Interfaces with Other Disciplines

A new approach to measure systemic risk: A bivariate copula model for dependent censored data

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ARTICLE INFO

ABSTRACT

We propose a novel approach based on the Marshall–Olkin (MO) copula to estimate the impact of systematic and idiosyncratic components on cross-border systemic risk. To use the data on non-failed banks in the suggested method, we consider the time to bank failure as a censored variable. Therefore, we propose a pseudo-maximum likelihood estimation procedure for the MO copula for a Type I censored sample. We derive the log-likelihood function, the copula parameter estimator and the bootstrap confidence intervals. Empirical data on the banking system of three European countries (Germany, Italy and the UK) shows that the proposed censored model can accurately estimate the systematic component of cross-border systemic risk.

1. Introduction

The 2007–2008 financial crisis has shown how a shock that originates in one country or asset class can quickly propagate to other markets and across borders. A key aspect of financial contagion is given by the linkages among banks. In the Euro area, the cross-border exposures arose as a prominent issue with the European sovereign debt crisis in 2011 and 2012, where large exposure of many EU banks to stressed sovereigns were revealed by the European Banking Authority (EBA, 2013). In a broader perspective, correlated exposures have recently been shown to be a major source of systemic risk.

Given the importance of this research field, this paper is focused on systemic risk in the European banking sector. By definition, systemic risk involves a collection of interconnected institutions that have mutually beneficial business relationships through which insolvency can quickly propagate during periods of financial distress Billio, Getmansky, Lo, and Pelizzon (2012). Systemic risk is mainly due to *idiosyncratic* and *systematic shocks* (see De Bandt, Hartmann, & Peydró, 2009 and European Central Bank, 2010). The former affects only the health of a single financial institution, while the latter affects the whole economy, e.g. all financial institutions together at the same time. The component of systemic risk due to idiosyncratic shocks is also known as *contagion risk* in the literature (De Bandt et al., 2009).

One of the the main aims of this paper is to propose a new methodological approach for the analysis of systemic risk to jointly model idiosyncratic and systematic shocks. We propose to apply the copula approach to measure systemic risk between the banking sectors of two countries. To our knowledge, the only papers that previously applied copulae to assess banking system stability are Baglioni and Cherubini (2013a), Baglioni and Cherubini (2013b), Straetmans (2010) and Weiss (2012). In other words, the approach is quite novel to the area of banking and systemic risk.

The contributions of this paper are twofold. The first of these is to apply the Marshall and Olkin (MO) copula for modelling systemic risk between two countries. The second innovative aspect is how time to failure is considered for non-failed banks as right-censored. As the MO copula is an extreme value copula, it is suitable to study the dependence between extreme events such as bank failures. Moreover, since the MO copula shows an upper tail dependence, in order to apply it to systemic risk, we suggest to consider the distribution function (df) of time to failure for each country as the marginal df of the MO copula. Coherently with expectations, the dependence is stronger for high values of probabilities of bank failure.

Another important advantage of the MO copula is that it has both an absolute continuous and a singular part. Thanks to the singular component, we can assign a non-null probability to the event that two banks in two countries show similar failure probabilities at the same time if the copula parameter is not null. Therefore, the singular part represents the systematic component of systemic risk. In other terms, it is given by the joint probabilities of failure due to simultaneous shocks on banks located in two different

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countries with similar marginal failure probabilities. In this paper, we consider the joint failure probability of two banks operating in two different countries as a linear combination of idiosyncratic and systematic shocks. The weights of these two kinds of shocks is a function of the parameter of the MO copula. To the best of our knowledge, this is the first paper that proposes an approach to estimate the contributions of idiosyncratic and systematic shocks to cross-border systemic risk. Baglioni and Cherubini (2013a) also use the MO copula to propose an index that represents the average impact of systematic and idiosyncratic risk, but they are focused on the financial system in a given country, not on cross-border systemic risk.

Regarding the second innovation of this paper, we apply a Type I censored sampling, i.e. the testing stops at a predetermined time, at which point any non-failed banks are right-censored. In this way all the information of non-failed banks can be used to estimate the parameter of the dependence structure. Finally, we suggest a pseudo- maximum likelihood method to estimate the parameter of the MO copula for the Type I censored sampling[1]. We derive the log-likelihood function, the copula parameter estimator and the bootstrap confidence intervals. The pseudo-maximum likelihood method handles the complexity given by the presence of both a continuous and a singular component of the MO copula. As far as we know, this is the first paper that applies the MO copula to censored data on systemic risk.

In this work, we apply the suggested model to balance sheet data on three of the most important banking systems in Europe: Germany, Italy and the UK. These countries present different characteristics. Germany and Italy are characterised by a large number of small banks, while the UK banking system is a concentrated banking system with a few large banks. We pair up banks in two European countries in terms of their probabilities of bank failure estimated by using the BGEVA model (Calabrese, Marra, & Osmetti, 2016 and Calabrese, Degl' Innocenti, & Osmetti, 2017). In order to estimate the marginal cumulative distribution function (cds) of the MO copula, we use the empirical cdf of time to failure for each country.

We apply the proposals of this paper to data over the period 1995–2012. The European sovereign debt crisis of 2009 is included in the empirical analysis. At first, we estimate the probability of failure for banks in each country using the BGEVA model (see Calabrese & Osmetti, 2013 and Calabrese & Osmetti, 2015) on a set of bank specific factors addressed by the CAMELS framework (e.g. Arena, 2008). To capture the economic cycle, we include macroeconomic variables in the BGEVA model. The estimates so obtained are used to pair up banks in two countries. In the country with the higher number of banks, we consider only the banks with higher risk failure.

We compare the MO copula with the copula models used in the literature Weiss (2012), such as the Gaussian copula, the Gumbel copula and a mixture of the Frank, Clayton and Gumbel copula. An important result of this empirical analysis is that the estimate of the upper tail dependence in the MO copula is higher due to the singular component. Moreover, according to a goodness-of-fit measure, the MO copula is the model that best fits the data. Finally, when we apply censored techniques to the data, coherently with our expectations, we find that the impact of the systematic component on systemic risk increases.

We organise the paper as follows. The next section describes the literature review. Section 3 explains the methodological proposal. Section 4 describes the dataset and reports the main results on cross-border systemic risk. Finally, the last section contains some concluding remarks. In the appendix, we report the score functions to obtain the pseudo-maximum likelihood estimator of the parameter of the MO copula for Type I censored sampling and a simulation study.

2. Literature review

The European Central Bank European Central Bank (2010) has identified three main approaches to analyse systemic risk. First, early warning signal models use information on current data to estimate the likelihood that intermediaries show financial deterioration (for example Calabrese & Giudici, 2015). Second, contagion and spillover models can be used to analyse the transmission of financial shocks across banks (see e.g. Billio et al., 2012). Third, stress testing models can assess the effects of macroeconomic shocks on the banking system (e.g. EBA, 2013). The first two strands are mainly focused on idiosyncratic shocks, while the last one primarily analyses the systematic component of systemic risk. A way of analysing systematic shocks in the first two groups of models is to include macroeconomic variables, analogous to Calabrese and Giudici (2015).

Another possible classification of the literature on systemic risk can be divided into two different strands: the first area of research uses financial market data, see e.g. Cappiello, Gerard, and Manganelli (2005), Engle and Manganelli (2004), Gropp, Lo Duca, and Vesala (2009), Hartmann, Straetmans, and de Vries (2004), Longin and Solnik (2001). The second approach is based on banks' balance sheet data to assess systemic risk, see e.g. Degryse and Nguyen (2007), Mistrulli (2011), van Lelyveld and Liedorp (2006) and Sormaki, Bech, Arnold, Glass, and Beyeler (2007).

Different methodologies have been applied to analyse contagion risk. Some studies assume that the presence of contagion risk can be detected by observing negative abnormal returns (see e.g. Akhigbe & Madura, 2001, Gropp & Moermann, 2004, Kabir & Hassan, 2005). Few authors have used extreme value theory to analyse the idiosyncratic shocks (see e.g. Gropp & Moermann, 2004 and Gropp et al., 2009), others have used a copula-based approach (see De Vries, 2005, Straetmans, 2010 and Weiss, 2012). Weiss (2012) captures the changes in the dependence structure of abnormal bank returns by analysing the changes in the parametric form and the parameters of various copulae. Specifically, the author analyses changes in the dependence structure of banks around bailout announcements. To cover a maximal variety of tail dependence structures, Weiss (2012) considers a convex combination over time of the Student's t, Frank, Clayton and Gumbel copula. The author uses the Akaike's Information Criterion (AIC) to choose the copula with the highest goodness of fit. He obtains that the Clayton-Frank-Gumbel mixture shows the best fit to the logarithmic stock returns of German banks. De Vries (2005) suggests the Gumbel copula with Pareto marginal dfs as a joint distribution of the returns on syndicated loans to obtain heavy tailed marginal dfs, positive correlation and asymptotic independence.

We highlight that all the previous copulae are absolutely continuous, this means that the impact of the systematic component on systemic risk could be underestimated. We overcome this drawback by applying the MO copula. There is a limited literature on the use of the MO copula for modelling systemic risk. Baglioni and Cherubini (2013a) propose a new financial stability index (named Cuadras and Augé index) to measure the fragility of the banking sector in a given country. Time to failure for a bank is assumed to follow an exponential distribution. Each bank shows the same intensity parameter of the exponential distribution, so the multivariate intensity based model is homogeneous. To model the dependence structure between banks the authors use a symmetric MO copula. Baglioni and Cherubini (2013b) extend this approach to marginal distributions with non-constant intensity parameters

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[1] Osmetti (2012) suggested an estimation technique for the Type II censored sampling, i.e. the testing stops after a given number of observations fails.

and to non-symmetric MO copula. Using a hierarchical approach, the authors model the systemic risk within the banking system at the lower level and the probability of a joint default of the banking system and the public sector at the higher level. In both papers the model parameters are estimated by non-parametric measures of association such as Spearman's rank correlation.

The main methodological differences between this paper and the previous works Baglioni and Cherubini (2013a) and Baglioni and Cherubini (2013b) are that we do not assume parametric distributions for the marginal default probabilities. We mainly focus our attention on the dependence structure and the impact of the systematic component on cross-border systemic risk. Baglioni and Cherubini (2013a) and Baglioni and Cherubini (2013b) use a non-parametric approach to estimate the copula parameter, instead we use a semiparametric technique based on the pseudo-maximum likelihood method. Furthermore, we use a censored sampling to include the observation of non-failed banks for the estimation of the copula parameter.

From an empirical point of view, most of the copula-based approaches cited in this section use financial market data. On the contrary, we use banks' balance sheet data in this paper as we analyse European countries such as Germany and Italy characterised by a high number of small banks, for which market data are not available.

3. A new copula model for estimating systemic risk

In this work we propose to model the dependence structure of cross-border bank failures using a copula approach. The concept of copula represents a flexible method since it does not require parametric assumptions on the marginal components (Nelsen, 2006, and Fisher, 1997). In this way, a general class of distributions can be expressed through a simple model specification.

There are several advantages in applying the copula approach to systemic risk. Firstly, the copula function is a suitable model to represent the dependence structure between rare events. As the percentage of bank failures is usually very low (lower than 5%), bank failure can be classified as a rare event. We propose in this work to use a model that better classifies rare events, such as the BGEVA model (Calabrese & Giudici, 2015), to estimate the empirical marginal cdfs in a copula framework.

Secondly, the copula model accounts for non-linear dependence and upper tail dependence. Few empirical studies, for example Weiss (2012), have shown that linear models, such as linear regression analysis, are usually unable to capture contagion effects. Therefore, to accurately assess systemic risk, we consider a copula that allows for tail dependence, analogously to Baglioni and Cherubini (2013a) and Baglioni and Cherubini (2013b) for CDS quotes and De Vries (2005) and Straetmans (2010) for bank stock returns.

Thirdly, the parametric specification of the marginal distributions is not required in the copula approach, only the characteristics of the dependence structure are defined. From the available copula families, we consider an extreme value copula with non-trivial tail dependence given by the Marshall–Olkin copula in order to represent the dependence structure between bank failures.

3.1. Copulae and tail dependence

Every bivariate and multivariate cdf F can be treated as the result of two components: the marginal distributions and the dependence structure. The copula describes the way that the two marginal distributions are put together into the bivariate cdf.

In mathematical terms, a bivariate copula is a function $C: I^2 \rightarrow I$, with $I^2 = [0, 1] \times [0, 1]$ and $I = [0, 1]$, which satisfies all the properties of a cdf. In particular, it is the bivariate cdf of a random vector

(U, V) with uniform marginal random variables (rvs) in [0,1]

$$C(u, v) = P(U \leq u, V \leq v), \quad 0 \leq u \leq 1 \quad 0 \leq v \leq 1.$$

To better understand the copula model we consider the Sklar's theorem Sklar (1959).

Theorem 3.1 (Sklar). *Let* (X, Y) *be a bivariate random variable with joint cdf* $F_{X,Y}(x, y)$ *and marginal cdfs* $F_X(x)$ *and* $F_Y(y)$. *It exists a copula function* $C: I^2 \rightarrow I$ *such that* $\forall x, y \in \mathcal{R}$

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)) \tag{3.1}$$

If $F_X(x)$ *and* $F_Y(y)$ *are continuous functions then the copula* $C(\cdot)$ *is unique. Otherwise,* $C(\cdot)$ *is uniquely determined on* $RanF_X \times RanF_Y$. *Conversely, if* $C(\cdot)$ *is a copula function and* $F_X(x)$ *and* $F_Y(y)$ *are marginal cdfs, then the* $F_{X,Y}(x, y)$ *in (3.1) is a bivariate cdf.*

If the marginal cdfs are continuous and strictly increasing functions, from (3.1) the copula function is

$$C(u, v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)) \tag{3.2}$$

where $u = F_X(x)$ and $v = F_Y(y)$ are the cdfs $F_X(\cdot)$ and $F_Y(\cdot)$, respectively.

However, if the marginal cdfs are not strictly increasing functions, then the inverse of the cdf does not exist. In this case, we can consider the quasi-inverse of a cdf defined as $F^{(-1)}(t) = inf\{x|F(x) \geq t\} = sup\{x|F(x) \leq t\}$ for all $t \in I$ (see Nelsen, 2006 for details).

Thus, a copula captures the dependence structure between the marginal probabilities $F_X(x)$ and $F_Y(y)$ and, consequently, between the marginal rvs X and Y.

A pivotal characteristic for systemic risk analysis is the upper tail dependence. An upper tail dependence parameter χ_u is defined as

$$\chi_u = \lim_{u \rightarrow 1^-} P[X > F_X^{-1}(u)|Y > F_Y^{-1}(u))]$$
$$= \lim_{u \rightarrow 1^-} P[Y > F_Y^{-1}(u)|X > F_X^{-1}(u))] \tag{3.3}$$

when the limit exists. Higher is the value of $\chi_u \in (0, 1]$, higher is the level of upper tail dependence. Analogously, the lower tail dependence parameter χ_l can be defined. See Nelsen (2006) for the expressions of the lower and upper tail dependence parameters for the main copula families.

3.2. The Marshall-Olkin copula

Let X and Y be the time to failure of two banks located in two countries and let $F_X(t) = P(X \leq t)$ and $F_Y(t) = P(Y \leq t)$ be their probabilities of failure over a given time period. We consider a copula function to analyse the dependence structure between the time to failure of banks situated in two different countries. Particularly, we suggest to use the Marshall and Olkin (MO) copula.

The MO bivariate exponential distribution was proposed by Marshall and Olkin in 1967 (Mai & Scherer, 2012). It is used in reliability analysis to model jointly failure time of two components in a system when the failure is due to both idiosyncratic shocks, given by the characteristics of the components, and shocks common to both the components. The MO copula models the dependence structure of the namesake probability distribution. The main advantage of our suggestion is that the dependence structure of time to bank failure could be due to both idiosyncratic and systematic shocks. As explained in Section 2, the literature shows that both these components are important to model systemic risk.

In the case of two exchangeable marginal rvs X and Y, the MO copula or Cuadras-Augé copula (see Nelsen, 2006 and Mai & Scherer, 2012) is defined as

$$C(u, v) = P(U \leq v, V \leq v) = uv \min(u^{-\theta}, v^{-\theta}) \tag{3.4}$$

where $\theta \in [0, 1]$ represents the intensity of the (positive) relationship between the marginals. If $\theta = 0$ then the rvs X and Y are stochastically independent and the MO copula becomes $C(u, v) = uv$. If $\theta = 1$ then there is a perfect positive association between the rvs X and Y and the MO copula becomes $C(u, v) = min(u, v)$. Furthermore, the MO copula is an extreme value copula with an upper right tail dependence where θ is the upper tail dependence parameter χ_u defined in Eq. (3.3).

An important characteristic of the MO copula (3.4) is that it has an absolute continuous part and a singularity for $u = v$ with positive probability (see Nelsen, 2006 and Osmetti, 2012). Thanks to the singular part, we can assign a non-null probability to the event $U = V$. This means that the failure of two banks (characterised by the same marginal cdf) located in two different countries at the same time has a non null probability. Hence, the MO copula can be considered as a linear combination of the absolute continuous part C_a and the singular part C_s

$$C(u, v) = \frac{2 - 2\theta}{2 - \theta} C_a(u, v) + \frac{\theta}{2 - \theta} C_s(u, v) \tag{3.5}$$

where $C_s(u, v) = [\min(u^\theta, v^\theta)]^{\frac{2 - \theta}{\theta}}$ for $u = v$ and $C_a(u, v)$ for $u \neq v$ is

$$C_a(u, v) = \frac{2 - \theta}{2 - 2\theta} [uv \min(u^{-\theta}, v^{-\theta})] - \frac{\theta}{2 - 2\theta} C_s(u, v).$$

As explained in Section 2, the systemic risk is due to both the idiosyncratic and the systematic shocks. The former is mainly characterised by banks' characteristics, the latter represents characteristics common to both the countries, such as macroeconomic conditions.

In Eq. (3.5) the idiosyncratic component is represented by the absolute continuous part C_a and the systematic component is given by the singular part C_s. The weights of these two components are a function of the copula parameter $\theta \in [0, 1]$. If $\theta = 0$, the systemic risk is given only by idiosyncratic shocks. This means that the copula function in Eq. (3.5) is given only by the absolutely continuous component $C(u, v) = C_a(u, v) = uv$. In this case, the marginal failure probabilities are independent, so the joint failure probability is given by the product of the marginal probabilities in the two countries. Instead, if the parameter θ is high ($\theta > 2/3$), then systematic shocks are more important than idiosyncratic shocks to explain systemic risk. For values of θ very close to 1, the idiosyncratic component is very small (Baglioni & Cherubini, 2013b).

As the copula defined in Eq. (3.4) is exchangeable (Baglioni & Cherubini, 2013a), this means that the dependence structure is symmetric $C(u, v) = C(v, u)$. In other terms, the order of the two analysed countries does not affect the cross-border measure. We obtain the same result for the pair given by the country A and B and for the pair given by the country B and A.

The cdf defined in (3.1) can be estimated by parametric or semiparametric approaches. The widely used parametric approaches are the maximum likelihood (ML) method and the two-stage inference function for margins (IFM) method proposed by Joe (1997). Important discussions about the properties of the two methods could be found in Joe (2005), Kojadinovic and Yan (2010) and Kim, Silvapulle, and Silvapulle (2007).

Beims, Bain, and Higgins (1972) and Bhattacharyya and Johnson (1973) use ML method to estimate a MO bivariate exponential distribution. When the marginal distributions are unknown, a semiparametric method is preferred. This is represented by the Pseudo Maximum Likelihood (PML) or the canonical maximum likelihood (see Cherubini, Luciano, & Vecchiato, 2004 and Genest, Ghoudi, & Rivest, 1995). In contrast to parametric methods such as ML and IFM, the PML method does not require that the user specifies the functional forms for the marginal distributions. In particular, the PML method is a two-step semiparametric estimation approach:

in the first step the marginal cdfs are estimated by the empirical cdf, in the second step the copula parameters are estimated by the maximum likelihood method. For more details on the properties of the method and on the comparison between the parametric and semiparametric methods see for example Kim et al. (2007), Kojadinovic and Yan (2010), Genest et al. (1995), Genest and Werker (2002), Kole, Koedijk, and Verbeek (2007), Durrleman, Nikeghbali, and Roncalli (2000), Mai and Scherer (2014), Mai and Scherer (2012). Other popular procedures for estimating the MO copula parameter are the method of moments (Hering & Mai, 2012) and an approach based on the inversion of Spearman's rho and Kendall's tau (Kojadinovic & Yan, 2010).

In this section we suggest to apply a PML to estimate the MO copula. In the first step, we consider the empirical cdf as a nonparametric estimator of the cdf of the time to bank failure for each country $\hat{u}_i = \hat{F}_X(x_i)$ and $\hat{v}_i = \hat{F}_Y(y_i)$. In the second step, we obtain the estimator of the parameter $\theta \in (0, 1)$ of the MO copula by maximising the conditional likelihood function as follows

$$\hat{\theta} = arg \max L(\theta | \hat{\underline{u}}, \hat{\underline{v}})$$

where

$$L(\theta | \hat{\underline{u}}, \hat{\underline{v}}) = \prod_{i=1}^{n} c_\theta(\hat{u}_i, \hat{v}_i) \tag{3.6}$$

To compute the probability density function $c_\theta(\cdot)$ in Eq. (3.6) we apply the procedure described in Proschan and Sullo (1976) for the MO exponential distribution. As shows in Eq. (3.5), the MO copula is not absolutely continuous respect to the two-dimensional Lebesgue measure (μ_2) and contains singularities. Consequently, the joint density function does not exist with respect to μ_2. Nevertheless, the copula is absolutely continuous with respect to a σ-finite measure $\mu(B)$ defined on the two-dimensional space as follows (see Beims et al., 1972; Osmetti, 2012; Proschan & Sullo, 1976):

$$\mu(B) = \mu_2(B) + \mu_1\left(B \cap \left\{x : (x, x) \in R_2^+\right\}\right) \tag{3.7}$$

for each $B \in B_2^+$ where μ_2 is a two-dimensional Lebesgue measure, B_2^+ is the Borel σ-algebra in R_2^+ and μ_1 is the Lebesgue measure on the real line.

It follows that we can define a probability density function $c_\theta(\cdot, \cdot)$ with respect to the measure $\mu(\cdot)$ defined in Eq. (3.7).

Theorem 3.2. *The MO copula density function $c_\theta(u, v)$ is defined as follows:*

$c_\theta(u, v)$

$$= \begin{cases} c_1(u, v) = (1 - \theta) \dfrac{1}{uv} C_\theta(u, v) = (1 - \theta)u^{-\theta} & if \ \{u > v\} \\[2mm] c_2(u, v) = (1 - \theta) \dfrac{1}{uv} C_\theta(u, v) = (1 - \theta)v^{-\theta} & if \ \{u < v\} \\[2mm] c_s(w) = \theta \dfrac{1}{w} C_\theta(w, w) = \theta w^{1-\theta} & if \ u = v = w \end{cases} \tag{3.8}$$

with $0 \leq v \leq 1$, $0 \leq u \leq 1$ and $0 < \theta < 1$.

Proof. We obtain $c_1(\cdot, \cdot)$ and $c_2(\cdot, \cdot)$ by computing the derivatives $\frac{\partial^2 C_\theta(u, v)}{\partial u \partial v}$ for $u > v$ and $v > u$, respectively. As we cannot obtain $c_s(\cdot)$ in a similar way, we follow the approach suggested by Sarhana and Balakrishnan (2007), Kundu and Gupta (2009) and Jamalizadeh and Kundu (2013) and we consider the following equation:

$$\int_0^1 \int_0^u c_1(u, v) dv du + \int_0^1 \int_0^v c_2(u, v) du dv + \int_0^1 c_s(w, w) dw = 1.$$

It follows that

$$I_1 = \int_0^1 \int_0^u c_1(u,v)dvdu = \int_0^1 \int_0^u (1-\theta)u^{-\theta}dvdu$$

$$= (1-\theta)\int_0^1 u^{1-\theta}du = (1-\theta)\int_0^1 w^{1-\theta}dw$$

$$I_2 = \int_0^1 \int_0^v c_1(u,v)dudv = \int_0^1 \int_0^v (1-\theta)v^{-\theta}dudv$$

$$= (1-\theta)\int_0^1 v^{1-\theta}dv = (1-\theta)\int_0^1 w^{1-\theta}dw$$

Since

$$\int_0^1 c_s(w,w) = 1 - (I_1 + I_2) = \theta \int_0^1 w^{1-\theta}dw = \frac{\theta}{(2-\theta)},$$

we have

$$c_s(w,w) = \theta w^{1-\theta} \qquad\qquad \square$$

The function $c_\theta(\cdot,\cdot)$ can be considered a probability density function if it is understood that the two terms $c_1(\cdot,\cdot)$ and $c_2(\cdot,\cdot)$ are probability density functions with respect to the two-dimensional Lebesgue measure and the third term $c_s(\cdot,\cdot)$ is a probability density function with respect to the one-dimensional Lebesgue measure (see Sarhana & Balakrishnan, 2007, Beims et al., 1972, Kundu & Gupta, 2009 and Proschan & Sullo, 1976). Therefore, even if the MO copula is not absolutely continuous with respect to the two-dimensional Lebesgue measure, we can specify the density function and derive the likelihood function $L(\theta|\underline{\hat{u}}, \underline{\hat{v}})$ as follows

$$L(\theta|\underline{\hat{u}}, \underline{\hat{v}}) \propto (1-\theta)^{n_1+n_2}\theta^{n_3}\prod_{i=1}^{n} C_\theta(\hat{u}_i, \hat{v}_i). \qquad (3.9)$$

The terms n_1, n_2 and n_3 are the number of observations such that $n_1 = \sharp\{\hat{u}_i < \hat{v}_i\}$, $n_2 = \sharp\{\hat{u}_i > \hat{v}_i\}$ and $n_3 = \sharp\{\hat{u}_i = \hat{v}_i\}$. Hence, the maximum likelihood estimator of θ is

$$\hat{\theta} = (1 + \exp(-\hat{\psi}))^{-1}$$

with

$$\hat{\psi} = -\ln\left[\frac{n - 2n_3 - S_{\min} + \sqrt{n^2 + S_{\min}^2 - S_{\min}(2n - 4n_3)}}{2n_3}\right]$$

with $n_3 > 0$ and $S_{\min} = \sum_{i=1}^{n} \min(-\ln(\hat{u}_i), -\ln(\hat{v}_i))$ (see Osmetti, 2012 for details). Ruiz-Rivas and Cuadras (1988) obtained a similar result.

3.3. Censored time of failure

The method described in the former section to estimate the MO copula allows to use only the information provided by failed banks that represents a very low percentage of the sample. To use also the characteristics of most of the banks that do not fail, we suggest to apply the Type I censored sampling on the right to the time to bank failure.

In the literature there are two main types of censored sampling: Type I and Type II censored sampling (Choen, 1991). The Type I censoring occurs when an experiment ends after a given time t^*. Hence, the number of censored observations is random. On the contrary, the Type II censored sampling occurs when an experiment ends after a specific number of observations has occurred. Therefore, the censoring time is random. Two different sample statistics are given by the estimation procedure for these sampling methods, as explained by Choen (1991) for the univariate context. While Osmetti (2012) proposed an estimator for the MO
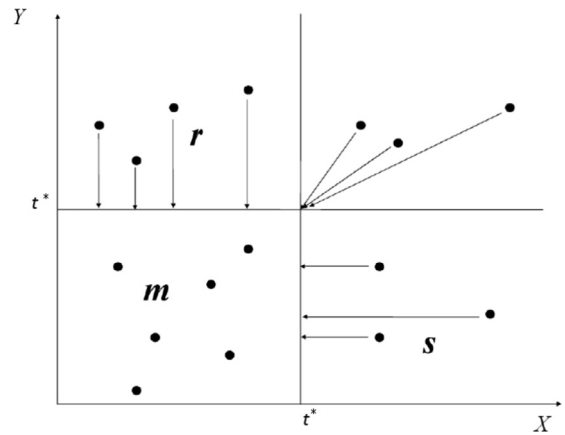

Fig. 1. Sample censored data.

copula with bivariate Type II censored sampling, in this section we suggest an estimation procedure for bivariate Type I censored data.

At the beginning, to pair up banks located in two different countries, we order banks in each country based on their failure probability. In the order created for each country, we consider the ith bank. Let x_i be the observed time to failure for the ith bank located in a given country and y_i the time to failure for the ith bank located in a different country. We define $m = \sharp\{x_i \le t^* \cap y_i \le t^*\}$ the number of pairs with both failed banks in the two countries. Furthermore, we define $r = \sharp\{x_i \le t^* \cap y_i > t^*\}$ the number of failed banks in the first country and of non-failed banks in the second country and $s = \sharp\{x_i > t^* \cap y_i \le t^*\}$ the number of non-failed banks in the first country and of failed banks in the second country. This means that $n - m = \sharp\{x_i > t^* \cap y_i > t^*\} + r + s$ is the number of pairs where at least one bank of the two countries is not failed. To apply a Type I censored sampling, we assign t^* to the time to failure for non-failed banks, as shown in the Fig. 1.

We modify the CLM procedure described in the previous section as follows. In the first step we estimate the marginal cdf using the Kaplan-Mayer estimator[2]: $\hat{u}_i = \hat{F}_X(x_i)$, $\hat{v}_i = \hat{F}_Y(y_i)$. Then, in the second step, we maximise the conditional likelihood function of the copula. We consider $(\Delta^X, \Delta^Y) = (I_{\{X \le t^*\}}(x), I_{\{Y \le t^*\}}(y))$, $\overline{\Delta}^X = 1 - \Delta^X$ and $\overline{\Delta}^Y = 1 - \Delta^Y$, where $I_A(\cdot)$ is the indicator function of the set A. Following Owzar and Sen (2003), we compute the conditional likelihood function for the copula

$$l(\theta|\hat{F}_X, \hat{F}_Y) = \sum_{i=1}^{n} \ln[c_\theta(\hat{F}_X(x_i), \hat{F}_Y(y_i))]^{\Delta_i^X \Delta_i^Y}$$

$$+ \sum_{i=1}^{n} \ln[C_\theta^1(\hat{F}_X(x_i), \hat{F}_Y(y_i))]^{\overline{\Delta}_i^X \Delta_i^Y}$$

$$+ \sum_{i=1}^{n} \ln[C_\theta^2(\hat{F}_X(x_i), \hat{F}_Y(y_i))]^{\overline{\Delta}_i^Y \Delta_i^X}$$

$$+ \sum_{i=1}^{n} \ln[C_\theta(\hat{F}_X(x_i), \hat{F}_Y(y_i))]^{\overline{\Delta}_i^X \overline{\Delta}_i^Y} \qquad (3.10)$$

where $c_\theta(u,v)$ is the copula density defined in (3.8), $C_\theta^1(u,v) = \frac{\partial C_\theta(u,v)}{\partial v}$ and $C_\theta^2(u,v) = \frac{\partial C_\theta(u,v)}{\partial u}$.

As described in Theorem 3.2 of the previous section, we can define the density function $c_\theta(\cdot,\cdot)$ in Eq. (3.8) with respect to $\mu(B)$ and derive the likelihood function in (3.10).

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[2] The Kaplan–Maier estimator is usually used to estimate the cdf for a censored sample (see Kaplan & Meier, 1958).

The maximum likelihood estimator of θ for Type I censored data is

$$\hat{\theta}_c = (1 + \exp(-\hat{\psi}_c))^{-1} \tag{3.11}$$

with

$$\hat{\psi}_c =$$
$$- \ln \left[\frac{m + r + s - 2m_3 - S_{\min} + \sqrt{(m + r + S_{\min} - 2m_3)^2 + 4m_3(m + r + s - m_3)}}{2m_3} \right] \tag{3.12}$$

where $m_1 = m - \sharp\{\hat{u}_i \geq \hat{v}_i\}$, $m_2 = m - \sharp\{\hat{u}_i \leq \hat{v}_i\}$, $m_3 = m - m_1 - m_2$ and

$$S_{\min} = \sum_{i=1}^{m} \min(-\ln(\hat{u}_i), -\ln(\hat{v}_i)) + \sum_{i=1}^{r} [-\ln(\hat{u}_i)]$$
$$+ \sum_{i=1}^{s} [-\ln(\hat{v}_i)] + (n - m - r - s)t^*.$$

The maximum likelihood estimator (3.11) is the unique and acceptable solution of this optimisation problem (see Appendix A.1 for details).

4. Empirical results

4.1. Dataset

The empirical analysis is based on annual data for the period 1995–2012 for the German (DE), the Italian (IT) and the UK banks. The data are from Bankscope, a comprehensive database of balance sheet and income statement data for individual banks across the world provided by the private company Bureau Van Dijk. The time horizon and the geographic area are important for the European sovereign debt crisis of 2009. We choose to analyse the cross-border bank interdependence between Italy, Germany and the UK since their banking systems are quite different. For example, most of the Italian and the German banks are quite small and they are cooperative or savings banks (around 90% in Germany). In the UK the average bank size is very large, there are not traditionally regional or state banks and only one cooperative bank.

All the three banking systems came under pressure during the financial and the sovereign debt crisis. The UK banks were significant exposed to toxic assets which originated in the US, the Italian and the German banks less. The impact of the sovereign debt crisis was stronger on the Italian and the German banking systems, even if the stability of the German system has been achieved in the short run in large part through substantial government support measures.

To analyse these banking systems, we choose a definition of bank failure in accord with Arena (2008) and Calabrese and Giudici (2015). A bank failure occurs when the bank is in at least one of the following statuses: bankruptcy, in liquidation, dissolved or under receivership. As mergers and acquisition could have been carried out for strategic reasons rather than insolvency aims Arena (2008), banks that are merged or acquired by another bank are not considered failed. All data are available for 1802 German banks, 602 Italian banks and 265 UK banks. These sample sizes are coherent with the characteristics of the banking systems of these countries. The number of failed banks are 72 for UK, 30 for IT and 86 for DE.

To pair up banks located in two countries, we order the banks in each country based on their failure risk. In particular, we apply the BGEVA model (see Calabrese & Osmetti, 2013 and Calabrese & Osmetti, 2015) to estimate the probability of failure for each bank in a given country. The BGEVA model is a semiparametric regression approach suitable to correctly classify binary rare events. As

explanatory variables in the BGEVA model, we follow the literature on bank failure: we consider two sets of variables, one is bank specific, i.e. the financial ratios associated with the CAMELS rating system Arena (2008), the latter is given by macroeconomic factors that affect the all banking system (Calabrese & Giudici, 2015). To measure the severity of multicollinearity we have computed the Variance Inflation Factor (VIF) for each explanatory variable. We consider 22 independent variables, we remove those with a VIF higher than 5 and we obtain the following 18 covariates: Total Assets, Loan Loss Reserve over Gross Loans, Equity over Total Assets, Return on Average Assets (ROAA), Return on Average Equity (ROAE), Net Loans over Total Assets, Liquid Assets over Cust& ST Funding, Interbank Assets over Interbank Liabilities, Liquid Assets over Tot Dep & Bor, Tier 1 Ratio, Total Capital Ratio, Equity over Liabilities, Equity over Net Loans, Net Interest Margin, Growth Rate of GDP, Inflation Rate, Unemployment Rate and Interest Rate.

4.2. Estimation results

After ordering the banks in each country based on their failure probability, to apply a bivariate copula we consider the same number of banks with higher failure risk in the two analysed countries. Then, we use the empirical cdfs of time to failure for each country as marginal cdfs of the MO copula. Therefore, we estimate the parameter θ of the MO copula both in the case of complete and censored sample following the procedures suggested in Section 3.2. and 3.3. The sample size of the complete data is given by the lowest number (30) of failed banks in the three countries UK, Italy and Germany. For the censored sample, the sample size is given by 265 failed and non-failed banks.

The singular component of the MO copula is obtained by the pairs of banks in two different countries that fail in the same year with similar risk of failure estimated using the empirical cdfs of time to failure.[3] If we consider the countries UK and Italy, the singular component is respectively given by 9 and 113 banks for the non-censored and censored sample. When we analyse the UK and Germany, the singularity is represented by 11 and 158 banks for non-censored and censored sample. Finally, Italy and Germany show 13 (in the non-censored sample) and 187 (in the censored sample) banks that fail in the same year with similar risk of failure.

We compare the MO copula with the copula models used in the literature (see Rodriguez, 2007 and Weiss, 2012), such as the Gaussian copula, the Gumbel copula and a finite mixture of the Frank C_F, Clayton C_C and Gumbel C_G copulae ($F + C + G$)

$$C(u, v) = \pi_F C_F(u, v; \alpha) + \pi_C C_C(u, v; \gamma) + (1 - \pi_F - \pi_C) C_G(u, v; r)$$

with weights $0 \leq \pi_i \leq 1$ for $i = F, C, G$. The MO, Gumbel and the mixture of copulae display asymptotic tail dependence and asymmetry, while the Gaussian copula is symmetric without tail dependence. The parameter $-1 < \rho < 1$ of the Gaussian copula represents the linear correlation coefficient. The parameter $r > 1$ of the Gumbel copula is a measure of positive association and represents the intensity of the upper tail dependence ($\chi_u = 2 - 2^{1/r}$). The Frank copula is a symmetric copula and it shows positive dependence for $\alpha \in (0, +\infty)$, negative dependence for $\alpha \in (-\infty, 0)$ and independence for $\alpha = 0$. The tail dependence in the Frank copula is null. The Clayton copula shows also a positive dependence. Its parameter γ represents the intensity of the lower tail dependence ($\chi_u = 2^{-1/\gamma}$). Hence, the mixture of the Frank, Clayton and Gumbel copulae can display lower tail dependence for the Clayton copula, and upper tail dependence for the Gumbel copula.

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[3] The number of bank pairs for the singular component is given by #{|$v_i - u_i$| < 0.001}.

Table 1
Copula parameter estimates and bootstrap confidence intervals.

Copula	IT-UK	IT-DE	UK-DE
Gaussian	$\hat{\rho} = 0.25$	$\hat{\rho} = 0.30$	$\hat{\rho} = 0.27$
	(0.03; 0.32)	(0.13; 0.37)	(0.08; 0.34)
Gumbel	$\hat{r} = 1.30$	$\hat{r} = 1.40$	$\hat{r} = 1.37$
	(1.02; 1.49)	(1.05; 1.59)	(1.05; 1.57)
	$\hat{\pi}_F = 0.31$	$\hat{\pi}_F = 0.21$	$\hat{\pi}_F = 0.25$
	(0.12; 0.50)	(0.13; 0.35)	(0.14; 0.33)
$F + C + G$	$\hat{\pi}_C = 0.15$	$\hat{\pi}_C = 0.14$	$\hat{\pi}_C = 0.13$
	(0.009; 0.22)	(0.01; 0.23)	(0.01; 0.22)
	$\hat{\alpha} = 0.01$	$\hat{\alpha} = 0.04$	$\hat{\alpha} = 0.03$
	(0.00; 0.18)	(0.03; 0.19)	(0.03; 0.18)
	$\hat{\gamma} = 0.23$	$\hat{\gamma} = 0.26$	$\hat{\gamma} = 0.25$
	(0.13; 0.45)	(0.12; 0.47)	(0.10; 0.47)
	$\hat{r} = 1.33$	$\hat{r} = 1.45$	$\hat{r} = 1.45$
	(1.03; 1.69)	(1.10; 0.53)	(1.08; 0.54)
MO	$\hat{\theta} = 0.37$	$\hat{\theta} = 0.55$	$\hat{\theta} = 0.45$
	(0.29; 0.52)	(0.33; 0.59)	(0.28; 0.54)

Table 2
Fit measures.

	AIC		
Copula	IT-UK	IT-DE	UK-DE
Gaussian	−4.32	−10.3	−9.18
Gumbel	−19.20	−22.33	−18.45
F+C+G	−24.87	−25.98	−18.45
MO copula	−34.44	−37.89	−21.67

For each copula we compute the PML estimate of the copula parameters and the bootstrap confidence intervals (see Efron & Tibshirani, 1993) on 1000 bootstrap samples randomly drawn. The results are reported in Table 1. The linear correlation coefficient $\hat{\rho}$ of the Gaussian copula is close to zero for all the pairs of countries. This result could be due to the fact that the Gaussian copula displays only a linear dependence and not a tail dependence. The latter is what we expect in the data. To verify this expectation we apply a Gumbel copula that shows upper tail dependence and a mixture of copulae that displays both upper and lower tail dependence. Since the parameter $\hat{r}$ is higher than 1 for all the three pairs of countries, this means that there is upper tail dependence. The intensity of this dependence is quite low since all the values of r are close to 1.

In agreement with the expectations, the Gumbel copula shows the highest weight in the mixture model for all the pairs of countries ($\hat{\pi}_G$=0.54 for IT-UK, $\hat{\pi}_G$=0.65 for IT-DE and $\hat{\pi}_G$=0.62 for UK-DE). We use Eq. (3.3) to compute the upper tail dependence parameter. We obtain χ_u=0.316 for IT-UK, χ_u=0.365 for UK-DE and χ_u=0.387 for IT-DE. This means that the intensity of the upper tail dependence in the mixture model is still low. We highlight that the orderings of the upper tail dependence parameter estimates in both the mixture and the Gumbel copulae are the same. Furthermore, these orderings correspond to the one of the linear correlation coefficients in the Gaussian copula. From this ordering we deduce that the systemic risk for IT-DE is higher than that for DE-UK that is finally higher than the one for IT-UK. This result is in line with expectations and with the outcomes obtained in Gropp et al. (2009). In Gropp et al. (2009) the authors estimate the contagion directions of banks that experience a large shock on the same day. They obtain a strong bilateral relationship between Italy and Germany and a weak bilateral contagion between the UK and Germany.

Finally, we apply the MO copula. Its parameter θ represents the upper tail dependence parameter. From Table 1 obtain that the MO model shows an higher tail dependence than those of the previous copula models. The tail dependence between the failed banks in Italy and Germany is medium-high (χ_u=0.55), the one between the UK and Germany is medium-low (χ_u=0.45).

The higher value of the upper tail dependence parameter in the MO copula could be due to include a singular part in the model to assign a non-null probability to the event that banks in two countries fail at the same time. In this way, we can accurately estimate the systematic component of systemic risk. On the contrary, in the Gumbel and in the mixture model this component could be underestimated, as the data show.

We explained in Section 3.2 the role of the copula parameter θ. The weights of idiosyncratic and systematic shocks are a function of θ as given by Eq. (3.5). If θ is very high (i.e. $\theta > 2/3$), the systematic component is more important than the idiosyncratic one to explain systemic risk. On the contrary if θ is equal to zero, the systemic risk is explained only by the idiosyncratic shocks. As Italian and German banks are under the same monetary policy of the European Central Bank, it is coherent that the systematic component for this pair of countries is more relevant than that for two banking systems with different monetary policies. Fig. 2 shows the estimated MO copula function and its contour levels for the couples IT-UK, UK-DE and IT-DE.

To identify the copula that best fits the data, we need to choose a criterion. As the models are non-nested, we use a modified version of the Akaike Information Criterion (AIC) associated with the PML (Chen & Fan, 2005; McNeil, Frey, & Embrechts, 2005), given by

$$AIC^* = 2k - 2l(\widehat{\theta}) + \frac{2k(k+1)}{n-k-1} \qquad (4.1)$$

where $l(\widehat{\theta})$ is the maximum of the log pseudo likelihood function, k is the number of estimated parameters, and n is the sample size. The last term in Eq. (4.1) is a correction for small sample bias (Breymann, Dias, & Embrechts, 2003). According to this criterion, the model with best fit is the one that minimises the AIC.

Grø nneberg and Hjort (2014) investigated the limitations of the AIC for copula model selection in semiparametric PML methods and they proposed the cross validation Copula Information Criteria (CIC) to overcome these drawbacks. However, Jordanger and Tjstheim (2014) compared the performance of the AIC and the CIC in a simulation study, obtaining minor differences between these two criteria and emphasising that the CIC is computational intensive. Given these results, we prefer to use the AIC instead of the CIC. Based on our knowledge, there is a lack of theoretical justification in the literature to use the AIC for comparing absolutely continuous and non-absolutely continuous copulae. As El-Bassiouny, Shahen, and Bouhawwash (2018), Hanagal and Sharma (2015) and Sarhan, Hamilton, Smith, and Kundu (2011) used the AIC for a copula function with a singular component, we calculate this criterion for the MO copula using the pseudo likelihood function (3.6) in Eq. (4.1). *We choose the MO copula based on the results in* Table 2 *and its characteristics described in Section 3.2.*

The results for a censored sampling are shown in Table 3. We obtain that the estimates of the copula parameter θ increase for all the three pairs of countries. This means that the systematic component becomes more important for all the pairs of countries when we consider the characteristics of all the sample. As θ is the upper tail dependence parameter, the most important result of this empirical analysis is that the intensity of the upper tail dependence increases if we consider a censored sampling. In other words, the contagion risk could be underestimated if we do not consider the characteristics of non-failed banks. Moreover, as the length of the bootstrap confident intervals in Table 3 decreases for the censored sample, the estimates of the copula parameter θ are more accurate.
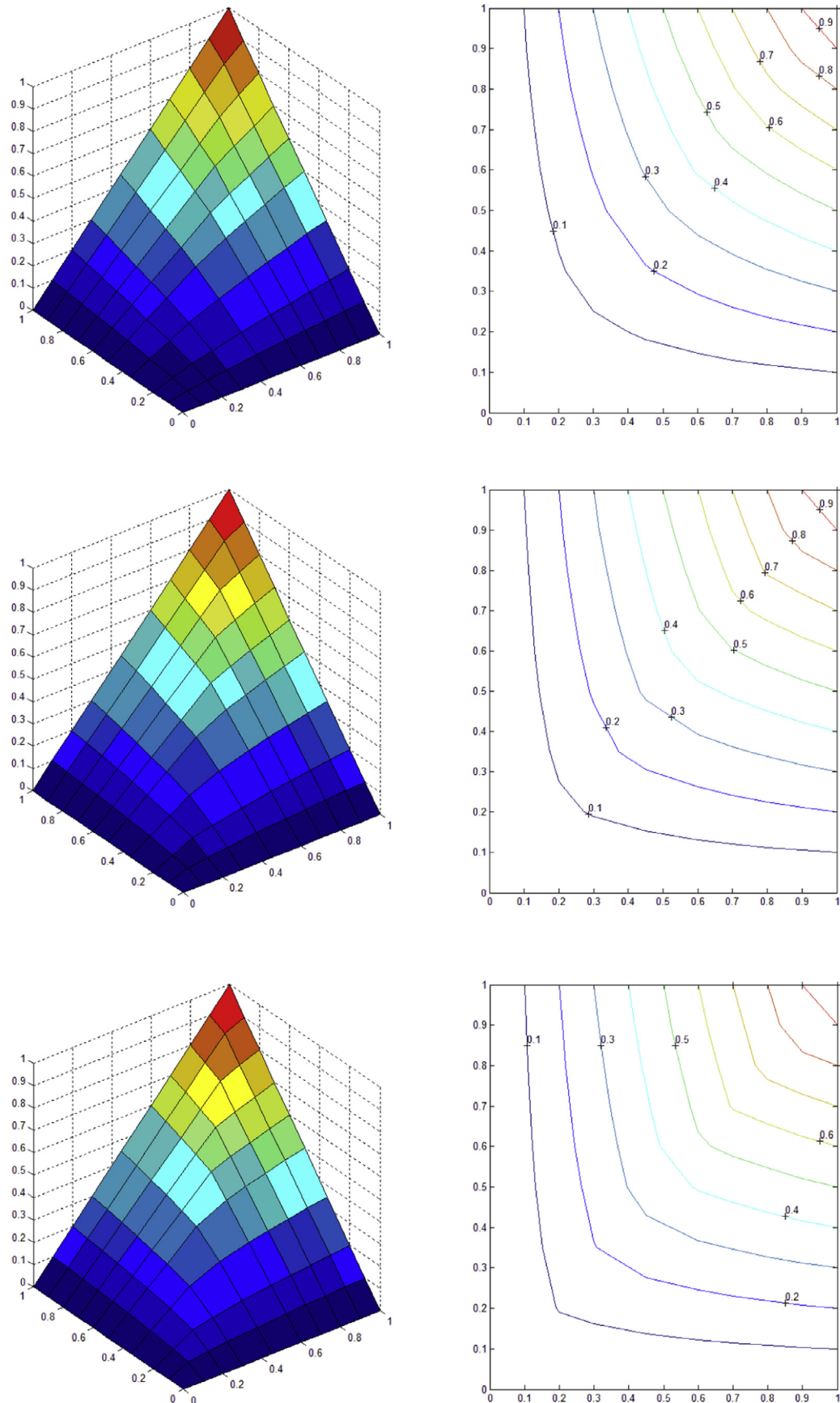
Fig. 2. The MO copula and the contour lines estimate for IT-UK (top), UK-DE (middle), and IT–DE (bottom).

Table 3
Copula parameter estimates and bootstrap confidence intervals for complete and censored sample.

	IT-UK	IT-DE	UK-DE
Complete sample	$\hat{\theta} = 0.37$	$\hat{\theta} = 0.55$	$\hat{\theta} = 0.45$
	(0.29; 0.52)	(0.33; 0.59)	(0.28; 0.54)
Censored sample	$\hat{\theta} = 0.50$	$\hat{\theta} = 0.83$	$\hat{\theta} = 0.76$
	(0.48; 0.51)	(0.82; 0.84)	(0.75; 0.77)

5. Conclusions

In this paper we propose a novel copula-based approach for modelling cross-border systemic risk. In particular, the MO copula is used to estimate the dependence between times to bank failures located in two different countries. The main advantage of this model is that the impact of the idiosyncratic and systematic components on the systemic risk can be measured. We highlight that the idiosyncratic component is represented by the continuous part of the copula, the systematic by the singular part. To include the information on non-failed banks in the estimation procedure, we consider a censoring mechanism. We propose a pseudo-maximum likelihood method to estimate the MO copula parameter for Type I censored samples.

Such a proposal is applied to data on three of the main banking systems in Europe (Germany, Italy and the UK). The first important result of this empirical analysis is that the MO copula is the copula that best fits the data according to the AIC measure. The second important result is that the impact of the systematic risk is higher if we consider a censored sample compared to that obtained for a complete sample (without a censoring technique). We hope that this work proposes a novel method that central banks can use to provide more accurate estimates of systemic risk.

This paper is focused on the analysis of cross-border systemic risk between two countries. From an empirical point of view, further work will extend the approach here proposed to analyse the systemic risk between more than two countries using a higher dimensional copula. Another further research from a methodological point of view is to provide the theoretical justification of using the AIC or its modification to compare absolutely continuous and non-absolutely continuous copula models.

Appendix A

A1. The estimator in Type I censored sampling

We suggest the maximum likelihood estimator (3.11) in the case of Type I censored sampling. We consider the observations as shown in Fig. 1, we apply the logit transformation $\theta = (1 + \exp(-\psi))^{-1}$ to the conditional log-likelihood function (3.10), so we obtain

$$l(\psi|\hat{u},\hat{v}) = k + (m_1 + m_2 + r + s)\ln[1 - (1 + \exp(-\psi))^{-1}]$$
$$+ m_3 \ln[(1 + \exp(-\psi))^{-1}]$$
$$- (1 - (1 + \exp(-\psi))^{-1})(S_1(t^*) + S_2(t^*))$$
$$- (1 + \exp(-\psi))^{-1}S_{\max}(t^*)$$

where k is a constant and

$$S_1(t^*) = \sum_{i=1}^{m+r}[-\ln(\hat{u}_i)] + (n - m - r)t^*,$$

$$S_2(t^*) = \sum_{i=1}^{m+s}[-\ln(\hat{v}_i)] + (n - m - s)t^*$$

and

$$S_{\max}(t^*) = \sum_{i=1}^{m}\max[-\ln(\hat{u}_i), -\ln(\hat{v}_i)] + rt^* + st^* + (n - m - r - s)t^*.$$

The previous equation can be simplified and it becomes

$$l(\psi|\hat{u},\hat{v}) = k + (m_1 + m_2 + r + s)(-\psi)$$
$$- (m + r + s)\ln[(1 + \exp(-\psi))]$$
$$- \frac{\exp(-\psi)}{(1 + \exp(-\psi))}(S_1(t^*) + S_2(t^*))$$
$$- (1 + \exp(-\psi))^{-1}S_{\max}(t^*)$$

By differentiating the log-likelihood function with respect to ψ, we obtain

$$\frac{\partial l(\psi|\hat{u},\hat{v})}{\partial \psi} = -(m_1 + m_2 + r + s) + (m + r + s)\frac{\exp(-\psi)}{(1 + \exp(-\psi))}$$
$$+ \frac{\exp(-\psi)}{(1 + \exp(-\psi))^2}[S_1(t^*) + S_2(t^*) - S_{max}(t^*)]$$

Setting $\frac{\partial l(\psi|\hat{u},\hat{v})}{\partial \psi} = 0$ we obtain

$$m_3 \exp(-2\psi) - (m + r + s - 2m_3 + S_{\min}(t^*))\exp(-\psi)$$
$$- (m + r + s - m_3) = 0,$$

where $S_{\min}(t^*) = S_1(t^*) + S_2(t^*) - S_{\max}(t^*)$.

By solving the previous equation with respect to $\exp(-\psi)$, we obtain two solutions

$$z_{1,2} = \frac{m + r + s - 2m_3 - S_{\min}(t^*) \pm \sqrt{(m + r + s - 2m_3 - S_{\min}(t^*))^2 + 4m_3(m + r + s - m_3)}}{2m_3}$$

Since only the solution

$$z_1 = \frac{m + r + s - 2m_3 - S_{\min}(t^*) + \sqrt{(m + r + s - 2m_3 - S_{min}(t^*))^2 + 4m_3(m + r + s - m_3)}}{2m_3}$$

has positive values, it is the unique accepted solution for $\exp(-\psi)$. Hence, the unique solution of the optimisation problem is

$$\hat{\psi}_c = -\ln(z_1) \qquad (A.1)$$

We obtain that the previous solution is a maximum from the sign of the second derivative. In (A.1) t^* is fixed and the number of failed banks in one or both countries (m, r and s) are random variables.

A2. Simulation study

In this section we perform a Monte Carlo simulation study to analyse the properties of the estimation procedures described in Sections 3.2 and 3.3 for finite samples. We generate 2000 samples with different sample size $n = 20, 50, 100, 500$ from a bivariate distribution with MO copula and two marginal exponential variables with parameter $\lambda = 2$. We consider only one marginal distribution function in the simulation studies as the copula parameter estimator is ranked-based, so it does not dependent on the marginal distribution. We choose different values of the copula parameter $\theta = 0.1, 0.7, 0.9$, corresponding to low, medium and
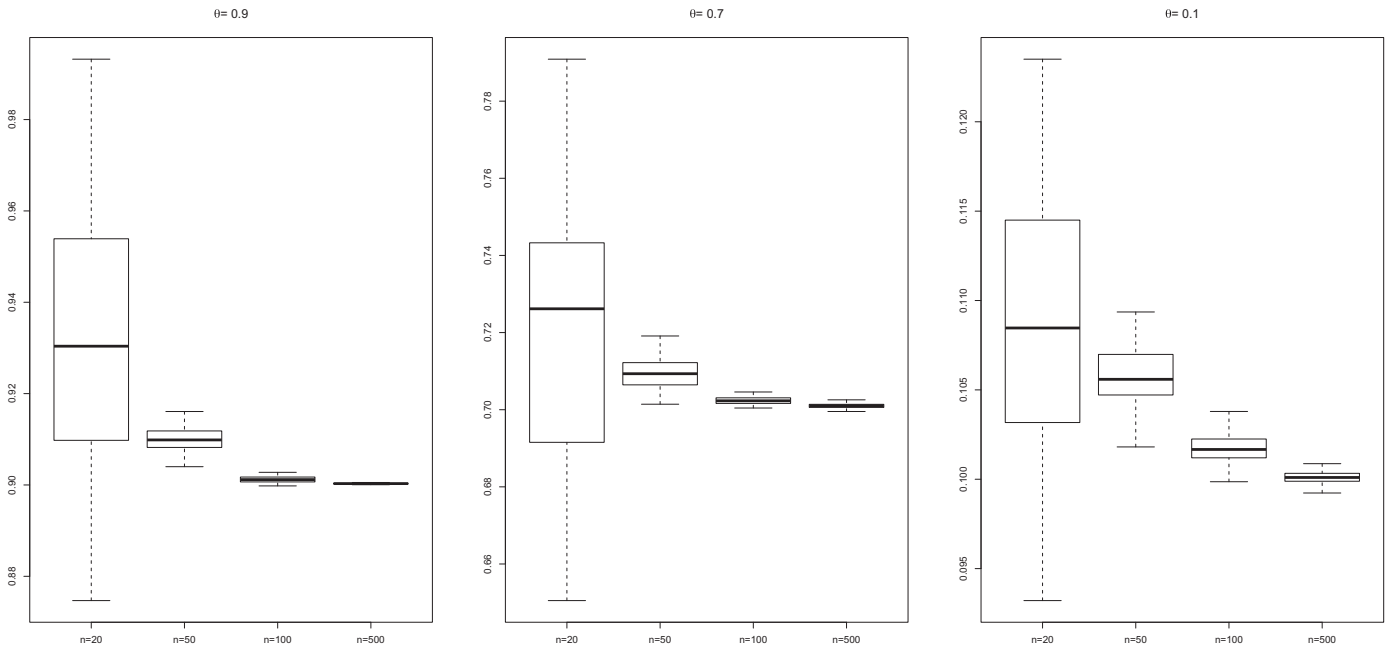
Fig. 3. Boxplot of the estimator distribution of the copula parameter θ for the complete sample with different values of n and θ.

Table 4
Bias and MSE of the copula parameter θ estimated using the pseudo maximum likelihood method for a complete sample.

$\hat{\theta}$	n	Bias	MSE	Time
0.9	20	0.0213	0.0360	2.467
	50	0.0102	0.0026	3.267
	100	0.0012	0.0006	3.761
	500	0.0003	0.0001	4.599
0.7	20	0.0114	0.0417	2.879
	50	0.0093	0.0037	3.403
	100	0.0024	0.0010	3.879
	500	0.0010	0.0006	4.991
0.1	20	0.0099	0.0095	2.956
	50	0.0056	0.0015	3.548
	100	0.0016	0.0009	3.954
	500	0.0001	0.0003	4.938

Table 5
Bias and MSE of the copula parameter θ estimated using the pseudo maximum likelihood method for a censored sampling.

$\hat{\theta}$	n	Bias	MSE	m
0.9	20	0.0221	0.0377	18
	50	0.0111	0.0029	42
	100	0.0009	0.0007	86
	500	0.0008	0.0003	400
0.7	20	0.0270	0.0515	18
	50	0.0099	0.0039	40
	100	0.0029	0.0019	78
	500	0.0008	0.0008	400
0.1	20	0.0168	0.0099	16
	50	0.0088	0.0019	39
	100	0.0025	0.0011	78
	500	0.0007	0.0008	380

high positive dependence. We analyse the bias (Bias) and the mean square error (MSE) of the parameter θ for the procedures proposed in Sections 3.2 and 3.3.

Table 4 reports the results for the complete sample using the PML estimation procedure explained in Section 3.2. The outcomes show that this technique is accurate in estimating the copula parameter as the bias and the MSE are usually lower than one tenth of the real value of the parameter even for a small sample size ($n = 20$). Moreover, the estimates are consistent as the bias and the MSE decrease when the sample size increases, for a given θ. The last column in Table 4 shows the time to end of study (Time).

Fig. 3 shows the boxplot of the estimator distribution of the copula parameter θ for different values of n and θ.

We also apply the estimation procedure for a censored sample described in Section 3.3. The results of the bias and the MSE for the copula parameter θ for I type censored sampling are reported in Table 5, where we choose $t^* = 2$ as time for censoring. Particularly, n is the size of the censored sample, with m observed units and $n - m$ not observed ones. Table 5 shows that both the bias and the MSE of the censored sample are higher than the corresponding

ones for a complete sample, for given θ and n, as the observation time $t^* = 2$ is lower than the time for the complete sample. Analogously to the results for a complete sample, the bias and the MSE decrease as the sample size n increase in Table 5.

Fig. 4 shows the boxplot of the estimator distribution for different values of n and different values of the true parameter θ.

Figs. 3 and 4 show that the estimation procedure slightly overestimates the parameter value for small sample size.

Finally, we generate 2000 random samples from a bivariate random variable with marginal exponential distributions of parameter $\lambda = 2$ and an exchangeable MO copula of parameter $\theta = 0.9$ and $\theta = 0.7$. Afterwards, we apply the estimation procedure described in Section 3.2 to a complete sample with m observations and the approach described in Section 3.3 to a censored sample with m observed units and $n - m$ not observed units.

We report the MSE of these two methods in Table 6. The MSE in the censored sample is lower than that in the complete one. Therefore, if we consider also the characteristics of non-observed units in the sample, the estimate of the dependence becomes more accuracy.
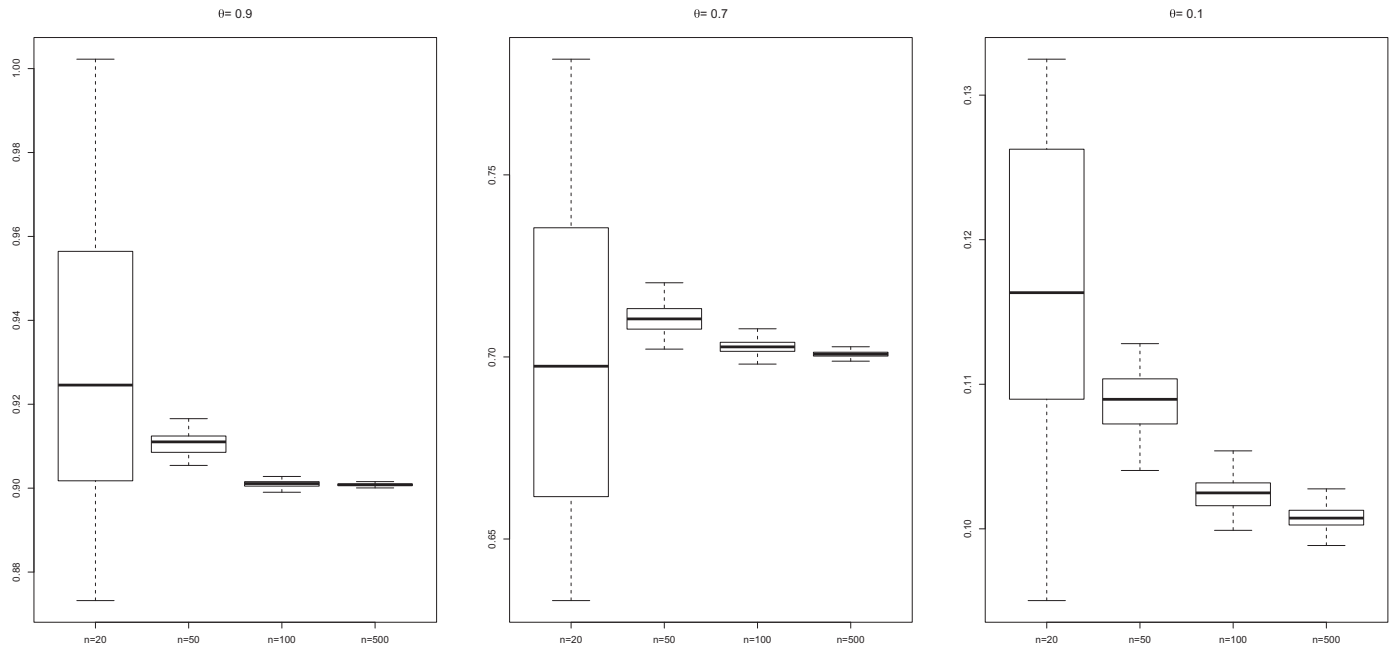
Fig. 4. Boxplot of the estimator distribution of the copula parameter θ for the censored sample with different values of n and θ.

Table 6
Complete sample vs. censored sample.

$\hat{\theta}$	m	$n-m$	MSE	
			Complete sample	Censored sample
0.9	30	20	0.0261	0.0057
		70	0.0261	0.0026
		470	0.0261	0.0008
	50	50	0.0026	0.0016
		450	0.0026	0.0006
0.7	30	20	0.0233	0.0055
		70	0.0233	0.0034
		470	0.0233	0.0009
	50	50	0.0037	0.0028
		450	0.0037	0.0008
0.1	30	20	0.0054	0.0024
		70	0.0054	0.0019
		470	0.0054	0.0009
	50	50	0.0015	0.0012
		450	0.0015	0.0008

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