $\bar{q}_{II}$ and $\bar{q}_{III}^{(1)}$ in Fig. 2 (left) along with the dual distortion function associated to the system without repairs. We observe that $T \le_{ST} T_{II} \le_{ST} T_I$ and $T_{II} \le_{ST} T_{III}^{(1)}$. In Fig. 2 (right) we represent the quotients $\bar{q}_{III}^{(1)}/\bar{q}_{II}$, $\bar{q}_I/\bar{q}_{II}$ and $\bar{q}_I/\bar{q}_{III}^{(1)}$. The first and second ratios are above the line $y = 1$ and they are decreasing, therefore $T_{II} \le_{HR} T_{III}^{(1)}$ and $T_{II} \le_{HR} T_I$. However, $\bar{q}_I/\bar{q}_{III}^{(1)}$ crosses the line $y = 1$ at the value $u_0 = 0.5862$ and thereby T_I and $T_{III}^{(1)}$ are not comparable in the ST order. As the ratio is bathtub, we have $T_{III}^{(1)} \le_{MRL} T_I$ whenever $E(T_{III}^{(1)}) \le E(T_I)$.

Proceeding as in the examples above, we can obtain the stochastic comparisons among the three policies considered in this paper for any coherent system. In particular, Table 2 provides the best replacement policy in terms of the usual stochastic order for all the coherent systems with 1–4 IID components. The coefficients c_i and d_j, associated to the distortion function $\bar{q}_{II}$ are given for each system as well. As one would expect in the case of IID components, the policy II induces a more reliable system in most of cases (see Theorem 4.2). However, there exist some systems where repairing a fix component is better than repairing the component which causes the failure of the system. In particular, the systems 7 and 24 in Table 2 satisfy that the system's reliability is improved in a higher level if we apply the policy III rather than the policies I or II. For both systems the first component is the most important component and its functioning implies the system functioning. Furthermore, the policies II and III are better than policy I for the systems 25 and 26 and both policies are not ordered. In this case, the optimal policy depends on if the decision maker is interested in improving the reliability of the system in an advanced or early age.

5. Conclusions

In the present paper we give a procedure to determine the reliability functions of coherent systems under a minimal repair maintenance and three different replacement policies. The components can be dependent or independent. In the first replacement policy, the first broken component is repaired. In the second case, a minimal repair is applied to the component which produces the failure of the system. In the third one, a fixed component is repaired in case of failure. Note that in the two first cases we do not know a priori which component will be repaired. In this context, we have proved that if the components are ID, then the reliability function associated to the lifetime of the repaired system in case I can be expressed as a distortion of the common component reliability function (see Theorem 3.2). This distortion depends on the structure of the system and on the underlying survival copula. We provide an explicit expression of this distortion in Theorem 3.3 for IID components. Analogously, we have proved that the reliability function for the case II can also be expressed using a distortion function when the components are exchangeable. This distortion is simplified for the IID case in Theorem 3.6. The new technique developed here can also be used to study other replacement policies. As an example, we provide an explicit expression for the dual distortion functions associated to the case of repairing the two first broken components in a general system or the k first broken components in a series system.

These representation results are used to compare the three replacement policies using the main stochastic orders. In this sense, our first comparison result shows that, for any coherent system with IID components, the case II is always a better strategy of replacement than the case I in the stochastic order (see Theorem 4.2). We prove with an example that this property is not true when the components are dependent. Furthermore, the previous result holds for the hazard rate order when we consider systems which preserve the IFR property (see Theorem 4.3). Unfortunately, the case III is not ST-ordered with neither case I nor case II, even assuming IID components. We provide both counterexamples as well as some interesting examples including the comparisons of all the coherent systems with 1–4 IID components (see Table 2).

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