IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received September 30, 2019, accepted October 8, 2019, date of publication October 22, 2019, date of current version October 31, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2948847

Constructing and Managing Multi-Granular Linguistic Values Based on Linguistic Terms and Their Fuzzy Sets

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This work was supported in part by the National Natural Science Foundation of China under Grant 61372187, and in part by the Scientific and Technological Project of Sichuan Province under Grant 2019YFG0100.

ABSTRACT Constructing and managing multi-granular linguistic values are more and more important for linguistic decision making in big data or social computing environments, linguistic variable is the fundamental of constructing and managing multi-granular linguistic values. Based on analysis of linguistic values and drawbacks of symbolic or fuzzy set methods in processing linguistic information, a linguistic value is expressed by a formal linguistic concept, which is constructed by a linguistic term and it's fuzzy sets, *i.e.*, intension (name) and extension (meaning) of the concept are a linguistic term and it's fuzzy sets. A new symbolic translation based on fuzzy sets is provided to obtain formal 2-tuple linguistic concepts, which are continuous formal linguistic concepts. By using linguistic hedges, the hierarchy of multi-granular formal linguistic concepts is constructed, and managing multi-granular linguistic values is carried out by a new transformation function between formal linguistic concepts of the hierarchy. Cases study shows that the proposed method combines advantages of symbolic approaches and fuzzy set methods in linguistic information processing and overcomes their drawbacks due to fuzzy sets and linguistic term as entity in linguistic information processing based on formal linguistic concepts, intensions are utilized to deal with linguistic information and extensions are used to represent meanings and obtain natural or artificial language concepts. It seems that constructing and managing multi-granular linguistic values via formal linguistic concepts is an useful and alternative method in linguistic information processing.

INDEX TERMS Linguistic variable, linguistic hedge, 2-tuple linguistic term, multi-granular linguistic values, linguistic decision making.

I. INTRODUCTION

The concept of linguistic variable plays a pivotal role in all applications of fuzzy logic, especially in computing with words or linguistic information processing [3]–[6]. Formally, linguistic variable is defined as [7]: A linguistic variable is characterized by a quintuple (L, H, U, G, M), in which L is the name of the variable; H denotes the term set of L, *i.e.*, the set of names of linguistic values of L, with each value being a fuzzy variable denoted generically by X and ranging across a universe of discourse U which is associated with the base variable u; G is a syntactic rule (which usually takes the form of a grammar) for generating the names of values of L; and M is a semantic rule for associating its meaning with each L, $M(X)$, which is a fuzzy subset of U. For example, *height* is a linguistic variable defined on the universe $(0, 2.5m]$ and *high* is a linguistic value of *height*, the trapezoidal fuzzy set $\mu_{high}(u) = (1.7, 1.9, 2.5, 2.5)$ on $(0, 2.5m]$ can be a semantic value or meaning of *high*. In practical applications, *high* can be utilized to express qualitative knowledge "Europeans are *high*" and meaning of *high* can be represented by $\mu_{high}(u)$, due to calculable character of fuzzy sets, linguistic knowledge "Europeans are *high*" can be further processed by using $\mu_{high}(u)$ in a knowledge system. All the time, fuzzy sets as meanings of linguistic values have been successfully applied to represent and handle imprecise or uncertain qualitative and

The associate editor coordinating the review of this manuscript and approving it for publication was Omar Khadeer Hussain.

quantitative knowledge in intelligent systems [8]–[13]. After 1996 year, Zadeh emphasized computing with words or linguistic information processing [6] based on linguistic variables, in which information is represented by names of linguistic values as well as fuzzy sets, more important, names of linguistic values are persisted in the process and result of information processing, in other words, each fuzzy set in information processing is described by name of linguistic value. Up to now, many researchers have studied computing with words or linguistic information processing methods, which roughly be categorized as follows:

A. VIA FUZZY SETS OF LINGUISTIC VALUES

In the category, meanings of linguistic values are represented by fuzzy sets on the universe U based on parameters and a semantic rule. Names of linguistic values are generated by a context-free grammar $G = (V_N, V_T, I, P)$, where V_N is the set of non-terminals, V_T is the set of terminals, I is the starting symbol and P the production rules, for example, let V_N be consisted by primary names of linguistic values {*low*, *medium*, *high*}, V_T be linguistic hedges {*slightly*, *more or less*, *rather*, *much*, *very*,$\cdots$, }, relations {*higher than*, *lower than*, *at least*, *at most*, *between*, $\cdots$, } or connectives {*but*, *and*, *or*}, then names of linguistic values, such as {*slightly high*, *more or less high*, *much high*, *very high*, *very very high*, *at least high*, *medium or high*, $\cdots$, }, can be generated by using the context-free grammar G. Theoretically, meanings of these linguistic values are represented by fuzzy sets on the universe U, then operations on fuzzy sets are used to deal with information described by names of linguistic values. By using approximation method, fuzzy set results are once again described by names of linguistic values [3]–[7], [14]–[18];

B. VIA FUZZY LOGIC OR ALGEBRA ON THE SET OF LINGUISTIC VALUES

In the category, names of linguistic values are embedded in fuzzy natural logic or algebraic system, then linguistic information are handled by logic inference or algebraic calculus. Theoretically, fuzzy natural logic is to develop a mathematical model of human reasoning whose typical feature is the use of natural or artificial language. Novak et al have proposed perception-based logic deduction to describe and process evaluative linguistic expressions [19]–[22]. By using linguistic hedges, Ho et al have presented hedge algebra to represent and reason linguistic human knowledge with particular truth values of vague sentences [23]–[27]. Pei et al provided linguistic formal concept lattice to analyze the relation and hierarchical structure of linguistic values [12];

C. VIA AN ORDERED STRUCTURE OF LINGUISTIC VALUES

In the category, any finite primary names of linguistic values, also called as linguistic terms, are embedded in a natural ordering, which roots in meanings of natural languages, such as *high*> *medium* > *low* in natural language because *high* possesses a meaning greater (or stronger) than *medium*, *medium* possesses a meaning greater (or stronger) than *low*. By this way, linguistic terms can be identified by natural numbers serving as their inferior indexes, such as the set of linguistic terms $H_g = \{s_0, s_1, \cdots, s_g\}$ with a linear order: $s_i \geq s_j$ if and only if $i \geq j$. The methods are also called as symbolic approaches or linguistic symbolic computational models [13], in which the 2-tuple fuzzy linguistic representation model [28] has been proposed as the main symbolic approach and widely applied in linguistic decision making [29]–[36]. In addition, linguistic hierarchy can be built on several sets of linguistic terms, which together with 2-tuple fuzzy linguistic representation model can be utilized to construct and manage multi-granular linguistic terms and apply in multi-granular linguistic decision making [37]–[52].

It can be noticed from a linguistic variable (L, H, U, G, M) that a linguistic value is characterized by a syntactic value (name or linguistic term) and a semantic value (a fuzzy set on the universe U), the linguistic term is utilized to describe the fuzzy set, conversely, the fuzzy set is used to represent meaning of the linguistic term because it is fuzziness or unsharp boundaries on U. In real world practices, if there is no fuzzy set, then meaning of linguistic term can not be represented. If there is no linguistic term, then fuzzy set can not be described. For example, in ''Europeans are *high*'', if there is no fuzzy set $\mu_{high}(u) = (1.7, 1.9, 2.5, 2.5)$ on $(0, 2.5m]$, then meaning of *high* can not be represented and one can not understand ''how height is *high*''. Conversely, if there is no linguistic term *high*, $\mu_{high}(u)$ can not be described and one can not understand ''what is $\mu_{high}(u)$'' by natural language. Accordingly, linguistic terms as well as fuzzy sets on U are necessary for linguistic values in computing with words or linguistic information process. In *Via fuzzy sets of linguistic values*, linguistic information process is concentrated on fuzzy sets on the universe U, it's drawbacks are computational complexity, a lack of accuracy, loss information and difficult understanding [13], [45], [48], [53], [54]. In *Via fuzzy logic or algebra on the set of linguistic values* and *Via an ordered structure of linguistic values*, fuzzy sets are unnecessary, linear ordered structure or algebraic system of linguistic terms are emphasized in linguistic information process, it's advantages are symbolic linguistic information, logic or algebraic calculus to deal with linguistic information, and no loss linguistic information. However, symbolic linguistic terms are man-made language, linguistic results are beyond comprehension because fuzzy sets are not utilized to represent meaning of them. It seems that linguistic term and fuzzy set of a linguistic value is entity, only using one of them maybe lead to drawbacks in computing with words or linguistic information process.

Recently, inspired by large-scale decision making problems in big data or social computing, constructing and managing multi-granular linguistic values become hot in linguistic information process [39]–[49]. In fact, constructing and managing multi-granular linguistic values are associated with two important researches: granular computing and multi-granular fuzzy linguistic model. On the one hand, granular computing is concerned with the development and processing

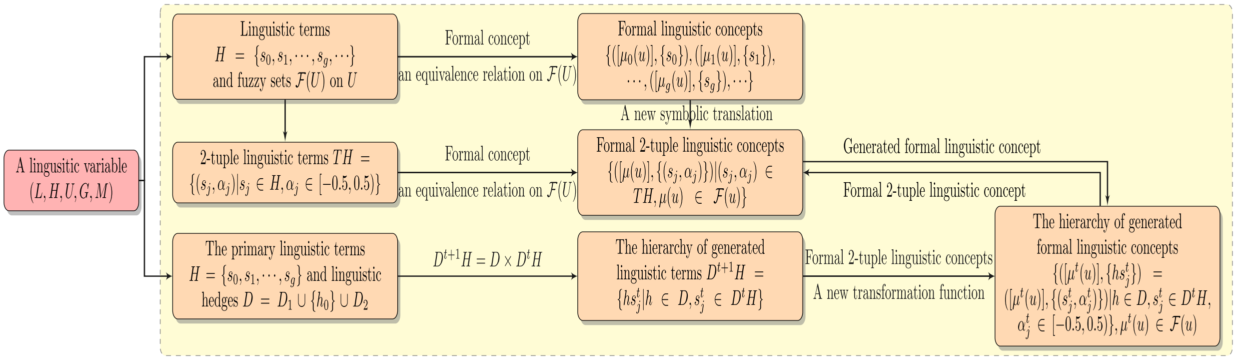

FIGURE 1. Constructing and managing multi-granular linguistic values via formal linguistic concepts.

information granules, which are formal entities and facilitate a way of organizing knowledge about the available data and relationships existing there. From the conceptual point of view, information granules are conceptually sound knowledge over which various models could be developed and utilized [55]. On the other hand, in real world practices, the same problem or system can be perceived at different levels of specificity (detail) depending on the complexity of the problem, available computing resources, and particular needs to be addressed, in such cases, using multi-granular linguistic values generated by syntactic rule G with different semantics becomes essential [56]–[58], for example ones also use *slightly high*, *more or less high*, *very high* or *very very high* to describe different precision of *high*. In many environments, multi-granular linguistic values help information providers with different knowledge levels and needs to grasp hierarchy and size of information granules.

In this paper, formal linguistic concepts are proposed to construct and manage multi-granular linguistic values in a linguistic variable (L, H, U, G, M), where each linguistic value is a formal linguistic concept, it's extension and intension are fuzzy sets and linguistic term of the linguistic value, respectively, then with the help of 2-tuple fuzzy linguistic representation model and linguistic hedges, constructing and managing multi-granular linguistic values are carried out via formal linguistic concepts. Major contributions of the paper (shown in Fig.(1)) are summarized as follows:

1) Due to different knowledge level, background or experience, ones provide different fuzzy sets to represent meaning of the same linguistic term, this means that the relation between meaning and name of a linguistic value is many-to-one in practical applications. In the paper, the centroid of uncertainty and certainty of fuzzy set is utilized to define an equivalence relation between two fuzzy sets on U. Then each linguistic value is explained by a formal linguistic concept, an equivalence class of fuzzy sets and linguistic term are it's extension and intension instead of meaning and name of the linguistic value, respectively. Intuitively, the linguistic term describes fuzzy sets with the same uncertainty and certainty (an equivalence class on fuzzy sets) and fuzzy sets with the same uncertainty and certainty represent meaning of the linguistic term, which is provided by different people;

2) Inspired by 2-tuple fuzzy linguistic representation model, a new symbolic translation based on the centroid of uncertainty and certainty of fuzzy set is presented, then formal 2-tuple linguistic concepts are proposed, intuitively, formal 2-tuple linguistic concept means that any fuzzy set on the universe U can be described by a 2-tuple linguistic term and meaning of any 2-tuple linguistic term can be represented by an equivalence class on fuzzy sets;

3) With the help of linguistic hedges, a new linguistic hierarchy of generated linguistic terms is constructed, by defining a new transformation function between two levels of the linguistic hierarchy, a hierarchy of generated multi-granular formal linguistic concepts is constructed. Then managing multi-granular linguistic values can be carried out by using formal 2-tuple linguistic concepts and the hierarchy of generated multi-granular formal linguistic concepts.

The rest of the paper is structured as follows: In Section II, 2-tuple linguistic term and linguistic hierarchy are reviewed. In Section III, an equivalence relation between fuzzy sets on U is defined by the centroid of uncertainty and certainty of fuzzy set, then the formal linguistic concept is presented. In Section IV, a new symbolic translation based on the centroid of uncertainty and certainty of fuzzy set is presented and formal 2-tuple linguistic concept is provided. In Section V, a new hierarchical structure of multi-granular linguistic terms is constructed and a new transformation function between two levels is defined, then the hierarchy of generated multi-granular formal linguistic concepts is constructed. In Section VI, cases study are utilized to show formal linguistic concepts in constructing and managing multi-granular linguistic values. The conclude of the paper is in Section VII.

II. PRELIMINARIES

In the section, we briefly review 2-tuple linguistic term and hierarchical structure of linguistic terms.

A. THE 2-TUPLE FUZZY LINGUISTIC REPRESENTATION MODEL

The 2-tuple fuzzy linguistic representation model [28] is an important tool of computing with words or linguistic information process, where the model is concentrated on processing 2-tuple linguistic terms instead of fuzzy sets. Formally, 2-tuple fuzzy linguistic representation model represents linguistic information by means of a 2-tuple (s_j, α), where s_j is a linguistic term and α is a numerical value that represents the value of the symbolic translation.

Definition 1 ([28]): Let $\beta \in [0, g]$ be the result of an aggregation of the indices of a set of linguistic terms assessed in an primary linguistic term set $H_g = \{s_0, s_1, \cdots, s_g\}$, *i.e.*, the result of a symbolic aggregation operation. Let j =round(β) and $\alpha = \beta - j$ be two values such that $j \in [0, g]$ and $\alpha \in [-0.5, 0.5)$, then α is called a symbolic translation. Intuitively, the symbolic translation α of linguistic term s_j supports "the difference of information" between a counting of information β obtained from a symbolic aggregation operator and the closest linguistic term $s_j \in H_g(j = round(\beta))$. Theoretically, the 2-tuple fuzzy linguistic representation model provides transformation from numerical values of $[0, g]$ to 2-tuple linguistic terms on the primary linguistic term set H_g:

$$\Delta : [0, g] \longrightarrow H_g \times [-0.5, 0.5), \beta \mapsto \Delta(\beta) = (s_j, \alpha). \quad (1)$$

where $j = round(\beta)$, $\alpha = \beta - j \in [-0.5, 0.5)$ and $round(\cdot)$ is the usual rounding operation, $s_j \in H_g$ is the linguistic term that is mostly close to β and α represents the symbolic translation value. Δ is an one-to-one mapping, its inverse function transforms 2-tuple linguistic terms to its equivalent numerical values, *i.e.*,

$$\begin{aligned}\Delta^{-1} &: H_g \times [-0.5, 0.5) \longrightarrow [0, g],\\ &(s_j, \alpha) \longmapsto \Delta^{-1}(s_j, \alpha) = \beta = j + \alpha \in [0, g]. \quad (2)\end{aligned}$$

In linguistic information processing, 2-tuple linguistic terms $TH = \{(s_j, \alpha) | s_j \in H_g, \alpha \in [-0.5, 0.5)\}$ own many advantages, such as continuous linguistic domain, easy computation, without loss of information and so on [13], [28], [38]. For example, suppose the primary linguistic term set $H_2 = \{s_0\ (low), s_1\ (medium), s_2\ (high)\}$, then 2-tuple linguistic term $(s_2, -0.3)$ is shown in Fig.2.

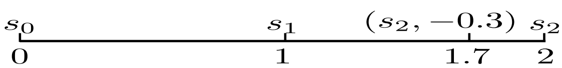

FIGURE 2. 2-tuple linguistic term $(s_2, -0.3)$ on $H_2 = \{s_0, s_1, s_2\}$.

B. LINGUISTIC HIERARCHY

Linguistic hierarchies arise quite naturally in problems for which one needs to deal with multiple sources of linguistic information, such as in the context of linguistic decision analysis, a linguistic hierarchy is necessary when linguistic assessments are assessed in linguistic term sets with different granularity of uncertainty or semantics.

1) BASED ON 2-TUPLE FUZZY LINGUISTIC REPRESENTATION MODEL

Based on 2-tuple linguistic terms, linguistic hierarchical structure has been discussed for aggregating multi-granular linguistic information [38]: A linguistic hierarchy is a set of levels, where each level is a linguistic term set with different granularity to the rest of levels of the hierarchy. Denote each level of a linguistic hierarchy as $l(t, n(t))$, where t is the level of the hierarchy and $n(t)$ is the granularity of the linguistic term set of the level t. The levels of the linguistic hierarchy are ordered according to their granularity, *i.e.*, for any t, $n(t+1) > n(t)$, this provides a linguistic refinement of the previous level. To build a linguistic hierarchy, the linguistic term set of level is obtained from its predecessor as $l(t, n(t)) \longrightarrow l(t+1, 2n(t) - 1)$. In practical applications, particularly in multi-expert decision-making problems, a problem may be defined over a multi-granular linguistic context where linguistic term sets from different linguistic hierarchies are utilized to represent assessments provided by different decision makers, to avoid the loss of information produced in the normalization process, transformation functions among linguistic term sets of the linguistic hierarchy have been provided, *i.e.*, for levels t and t' and 2-tuple linguistic terms (s_j^t, α_t) on $l(t, n(t)) = \{s_0^t, s_1^t, \cdots, s_{n(t)-1}^t\}$, $TF_{t'}^t : l(t, n(t)) \longrightarrow l(t', n(t'))$ such that

$$TF_{t'}^t(s_j^t, \alpha_t) = \Delta(\frac{\Delta^{-1}(s_j^t, \alpha_t)(n(t') - 1)}{n(t) - 1}).$$

For example, suppose $l(t, n(t)) = l(1, 3) = \{s_0^1, s_1^1, s_2^1\}$, then $l(t+1, 2n(t) - 1) = l(2, 5) = \{s_0^2, s_1^2, s_2^2, s_3^2, s_4^2\}$. For $(s_2^1, -0.3)$ on $l(1, 3)$,

$$\begin{aligned}TF_2^1(s_2^1, -0.3) &= \Delta(\frac{\Delta^{-1}(s_2^1, -0.3)(5-1)}{3-1})\\ &= \Delta(\frac{1.7 \times 4}{2}) = \Delta(3.4) = (s_3^2, 0.4).\end{aligned}$$

2) BASED ON THE ORDERED STRUCTURE OF LINGUISTIC TERMS

A linguistic hierarchy HL of a linguistic variable L is a hierarchical tree consisting of a finite number of levels, the t-th linguistic hierarchy is denoted by $L_t (t = 0, 1, \cdots, T)$ and defined as follows [48]:

- Level L_0 is the root of the tree labeled by the name of the linguistic variable, *i.e.*, $L_0 = L$.
- Each level $L_t = \{s_0^t, s_1^t, \cdots, s_{g(t)}^t\} (t = 1, \cdots, T)$ is a finitely linguistic term set accompanied with a total order such that:
  - $|L_t| = g(t) + 1 < |L_{t+1}| = g(t+1) + 1$ for any $t = 1, 2, \cdots, T-1$;
  - For each $t = 1, 2, \cdots, T-1$, there exists only one mapping $\Gamma_t : L_t \longrightarrow 2^{L_{t+1}} - \{\emptyset\}$ such that for any $s_j^t \neq s_{j'}^t \in L_t$, $\Gamma_t(s_j^t) \cap \Gamma_t(s_{j'}^t) = \emptyset$ and $\cup_{s_j^t \in L_t} \Gamma_t(s_j^t) = L_{t+1}$;
  - If $s_j^t \neq s_{j'}^t$ in L_t, then $s_i^{t+1} \neq s_{i'}^{t+1}$ in L_{t+1} for any $s_i^{t+1} \in \Gamma_t(s_j^t)$ and $s_{i'}^{t+1} \in \Gamma_t(s_{j'}^t)$.

Linguistic terms in the hierarchical tree are $HL = \bigcup_{t=1}^T L_t$ and each $s_j^t \in L_t$ is refined by linguistic terms $\Gamma_t(s_j^t)$ of L_{t+1}. For the mapping Γ_t, there exists a pseudo-inversion

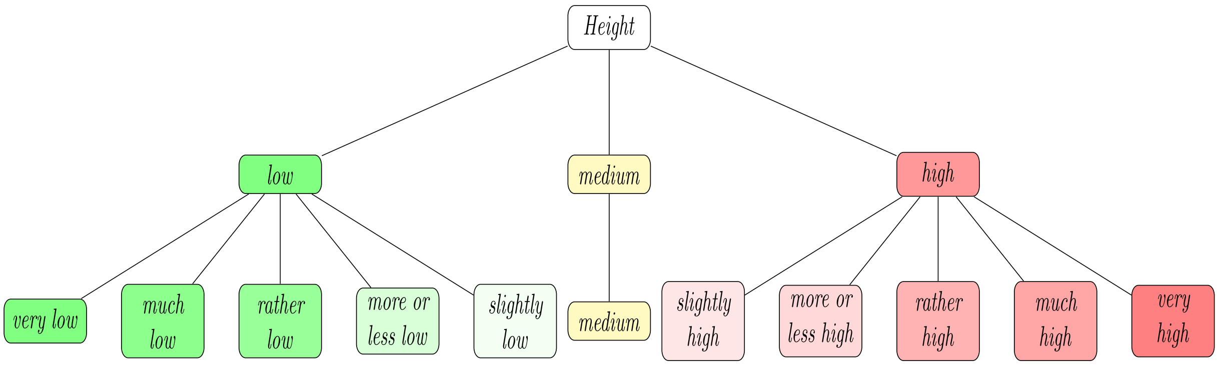

FIGURE 3. The hierarchical tree of *height*.

$\Gamma_t^- : L_{t+1} \longrightarrow L_t$ such that $\Gamma_t^-(s_i^{t+1}) = s_j^t$ if $s_i^{t+1} \in \Gamma_t(s_j^t)$. For example, Fig.(3) shows hierarchical tree of *height*, where

$$\begin{aligned} L_1 &= \{low, medium, high\}, \\ L_2 &= \{slightly\ low, more\ or\ less\ low, low, rather\ low, \\ &\quad much\ low, medium, slightly\ high, more\ or\ less\ high, \\ &\quad rather\ high, much\ high, very\ high\}. \end{aligned}$$

III. FORMAL LINGUISTIC CONCEPT OF LINGUISTIC VALUE

From the concept point of view, each linguistic value of a linguistic variable (L, H, U, G, M) is entity, which is consisted by fuzzy set and linguistic term, fuzzy set represents meaning of linguistic term and linguistic term as natural or artificial language describes fuzzy set, this is same with extension and intension of a formal concept, where extension is consisted by objects to understand intension and intension is the general character of objects and utilized to describe objects. In the section, an equivalence relation on fuzzy sets is analyzed, then formal linguistic concept is presented to represent linguistic value of (L, H, U, G, M).

A. AN EQUIVALENCE RELATION ON FUZZY SETS

In each linguistic value, general character described by linguistic term is fuzziness or un-sharp boundaries on U, due to different knowledge level, background or experience, there exists difference for different people to comprehend fuzziness or un-sharp boundaries on U, this means that different people provide different fuzzy set on U to understand the same linguistic term in real world practices. Hence an equivalence relation on fuzzy sets is necessary, the relation is used to evaluate which fuzzy sets on U can be described by the same linguistic term. To this end, uncertainty and certainty of a fuzzy set on the universe $U \subseteq \mathbb{R}$ are analyzed, then an equivalence relation on fuzzy sets is provided in the subsection, where concepts and notations concerning with fuzzy set on U are referred to [59]–[61].

In the paper, fuzzy set $\mu(u)$ described by a linguistic term s_i is such that it's all γ-cuts ($\gamma \in [0, 1]$) are a nest of nonempty closed intervals on U, *i.e.*, the γ-cut of $\mu(u)$ is $\{u|\mu(u) \geq \gamma\}$ $= [(\mu)_\gamma^l, (\mu)_\gamma^r] \subset U$ and $[(\mu)_{\gamma_1}^l, (\mu)_{\gamma_1}^r] \subseteq [(\mu)_{\gamma_2}^l, (\mu)_{\gamma_2}^r]$ if $\gamma_1 \geq \gamma_2$, in which, 1-cut of $\mu(u)$ is also said to be the kernel of $\mu(u)$, *i.e.*, $ker(\mu) = \{u|\mu(u) = 1\} = [(\mu)_1^l, (\mu)_1^r]$, denote all of these fuzzy sets as $\mathcal{F}(U)$. By membership degrees of a fuzzy set, the γ-cut of $\mu(u)$ can also be interpreted as in γ credible level, $[(\mu)_\gamma^l, (\mu)_\gamma^r]$ can be described by the linguistic term s_i, especially, the kernel $ker(\mu)$ is absolutely described by the linguistic term s_i. In other words, γ-cut ($\gamma \in [0, 1)$) of $\mu(u)$ are uncertain information described by the linguistic term s_i with γ credible degree, and the kernel $ker(\mu(u))$ is certain information described by the linguistic term s_i with 1 credible degree or absolute credibility. With the help of the centroid of the fuzzy set $\mu(u)$, uncertainty of $\mu(u)$ described by the linguistic term s_i can be further evaluated by the centroid of $\mu(u)$, *i.e.*, $U(\mu) = (x, y)$, x and y are computed by [62]

$$x = \frac{\int_{(\mu)_0^l}^{(\mu)_0^r} u\mu(u)du}{\int_{(\mu)_0^l}^{(\mu)_0^r} \mu(u)du}, \tag{3}$$

$$y = \frac{\int_0^1 \gamma((\mu^-(\gamma))^r - (\mu^-(\gamma))^l)d\gamma}{\int_0^1((\mu^-(\gamma))^r - (\mu^-(\gamma))^l)d\gamma}, \tag{4}$$

in which $\mu^-(\gamma)$ is the inverse function of $\mu(u)$ when $(\mu)^l(u)$ and $(\mu)^r(u)$ of $\mu(u)$ are both strictly monotone and continuous function. Intuitively, $U(\mu) = (x, y)$ interprets that x is the center of uncertainty described by the linguistic term s_i with y credible degree. Similarly, certainty of $\mu(u)$ described by the linguistic term s_i can be further evaluated by the centroid of $ker(\mu)$, *i.e.*, $C(\mu) = (z, 1)$, z is computed by

$$z = \frac{(\mu)_1^l + (\mu)_1^r}{2}, \tag{5}$$

$C(\mu) = (z, 1)$ is that z is the center of certainty absolutely described by the linguistic term s_i. Denote centers of uncertainty and certainty of $\mu(u)$ described by s_i as

$$(U(\mu), C(\mu)) = ((x, y), (z, 1)). \tag{6}$$

Accordingly, fuzziness or un-sharp boundaries on U described by the linguistic term s_i can also be interpreted by fuzzy sets with the same $(U(\mu), C(\mu))$. In real world practices, credible degrees y and 1 of x and z can be explained by weights of x and z, hence fuzziness or un-sharp boundaries on U described by s_i can be further simplified by

$$E(\mu) = \frac{yx}{1+y} + \frac{z}{1+y}. \tag{7}$$

Intuitively, $E(\mu)$ is the weighted mean of x and z such that $\frac{y}{1+y} + \frac{1}{1+y} = 1$ and $min\{x, z\} \leq AE(\mu) = \frac{yx}{1+y} + \frac{z}{1+y} \leq max\{x, z\}$, respectively. From the algebra point of view, $E(\mu)$ of fuzzy set $\mu(u)$ can be used to construct a relation between two fuzzy sets on the universe U, *i.e.*, for any fuzzy sets $\mu(u)$ and $\mu'(u)$ on U, $\mu(u)$ and $\mu'(u)$ have the relation $\equiv$ if and only if $E(\mu) = E(\mu')$,

$$\mu(u) \equiv \mu'(u) \Longleftrightarrow E(\mu) = E(\mu'). \tag{8}$$

It can be easily proved that the relation $\equiv$ is an equivalence relation between two fuzzy sets on U, *i.e.*, the relation satisfies reflexive property, symmetry and transitivity. Accordingly, a partition of $\mathcal{F}(U)$ on U can be obtained, *i.e.*,

$$\mathcal{F}(U)/\equiv = \{[\mu(u)] | u \in U, \mu(u) \in \mathcal{F}(U)\} \tag{9}$$

such that $\bigcup_{\mu(u)\in\mathcal{F}(U)}[\mu(u)] = \mathcal{F}(U)$ and $[\mu(u)] \cap [\mu'(u)] = \emptyset$ if $E(\mu) \neq E(\mu')$. In real world practices, each equivalence class $[\mu(u)]$ can be interpreted as fuzzy sets of $[\mu(u)]$ are with the same fuzziness or un-sharp boundaries on U and described by the same linguistic term s_i.

Example 1: For "height" on $(0, 2.5m]$, linguistic term "high" describes trapezoidal fuzzy set $\mu_{high}(u) = (1.7, 1.9, 2.5, 2.5)$, by using Eqs.(3)-(6), the center of uncertainty and certainty of $\mu_{high}(u)$ is $(U(\mu_{high}), C(\mu_{high})) = ((2.2, 0.48), (2.2, 1))$ and $E(\mu_{high}) = 2.2$. Fig.(4) shows several triangular or trapezoidal fuzzy sets of $[\mu_{high}(u)]$.

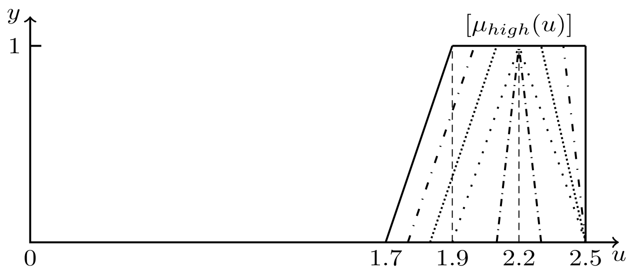

FIGURE 4. Several triangular or trapezoidal fuzzy sets of $[\mu_{high}(u)]$.

B. THE FORMAL LINGUISTIC CONCEPT

Based on the equivalence relation $\equiv$, each linguistic value of a linguistic variable (L, H, U, G, M) can be formalized as a formal linguistic concept, *i.e.*, the formal linguistic concept consists of fuzzy sets and linguistic term of the linguistic value, where fuzzy sets compose an equivalence class on $\mathcal{F}(U)/\equiv$ and represent meaning of linguistic term, which is also considered as different one provides different fuzzy set to represent meaning of the same linguistic term in practical applications. Linguistic term describe the general character of fuzzy sets, *i.e.*, fuzziness or un-sharp boundaries on U.

Definition 2: In a linguistic variable (L, H, U, G, M), each linguistic value is a formal linguistic concept, which has the form $([\mu(u)], \{s_i\})$, where $[\mu(u)] \in \mathcal{F}(U)/\equiv$ is extension and $s_i \in H$ is intension of the linguistic value.

According to formal linguistic concept $([\mu(u)], \{s_i\})$ of Definition 2, the existed linguistic information processing methods via fuzzy sets of linguistic values can be regarded as the methods based on extensions of linguistic values, the methods via fuzzy logic (algebra) or ordered structure of linguistic values can be regarded as the methods based on intensions of linguistic values. Due to many-to-one correspondence between extension and intension, the methods based on extensions of linguistic values cause computational complexity, lack accuracy, loss information or difficult linguistic description. The methods based on intensions of linguistic values cause fuzziness or un-sharp boundaries on U (or meaning of linguistic term) beyond comprehension.

Based on formal linguistic concepts, drawbacks of methods based on extensions or intensions of linguistic values can be overcome, because $([\mu(u)], \{s_i\})$ is entity at any time, s_i describes $[\mu(u)]$ and $[\mu(u)]$ represents meaning of s_i. Generally, in linguistic information process, especially in linguistic decision making, primary linguistic terms $H = \{s_0, s_1, \cdots, s_g\}$ with their fuzzy sets on the universe U are always provided, such as triangular or trapezoidal fuzzy sets, which are mostly utilized in the existed linguistic information processing methods [3]–[5], [8]–[10], [59]–[61]. According to Definition 2, the following definition can be obtained.

Definition 3: For primary linguistic term set $H = \{s_0, \cdots, s_g\}$ with their triangular or trapezoidal fuzzy sets $\{\mu_0(u), \cdots, \mu_g(u)\}$ on U, $H = \{([\mu_0(u)], \{s_0\}), ([\mu_1(u)], \{s_1\}), \cdots, ([\mu_g(u)], \{s_g\})\}$ is called as primary formal linguistic concepts, triangular or trapezoidal fuzzy set $\mu_j(u)$ $(j = 0, 1, \cdots, g)$ is called as prototype of $[\mu_j(u)]$.

Remark 1: In the paper, because $E(\mu_j)$ and $E(\mu_{j'})$ of fuzzy sets $\mu_j(u)$ and $\mu_{j'}(u)$ on U are comparable, hence in $H = \{([\mu_0(u)], \{s_0\}), ([\mu_1(u)], \{s_1\}), \cdots, ([\mu_g(u)], \{s_g\})\}$, $s_j < s_{j'}$ if and only if $E(\mu_j) < E(\mu_{j'})$. The order is different with "$s_j \leq s_{j'}$ if and only if $j \leq j'$", which is widely adopted in 2-tuple fuzzy linguistic representation model [13].

Example 2: For "height" on $(0, 2.5m]$, let primary linguistic terms $H_2 = \{s_0$ (low), s_1 (medium), s_2 (high)$\}$ with their triangular or trapezoidal fuzzy sets $\{\mu_0(u) = (0, 0, 1.4, 1.6), \mu_1(u) = (1.4, 1.65, 1.9), \mu_2(u) = (1.7, 1.9, 2.5, 2.5)\}$ on $(0, 2.5m]$. Then primary formal linguistic concepts $H_2 = \{([\mu_0(u)], \{s_0\}), ([\mu_1(u)], \{s_1\}), ([\mu_2(u)], \{s_2\})\}$ are shown in Fig.(5).

IV. THE FORMAL 2-TUPLE LINGUISTIC CONCEPT

Theoretically, there are infinite fuzzy sets on the universe U and infinite equivalent classes of $\mathcal{F}(U)$, however, finite linguistic terms are always utilized to describe imprecise or uncertain qualitative and quantitative information in practical applications, such as the primary formal linguistic concept $H = \{([\mu_j(u)], \{s_j\}) | j = 0, \cdots, g\}$ are always utilized to describe and represent evaluation information of

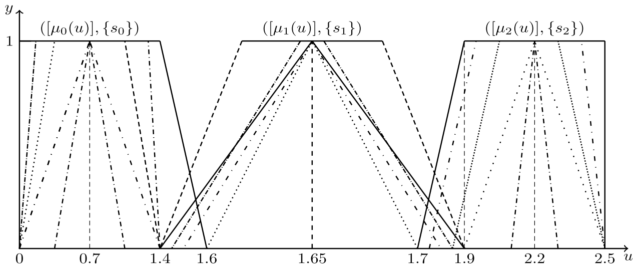

FIGURE 5. Primary formal linguistic concepts of linguistic variable *height*.

decision makers in linguistic decision making. This mans that there exist many equivalent classes of $\mathcal{F}(U)/\equiv$ which can not be described by primary linguistic terms of H. Inspired by 2-tuple fuzzy linguistic representation model, in this section, a new symbolic translation based on $E(\mu)$ of fuzzy set $\mu(u)$ is provided, then formal 2-tuple linguistic concept is constructed, *i.e.*, it's extension is an equivalence class of fuzzy sets and intension is a 2-tuple linguistic term.

A. FORMAL 2-TUPLE LINGUISTIC CONCEPT OF FUZZY SET

For any fuzzy sets $\mu(u), \mu'(u) \in \mathcal{F}(U)$ with $E(\mu) = \frac{yx}{1+y} + \frac{z}{1+y}$ and $E(\mu') = \frac{y'x'}{1+y'} + \frac{z'}{1+y'}$, denote

$$|\mu - \mu'| = |E(\mu) - E(\mu')|, \tag{10}$$

$$S_{\mu\mu'} = \begin{cases} 1, & \text{if } E(\mu') - E(\mu) \geq 0, \\ -1, & \text{if } E(\mu') - E(\mu) < 0. \end{cases} \tag{11}$$

Intuitively, $|\mu - \mu'|$ defined by $E(\mu)$ and $E(\mu')$ is similarity between two $\mu(u)$ and $\mu'(u)$, *i.e.*, the smaller $|\mu - \mu'|$ is, the more similar $\mu(u)$ and $\mu'(u)$ are. Especially, if $|\mu - \mu'| = |E(\mu) - E(\mu')| = 0$, then $E(\mu) = E(\mu')$, $\mu(u)$ and $\mu'(u)$ are in the same equivalence class of fuzzy sets. $S_{\mu\mu'}$ is the sign function of $E(\mu') - E(\mu)$.

Definition 4: Let a primary formal linguistic concepts $H = \{([\mu_j(u)], \{s_j\})|j = 0, \cdots, g\}$. For any fuzzy set $\mu(u)$ on the universe U, the 2-tuple linguistic term (s_j, α) can describe $\mu(u)$ if s_j and α are such that $|\mu_j - \mu|$, as shown at the bottom of the next page, furthermore $([\mu(u)], \{(s_j, \alpha)\})$ is called as a formal 2-tuple linguistic concept.

Property 1: In Definition 4, $\alpha \in [-0.5, 0.5]$.

Proof: (1) If $S_{\mu_j\mu} = 1$, *i.e.*, $E(\mu) - E(\mu_j) \geq 0$, then

$$\alpha = \frac{|\mu_j - \mu|}{|\mu_j - \mu| + |\mu_{j+1} - \mu|}.$$

Due to $|\mu_j - \mu| = min\{|\mu_i - \mu||i = 0, 1, \cdots, g\}$, $|\mu_j - \mu| \leq |\mu_{j+1} - \mu|$ can be obtained and

$$\alpha = \frac{|\mu_j - \mu|}{|\mu_j - \mu| + |\mu_{j+1} - \mu|} \leq \frac{|\mu_j - \mu|}{2|\mu_j - \mu|} = 0.5.$$

Especially, if $|\mu_j - \mu| = 0$, then $|E(\mu_j) - E(\mu)| = 0$, *i.e.*, $E(\mu_j) = E(\mu)$, this means $\mu \in [\mu_j]$ and $\mu \notin [\mu_{j+1}]$, in the case, $|\mu_{j+1} - \mu| \neq 0$ and

$$\alpha = \frac{|\mu_j - \mu|}{|\mu_j - \mu| + |\mu_{j+1} - \mu|} = \frac{0}{0 + |\mu_{j+1} - \mu|} = 0,$$

i.e., $\mu(u)$ is described by linguistic term $(s_j, 0) = s_j$.

(2) If $S_{\mu_j\mu} = -1$, *i.e.*, $E(\mu) - E(\mu_j) < 0$, then

$$\alpha = \frac{|\mu_j - \mu|}{-(|\mu_j - \mu| + |\mu_{j-1} - \mu|)}.$$

Due to $|\mu_j - \mu| = min\{|\mu_i - \mu||i = 0, 1, \cdots, g\}$, $|\mu_j - \mu| \leq |\mu_{j-1} - \mu|$ can be obtained and

$$\alpha = -\frac{|\mu_j - \mu|}{|\mu_j - \mu| + |\mu_{j-1} - \mu|} \geq -\frac{|\mu_j - \mu|}{2|\mu_j - \mu|} = -0.5.$$

Hence $\alpha \in [-0.5, 0.5]$ based on (1) and (2).

Remark 2: By Property 1, if fuzzy set $\mu(u)$ can be described by $(s_j, 0.5)$, then $E(\mu) - E(\mu_j) \geq 0$, $|\mu_j - \mu| = min\{|\mu_i - \mu||i = 0, 1, \cdots, g\}$ and

$$\alpha = \frac{|\mu_j - \mu|}{|\mu_j - \mu| + |\mu_{j+1} - \mu|} = 0.5,$$

hence $|\mu_j - \mu| = |\mu_{j+1} - \mu|$, *i.e.*, $|\mu_{j+1} - \mu| = min\{|\mu_i - \mu||i = 0, 1, \cdots, g\}$. Due to $E(\mu_j) < E(\mu_{j+1})$, hence $E(\mu) - E(\mu_{j+1}) < 0$ and $\mu(u)$ is also described by $(s_{j+1}, -0.5)$. In practical applications, formal 2-tuple linguistic concept employs $([\mu(u)], \{(s_{j+1}, -0.5)\})$ instead of $([\mu(u)], \{(s_j, 0.5)\})$.

Example 3: For linguistic variable "height" on $(0, 2.5m]$, let primary linguistic terms $H_2 = \{s_0$ (low), s_1 (medium), s_2 (high)$\}$ with their triangular or trapezoidal fuzzy sets $\{\mu_0(u) = (0, 0, 1.4, 1.6), \mu_1(u) = (1.4, 1.65, 1.9), \mu_2(u) = (1.7, 1.9, 2.5, 2.5)\}$ on $(0, 2.5m]$. Suppose triangular fuzzy set $\mu(u) = (1.2, 1.5, 1.8)$, according to Eqs.(3)-(6), $E(\mu) = (1.5, 1.5)$ can be obtained. According to Eqs.(10) and (11) and Definition 4, $|\mu_0 - \mu| = 0.8$, $|\mu_1 - \mu| = 0.15$ and $|\mu_2 - \mu| = 0.7$, i.e., $|\mu_1 - \mu| = min\{|\mu_0 - \mu|, |\mu_1 - \mu|, |\mu_2 - \mu|\}$, $S_{\mu_1\mu} = -1$ due to $1.5 - 1.65 < 0$, $\alpha = -\frac{0.15}{0.15+0.8} \doteq -0.16$, hence $\mu(u) = (1.2, 1.5, 1.8)$ can be described by 2-tuple linguistic term $(s_1, -0.16)$ and $([\mu(u)], \{(s_1, -0.16)\})$ is a formal 2-tuple linguistic concept shown in Fig.(6).

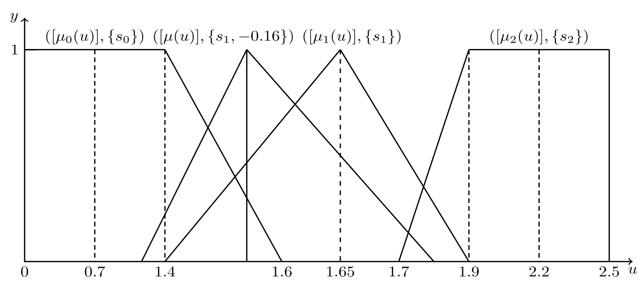

FIGURE 6. The formal 2-tuple linguistic concept $([\mu(u)], \{(s_1, -0.16)\})$.

Property 2: Let a primary formal linguistic concepts $H = \{([\mu_j(u)], \{s_j\})| j = 0, \cdots, g\}$. If fuzzy sets $\mu(u)$ and $\mu'(u)$ on U are described by 2-tuple linguistic term (s_j, α), then $E(\mu) = E(\mu')$, *i.e.*, $[\mu(u)] = [\mu'(u)]$ and $([\mu(u)], \{(s_j, \alpha)\})$ is a formal 2-tuple linguistic concept.

Proof: According to Definition 4, if $\alpha \in [-0.5, 0)$, then

$$\frac{|\mu_j - \mu|}{-(|\mu_j - \mu| + |\mu_{j-1} - \mu|)} = \alpha = \frac{|\mu_j - \mu'|}{-(|\mu_j - \mu'| + |\mu_{j-1} - \mu'|)}$$

According to Eq.(10), we have

$$\frac{|E(\mu_j)-E(\mu)|}{|E(\mu_j)-E(\mu)|+|E(\mu_{j-1})-E(\mu)|} = \frac{|E(\mu_j)-E(\mu')|}{|E(\mu_j)-E(\mu')|+|E(\mu_{j-1})-E(\mu')|}.$$

Due to $|\mu_j-\mu| = min\{|\mu_i-\mu||i=0,1,\cdots,g\}$, $|\mu_j-\mu'| = min\{|\mu_i-\mu'||i=0,1,\cdots,g\}$ and $E(\mu_{j-1}) < E(\mu_j)$, it can be obtained that

$$\begin{aligned}|E(\mu_j)-E(\mu)| &= E(\mu_j)-E(\mu),\\ |E(\mu_{j-1})-E(\mu)| &= E(\mu)-E(\mu_{j-1}),\\ |E(\mu_j)-E(\mu')| &= E(\mu_j)-E(\mu'),\\ |E(\mu_{j-1})-E(\mu')| &= E(\mu')-E(\mu_{j-1}),\end{aligned}$$

hence $E(\mu_j)-E(\mu) = E(\mu_j)-E(\mu')$, *i.e.*, $E(\mu) = E(\mu')$, $[\mu(u)] = [\mu'(u)]$ and $([\mu(u)], \{(s_j,\alpha)\})$ is a formal 2-tuple linguistic concept.

Property 2 means that fuzzy sets described by the same 2-tuple linguistic term must be in the same equivalence class and extension $[\mu(u)]$ can be utilized to decide intension (s_j,α) of the formal 2-tuple linguistic concept $([\mu(u)], \{(s_j,\alpha)\})$.

B. FORMAL 2-TUPLE LINGUISTIC CONCEPT OF 2-TUPLE LINGUISTIC TERM

In the subsection, it is discussed that intension (s_j,α) can be utilized to decide extension $[\mu(u)]$ of the formal 2-tuple linguistic concept $([\mu(u)], \{(s_j,\alpha)\})$.

Property 3: Let a primary formal linguistic concepts $H = \{([\mu_j(u)], \{s_j\})|\, j=0,\cdots,g\}$ and any 2-tuple linguistic term (s_j,α). If $\alpha\in[-0.5,0)$ and fuzzy set $\mu(u)$ with $E(\mu) = (1+\alpha)E(\mu_j)-\alpha E(\mu_{j-1})$, then $\mu(u)$ is described by (s_j,α). If $\alpha\in[0,0.5)$ and $\mu(u)$ with $E(\mu) = (1-\alpha)E(\mu_j)+\alpha\, E(\mu_{j+1})$, then $\mu(u)$ is described by (s_j,α).

Proof: According to Definition 4 and Eqs.(10) and (11),

$$\begin{aligned}|\mu_j-\mu| &= |E(\mu_j)-E(\mu)|\\ &= |E(\mu_j)-((1+\alpha)E(\mu_j)-\alpha E(\mu_{j-1}))|\\ &= -\alpha(E(\mu_j)-E(\mu_{j-1})),\\ |\mu_{j-1}-\mu| &= |E(\mu_{j-1})-E(\mu)|\\ &= |E(\mu_{j-1})-((1+\alpha)E(\mu_j)-\alpha E(\mu_{j-1}))|\\ &= (1+\alpha)(E(\mu_j)-E(\mu_{j-1})),\end{aligned}$$

due to $\alpha\in[-0.5,0)$, $|\mu_j-\mu| < |\mu_{j-1}-\mu|$ is obtained. For any $E(\mu_{j'}) < E(\mu_{j-1})$,

$$\begin{aligned}|\mu_{j'}-\mu| &= |E(\mu_{j'})-E(\mu)|\\ &= |E(\mu_{j'})-((1+\alpha)E(\mu_j)-\alpha E(\mu_{j-1}))|\\ &= (E(\mu_j)-E(\mu_{j'}))-\alpha(\mu_j-\mu_{j-1})\\ &> -\alpha(\mu_j-\mu_{j-1}) = |\mu_j-\mu|.\end{aligned}$$

For any $E(\mu_{j'}) > E(\mu_j)$,

$$\begin{aligned}|\mu_{j'}-\mu| &= |E(\mu_{j'})-E(\mu)|\\ &= |E(\mu_{j'})-((1+\alpha)E(\mu_j)-\alpha E(\mu_{j-1}))|\\ &= (E(\mu_{j'})-E(\mu_j))-\alpha(\mu_j-\mu_{j-1})\\ &> -\alpha(\mu_j-\mu_{j-1}) = |\mu_j-\mu|.\end{aligned}$$

Hence, it can be obtained that

$$|\mu_j-\mu| = min\{|\mu_i-\mu||i=0,1,\cdots,g\}.$$

In addition, $E(\mu)-E(\mu_j) = (1+\alpha)E(\mu_j)-\alpha E(\mu_{j-1})-E(\mu_j) = \alpha(E(\mu_j)-E(\mu_{j-1})) < 0$, hence $S_{\mu_j\mu} = -1$ and as shown at the bottom of the next page. Accordingly, $\mu(u)$ is described by 2-tuple linguistic term $(s_j,\ \alpha)$ if $\alpha\in[-0.5,0)$ and the fuzzy set $\mu(u)$ with $E(\mu) = (1+\alpha)\,E(\mu_j)-\alpha E(\mu_{j-1})$. Similarly, it can be proved that $\mu(u)$ is described by 2-tuple linguistic term (s_j,α) if $\alpha\in[0,0.5)$ and $\mu(u)$ with $E(\mu) = (1-\alpha)E(\mu_j)+\alpha E(\mu_{j+1})$.

Property 3 means that 2-tuple linguistic term (s_j,α) can be utilized to decide extension $[\mu(u)]$ of formal 2-tuple linguistic concept $([\mu(u)], \{(s_j,\alpha)\})$.

Corollary 1: Let a primary formal linguistic concepts $H = \{([\mu_j(u)], \{s_j\})|\ j = 0,\cdots,g\}$. For any equivalence class $[\mu(u)] \in \mathcal{F}(U)/\equiv$ and 2-tuple linguistic term (s_j,α), $([\mu(u)], \{(s_j,\alpha)\})$ is a formal 2-tuple linguistic concept if and only if $E(\mu) = (1+\alpha)E(\mu_j)-\alpha E(\mu_{j-1})$ when $\alpha\in[-0.5,0)$ and $E(\mu) = (1-\alpha)E(\mu_j)+\alpha E(\mu_{j+1})$ when $\alpha\in[0,0.5)$.

According to Corollary 1, extension $[\mu(u)]$ and intension 2-tuple linguistic term (s_j,α) of any formal 2-tuple linguistic concept $([\mu(u)], \{(s_j,\alpha)\})$ can be decided by each other. In addition, the symbolic translation α of 2-tuple linguistic term (s_j,α) owns a new meaning, *i.e.*, it is not only "the difference of information" between a symbolic aggregation result and linguistic term s_j, but also the weight which can be utilized to obtain meaning $[\mu(u)]$ of the 2-tuple linguistic term (s_j,α). From 2-tuple linguistic term (s_j,α) to formal 2-tuple linguistic concept $([\mu(u)], \{(s_j,\alpha)\})$, 2-tuple fuzzy linguistic representation model is no longer symbolic linguistic terms or man-made language, it is formal linguistic values of a linguistic variable and becomes a natural or artificial language, which can be understood step by step in the hierarchy of formal linguistic concepts, all of these are discussed in the next section. In the follows, denote $TH = \{([\mu(u)], \{(s_j,\alpha)\})|\mu(u)\in\mathcal{F}(U),\ s_j\in H,\alpha\in[-0.5,0.5)\}$ as all formal 2-tuple linguistic concepts on primary formal linguistic concepts H.

V. THE HIERARCHY OF FORMAL LINGUISTIC CONCEPTS

In this section, we analyze a new hierarchy of linguistic terms based on primary linguistic terms and linguistic hedges, then

$$|\mu_j-\mu| = min\{|\mu_i-\mu||i=0,1,\cdots,g\},$$
$$\alpha = \frac{2|\mu_j-\mu|}{(S_{\mu_j\mu}-1)(|\mu_j-\mu|+|\mu_{j-1}-\mu|)+(S_{\mu_j\mu}+1)(|\mu_j-\mu|+|\mu_{j+1}-\mu|)},$$

a new transformation function between consecutive levels of the linguistic term hierarchy is proposed. Based on formal 2-tuple linguistic concepts, the hierarchy of generated formal linguistic concepts is constructed to manage multi-granular linguistic values in linguistic information processing.

A. THE HIERARCHY OF GENERATED LINGUISTIC TERMS

Multi-granular linguistic terms are to satisfy the need of multi-experts or multiple sources of linguistic information in practical applications, which are managed by the hierarchy of multi-granular linguistic terms, such as a linguistic hierarchy [38] or linguistic hierarchical tree [48] have be constructed to manage multi-granular linguistic terms. In [12], linguistic truth algebra based on linguistic truth {*true*, *false*} and linguistic hedges, such as {*little*, *more or less*, *possibly*, *approximately*, *more*, *very*}, has been constructed for linguistic truth inference. In [20], evaluative linguistic expressions have been presented for advanced modeling of linguistic semantics, formally, an evaluative linguistic expression has the form *<linguistic hedge><primary linguistic term>*, such as *<very><low>* (=*very low*) or *<more or less><high>* (=*more or less high*), generally, linguistic expression triple is widely used as primary linguistic terms with two antonyms and a middle member, such as {*low*, *medium*, *high*}, the main linguistic hedges are widening hedges (their effects are to increase fuzziness, such as {*slightly*, *more or less*, *roughly*, *a sort of*}, and narrowing hedges (their effects are to decrease fuzziness, such as {*rather*, *very*, *extremely*, *significantly*}, it is important that linguistic hedges modify the meaning of linguistic term but do not replace it by new meaning.

In the paper, suppose linguistic hedges are $D = D_1 \cup \{h_0\} \cup D_2$, widening hedges are $D_1 = \{h_1^1, \cdots, h_{r_1}^1\}$ and narrowing hedges are $D_2 = \{h_1^2, \cdots, h_{r_2}^2\}$, h_0 is a specifying hedge which are neither narrowing nor widening, in hedge algebras [23]–[27], h_0 is also called as identity, *i.e.*, $< h_0 >$*<primary linguistic term>=primary linguistic term*. According to effects of hedges on fuzziness, a linear order on linguistic hedges can be defined as follows:

- For any $h^t_{r'_t}$ and $h^t_{r''_t} (t = 1, 2)$ in D_t, $h^t_{r'_t} \prec h^t_{r''_t}$ if and only if $r'_t < r''_t$, where $\prec$ is decided by effects of hedges on fuzziness, such as *slightly*$\prec$*roughly* or *rather*$\prec$*very*;
- In D, $h^1_{r_1} \prec h_0 \prec h^2_1$.

Definition 5: Let primary linguistic terms $H = \{s_0, s_1, \cdots, s_g\}$ and linguistic hedges D, a generated linguistic term set is defined by $DH = \{hs_i = (h, s_i) \in D \times H | h \in D, s_i \in H\}$.

In primary linguistic terms $H = \{s_j | j = 0, \cdots, g\}$, if g is even number, s_j and s_{g-j} are two antonyms and $s_{\frac{g}{2}}$ is the middle member, then H is called as balanced linguistic terms. An unbalanced linguistic term set is such that there exists $s_j \in H$, antonym of s_j is not in H. By using linguistic hedges in D on balanced linguistic terms H, then hs_j and hs_{g-j} are two antonyms for any $h \neq h_0$ in D and s_j in H.

Definition 6: For primary linguistic term set $H = \{s_0, s_1, \cdots, s_g\}$ and linguistic hedges in D, a linear order on generated linguistic terms $DH = \{hs_i = (h, s_i) \in D \times H | h \in D, s_i \in H\}$ is defined as follows;

- for any $hs_i, h's_{i'} \in DH$, if $s_i < s_{i'}$, then $hs_i < h's_{i'}$;
- for any $hs_i, h's_i \in DH$, if $s_i < s_{\frac{g}{2}}$ and $h \prec h'$, then $h's_i < hs_i$; If $s_i > s_{\frac{g}{2}}$ and $h \prec h'$, then $hs_i < h's_i$.

Based on Definitions 5 and 6, a new and alternative linguistic hierarchy of linguistic terms of a linguistic variable can be provided as follows, which is generated by primary linguistic term set H and linguistic hedges in D, each linguistic term has the form *<linguistic hedge><linguistic term>*.

- The root of the hierarchy labeled by the name L of the linguistic variable;
- The 0-level is the primary linguistic term set $H = \{s_0, s_1, \cdots, s_g\}$, where $s_{\frac{g}{2}}$ is the middle member, H is a balanced or unbalanced linguistic.
- The 1-level is the generated linguistic term set $DH = \{hs_i = (h, s_i) \in D \times H | h \in D, s_i \in H\}$, which is a finitely linguistic term set with a linear order defined by Definition 6 and $h_0 s_{\frac{g}{2}} = s_{\frac{g}{2}}$ is the middle member;
- The t-level ($t = 2, 3, \cdots, T$) is the generated linguistic term set $D^t H = D \times D^{t-1} H = \{hs_i^{t-1} | h \in D, s_i^{t-1} \in D^{t-1} H\}$, which is a finitely linguistic term set with a linear order defined by Definition 6 and $h_0 s^{t-1}_{\frac{g}{2}} = s_{\frac{g}{2}}$ is the middle member.

Generated linguistic terms in the hierarchy are such that

1) $D^{t-1}H = \{h_0\} \times D^{t-1}H \subset D^t H$ for any $t \in \{1, 2, \cdots, T\}$, in which $D^0 H = H$;
2) $|D^t H| \leq |D| \times |D^{t-1}H|$. If $D^{t-1}H$ is a balanced linguistic term set, $D' \subseteq D$ is a subset of linguistic hedges D and $D' \times \{s_i^{t-1}\} \subset D^t H$ for each linguistic term $s_i^{t-1} \in D^{t-1}H$, then $D^t H$ is also a balanced linguistic term set;
3) For each $t = 0, 1, \cdots, T-1$, there exists only one mapping $\Gamma_t : D^t H \longrightarrow 2^{D^{t+1}H} - \{\emptyset\}$ such that for each $s_i^t \in D^t H$, $\Gamma_t(s_i^t) = \{hs_i^t | h \in D\}$, $\Gamma_t(s_i^t) \neq \Gamma_t(s_{i'}^t)$ and $hs_i^t \neq hs_{i'}^t$ if $s_i^t \neq s_{i'}^t$ in $D^t H$. In addition, $t+1$-level is $D^{t+1}H = \bigcup_{s_i^t \in D^t H} \Gamma_t(s_i^t) = \{hs_i^t | h \in D\}$.

Denote all generated linguistic terms by using primary linguistic terms H and linguistic hedges D as $GH = \bigcup_{t=0}^{T} D^t H$, in which each $s_i^{t-1} \in D^{t-1}H$ is refined by the linguistic term

$$\frac{2|\mu_j - \mu|}{(S_{\mu_j\mu} - 1)(|\mu_j - \mu| + |\mu_{j-1} - \mu|) + (S_{\mu_j\mu} + 1)(|\mu_j - \mu| + |\mu_{j+1} - \mu|)}$$
$$= \frac{|\mu_j - \mu|}{-(|\mu_j - \mu| + |\mu_{j-1} - \mu|)} = \frac{-\alpha(E(\mu_j) - E(\mu_{j-1}))}{-(E(\mu_j) - E(\mu_{j-1}))} = \alpha.$$

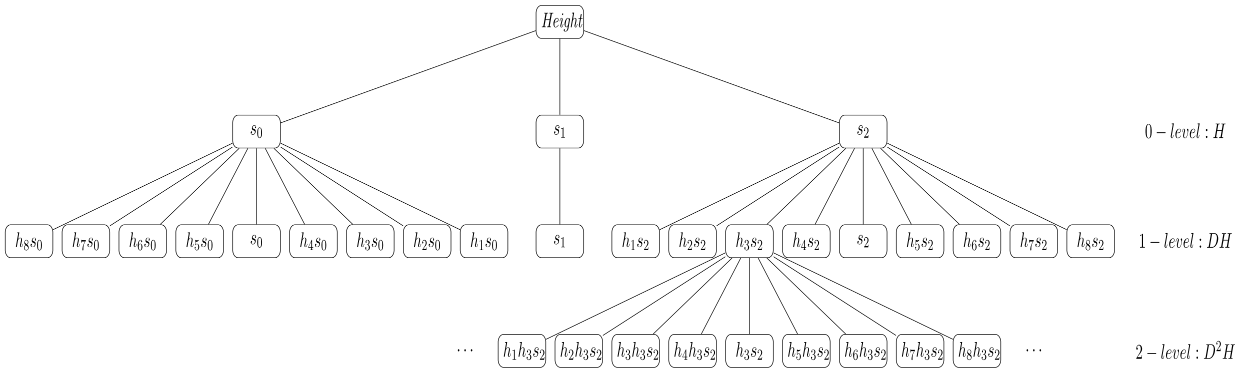

FIGURE 7. A hierarchy of generated linguistic terms of *height*.

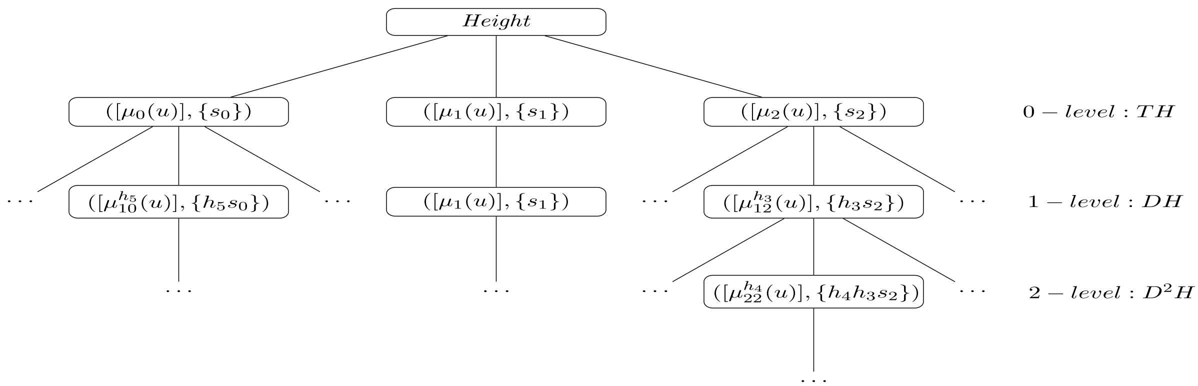

FIGURE 8. A hierarchy of generated formal linguistic concepts of *height*.

set $\Gamma_t(s_i^{t-1})$ of D^tH. Formally, for each mapping Γ_t, there exists a pseudo-inversion $\Gamma_t^- : D^{t+1}H \longrightarrow D^tH$ such that $\Gamma_t^-(hs_i^t) = s_i^t$ for each $hs_i^t \in D^{t+1}H$. Fig.(7) shows a hierarchy of generated linguistic terms of linguistic variable *height*, where primary linguistic terms is $H = \{low(s_0), medium(s_1), high\ (s_2)\}$, s_1 is the the middle member, linguistic hedges is $D = D_1 \cup \{h_0\} \cup D_2 = \{slightly(h_1), more\ or\ less(h_2), roughly(h_3), a\ sort\ of(h_4)\} \cup \{h_0\} \cup \{rather\ (h_5), very(h_6), extremely(h_7), significantly(h_8)\}$.

B. THE HIERARCHY OF GENERATED FORMAL LINGUISTIC CONCEPTS

From the formal concept point of view, the hierarchy of generated formal linguistic concepts is same with the hierarchy of generated linguistic terms, because generated linguistic terms are intensions of generated formal linguistic concepts, *i.e.*, the hierarchy of generated formal linguistic concepts is obtained by replacing linguistic terms in the hierarchy of generated linguistic terms as formal linguistic concepts, such as by using Fig.(7), a hierarchy of generated formal linguistic concepts of linguistic variable *height* is shown in Fig.(8).

In the hierarchy of generated formal linguistic concepts, because linguistic hedge modifies meaning of linguistic term but do not replace it by new meaning, each generated formal linguistic concept is refined by generated formal linguistic concepts at next level. Inspired by formal 2-tuple linguistic concept, a new transformation function is proposed to obtain formal 2-tuple linguistic concept representation of generated formal linguistic concept, *i.e.*, formal 2-tuple linguistic concept representation of generated formal linguistic concept in DH is obtained by the follow transformation function:

$$TF_1^0 : DH \longrightarrow TH, ([\mu(u)], \{hs_j\}) \mapsto ([\mu(u)], \{(s_j,\alpha)\}) \quad (12)$$

where $[\mu(u)]$ and α are decided by

- If $s_j < s_{\frac{g}{2}}$ and $h^2_{r_2'} \in D_2$, then $TF_1^0([\mu_{1j}^{r_2'}(u)], \{h^2_{r_2'}s_j\}) = ([\mu_{1j}^{r_2'}(u)], \{(s_j, \alpha_{r_2'})\})$ and

$$\alpha_{r_2'} = -\frac{r_2'}{2(r_2+1)}, r_2' = 1, \cdots, r_2. \quad (13)$$

$([\mu_{1j}^{r_2'}(u)], \{h^2_{r_2'}s_j\})$ is a generated formal linguistic concept. By Property 3, equivalence class $[\mu_{1j}^{r_2'}(u)]$ with $E(\mu_{1j}^{r_2'})$ is described by $h^2_{r_2'}s_j$, in which

$$\begin{aligned} E(\mu_{1j}^{r_2'}) &= (1+\alpha_{r_2'})E(\mu_j) + \alpha_{r_2'}E(\mu_{j-1}) \\ &= E(\mu_j) - \frac{r_2'}{2(r_2+1)}(E(\mu_j) - E(\mu_{j-1})). \end{aligned}$$

- If $s_j < s_{\frac{g}{2}}$ and $h^1_{r'_1} \in D_1$, then $TF^0_1([\mu^{r'_1}_{1j}(u)], \{h^1_{r'_1}s_j\}) = ([\mu^{r'_1}_{1j}(u)], \{(s_j, \alpha_{r'_1})\})$ and
$$\alpha_{r'_1} = 0.5 - \frac{r'_1}{2(r_1+1)}, r'_1 = 1, \cdots, r_1. \quad (14)$$
$([\mu^{r'_1}_{1j}(u)], h^1_{r'_1}s_j)$ is a generated formal linguistic concept. The equivalence class $[\mu^{r'_1}_{1j}(u)]$ with $E(\mu^{r'_1}_{1j})$ is described by $h^1_{r'_1}s_j$, in which
$$\begin{aligned} E(\mu^{r'_1}_{1j}) &= (1-\alpha_{r'_1})E(\mu_j) + \alpha_{r'_1}E(\mu_{j+1}) \\ &= (0.5 + \frac{r'_1}{2(r_1+1)})E(\mu_j) \\ &\quad +(0.5 - \frac{r'_1}{2(r_1+1)})E(\mu_{j+1}). \end{aligned}$$
- If $s_j \geq s_{\frac{g}{2}}$ and $h^2_{r'_2} \in D_2$, then $TF^0_1([\mu^{r'_2}_{1j}(u)], \}h^2_{r'_2}s_j\}) = ([\mu^{r'_2}_{1j}(u)], \{(s_j, \alpha_{r'_2})\})$ and
$$\alpha_{r'_2} = \frac{r'_2}{2(r_2+1)}, r'_2 = 1, \cdots, r_2. \quad (15)$$
$([\mu^{r'_2}_{1j}(u)], h^2_{r'_2}s_j)$ is a generated formal linguistic concept. The equivalence class $[\mu^{r'_2}_{1j}(u)]$ with $E(\mu^{r'_2}_{1j})$ is described by $h^2_{r'_2}s_j$, in which
$$\begin{aligned} E(\mu^{r'_2}_{1j}) &= (1-\alpha_{r'_2})E(\mu_j) + \alpha_{r'_2}E(\mu_{j+1}) \\ &= E(\mu_j) - \frac{r'_2}{2(r_2+1)}(E(\mu_j) - E(\mu_{j+1})). \end{aligned}$$
- If $s_j \geq s_{\frac{g}{2}}$ and $h^1_{r'_1} \in D_1$, then $TF^0_1([\mu^{r'_1}_{1j}(u)], \{h^1_{r'_1}s_j\}) = ([\mu^{r'_1}_{1j}(u)], \{(s_j, \alpha_{r'_1})\})$,
$$\alpha_{r'_1} = \frac{r'_1}{2(r_1+1)} - 0.5, r'_1 = 1, \cdots, r_1. \quad (16)$$
$([\mu^{r'_1}_{1j}(u)], h^1_{r'_1}s_j)$ is a generated formal linguistic concept. The equivalence class $[\mu^{r'_1}_{1j}(u)]$ with $E(\mu^{r'_1}_{1j})$ is described by $h^1_{r'_1}s_j$, in which
$$\begin{aligned} E(\mu^{r'_1}_{1j}) &= (1+\alpha_{r'_1})E(\mu_j) - \alpha_{r'_1}E(\mu_{j-1}) \\ &= (0.5 + \frac{r'_1}{2(r_1+1)})E(\mu_j) \\ &\quad +(0.5 - \frac{r'_1}{2(r_1+1)})E(\mu_{j-1}). \end{aligned}$$

Suppose that generated formal linguistic concepts at t-level ($t = 1, \cdots, T-1$) is $D^tH = \{([\mu^t_0(u)], \{s^t_0\}), ([\mu^t_1(u)], \{s^t_1\}), \cdots, ([\mu^t_{g(t)}(u)], \{s^t_{g(t)}\})\}$, denote
$$TH^t = \{([\mu(u)], \{(s^t_j, \alpha)\}) | \mu(u) \in \mathcal{F}(U), \alpha \in [-0.5, 0.5)\}$$
as all formal 2-tuple linguistic concepts on D^tH, then the transformation function from $D^{t+1}H$ to TH^t can be provided to obtain formal 2-tuple linguistic concept representation of generated formal linguistic concept in $D^{t+1}H$:
$$\begin{aligned} TF^t_{t+1} : D^{t+1}H &\longrightarrow TH^t, \\ ([\mu(u)], \{hs^t_j\}) &\longmapsto ([\mu(u)], \{(s^t_j, \alpha)\}) \end{aligned} \quad (17)$$
where $s^{t+1}_{\frac{g}{2}} = h_0 s^t_{\frac{g}{2}} = \cdots = s_{\frac{g}{2}}$ is the middle member, $\mu(u)$ and α can be obtained by Eqs.(13)-(16).

Intuitively, by using transformation functions $TF^t_{t+1}(t = 0, 1, \cdots, T-1)$, each generated formal linguistic concept at $t+1$-level has a formal 2-tuple linguistic concept representation on t-level. Based on Eqs.(13)-(16) and Property 3, the equivalence class of fuzzy sets on $\mathcal{F}(U)$ described by the 2-tuple linguistic term or generated linguistic term can be obtained, and generated formal linguistic concepts in $t+1$-level are constructed. Then the hierarchy of generated formal linguistic concepts by primary formal linguistic concepts H and linguistic hedges $D = D_1 \cup \{h_0\} \cup D_2$ is as follows:

- The root of the hierarchy labeled by the name L of the linguistic variable;
- The 0-level is primary formal linguistic concepts $H = \{([\mu_0(u)], \{s_0\}), \cdots, ([\mu_g(u)], \{s_g\})\}$, where $([\mu_{\frac{g}{2}}(u)], \{s_{\frac{g}{2}}\})$ is the middle member.
- The 1-level is generated formal linguistic concepts $DH = \{([\mu^h_{1j}(u)], \{hs_j\}) = ([\mu^h_{1j}(u)], \{(s_j, \alpha)\}) | h \in D, s_j \in H, \alpha \in [-0.5, 0.5)\}$, which is a finitely formal linguistic concept sets and $([\mu_{\frac{g}{2}}(u)], \{s_{\frac{g}{2}}\})$ is the middle member;
- The t-level ($t = 2, 3, \cdots, T$) is generated formal linguistic concepts $D^tH = D \times D^{t-1}H = \{([\mu^h_{tj}(u)], \{hs^{t-1}_j\}) = ([\mu^h_{tj}(u)], \{(s^{t-1}_j, \alpha)\}) | h \in D, s^{t-1}_j \in D^{t-1}H, \alpha \in [-0.5, 0.5)\}$, which is a finitely formal linguistic concept set and $([\mu_{\frac{g}{2}}(u)], \{s_{\frac{g}{2}}\})$ is the middle member.

Denote all generated formal linguistic concepts as $GH = \bigcup_{t=0}^{T} D^tH$, where each formal linguistic concept at t-level is refined by formal linguistic concepts at $t+1$-level and each formal linguistic concept at $t+1$-level is a formal 2-tuple linguistic concept representation on t-level, *i.e.*, each formal linguistic concept $([\mu^t_j(u)], \{s^t_j\}) \in D^tH$ is refined by formal linguistic concepts $\{([\mu^h_{tj}(u)], \{hs^t_j\}) | h \in D, \mu(u) \in \mathcal{F}(U)\}$ of $D^{t+1}H$, conversely, each formal linguistic concept $([\mu^h_{tj}(u)], \{hs^t_j\}) \in D^{t+1}H$ is a formal 2-tuple linguistic concept representation $([\mu^h_{tj}(u)], \{(s^t_j, \alpha)\})$ on t-level.

Corollary 2: Let a primary formal linguistic concepts $H = \{([\mu_j(u)], \{s_j\}) | j = 0, \cdots, g\}$ and linguistic hedges $D = D_1 \cup \{h_0\} \cup D_2 = \{h^1_1, \cdots, h^1_{r_1}\} \cup \{h_0\} \cup \{h^2_1, \cdots, h^2_{r_2}\}$. A generated formal linguistic concept $([\mu^h_{tj}(u)], \{hs^t_j\}) \in D^{t+1}H$ is a formal 2-tuple linguistic concept representation $([\mu(u)], \{(s_j, \alpha)\}) \in TH$ if $[\mu(u)] = [\mu^h_{tj}(u)]$.

Corollary 2 means that on the one hand, in the hierarchy of formal linguistic concepts, primary formal linguistic concepts $H = \{([\mu_j(u)], \{s_j\}) | j = 0, \cdots, g\}$ are step by step refined by generated formal linguistic concepts $\{([\mu^h_{tj}(u)], \{hs^t_j\})|h \in D, \mu(u) \in \mathcal{F}(U)\}$ of $D^{t+1}H$ and each generated formal linguistic concept $([\mu^h_{tj}(u)], \{hs^t_j\})$ is formal 2-tuple linguistic concept representation $([\mu(u)], \{(s_j, \alpha)\})$ on H, *i.e.*, primary formal linguistic concepts is also refined by formal 2-tuple linguistic concepts on primary formal linguistic concepts H. On the other hand, formal 2-tuple linguistic concepts on H as man-made language are step by step formalized by generated formal linguistic concepts in the hierarchy, formal 2-tuple linguistic concepts are not only continuous linguistic concepts but also become natural or artificial language concepts in the hierarchy of formal linguistic concepts.

VI. ILLUSTRATIVE EXAMPLES

In the section, we utilize two examples to illustrate constructing and managing multi-granular linguistic values via formal linguistic concepts and it's application in linguistic decision making. The first example utilizes linguistic variable *height* to show constructing and managing multi-granular linguistic values via formal linguistic concepts of *height*. The second example is a mixed linguistic decision making, in which evaluation information provided by decision makers is fuzzy sets or linguistic values, here 2-tuple linguistic term method, fuzzy set method and formal linguistic concept method are utilized in the example.

Example 4: Let the primary formal linguistic concepts $H = \{([\mu_0(u)], \{s_0\}), ([\mu_1(u)], \{s_1\}), ([\mu_2(u)], \{s_2\})\}$ of linguistic variable "height" on $(0, 2.5m]$, s_0, s_1 and s_2 be low, medium and high, respectively. Prototypes of $[\mu_0(u)]$, $[\mu_1(u)]$ and $[\mu_2(u)]$ are shown in Fig.(5), in which $E(\mu_0) = 0.7$, $E(\mu_1) = 1.65$ and $E(\mu_2) = 2.2$. Suppose linguistic hedges $D = D_1 \cup \{h_0\} \cup D_2 = \{\text{slightly } (h_1)\} \cup \{h_0\} \cup \{\text{very } (h_2)\}$. Then generated linguistic terms is $DH = \{h_2s_0 = s^1_0, s_0 = s^1_1, h_1s_0 = s^1_2, s_1 = s^1_3, h_1s_2 = s^1_4, s_2 = s^1_5, h_2s_2 = s^1_6\}$, which are very low, low, slightly low, medium, slightly high, high and very high, respectively.

According to Eq.(13), generated formal linguistic concept $([\mu_{s^1_0}(u)], \{h_2s_0\})$ has formal 2-tuple linguistic concept representation $([\mu_{s^1_0}(u)], \{(s_0, \alpha_2)\})$, in which

$$\alpha_2 = -\frac{1}{2(1+1)} = -0.25.$$

By $E(\mu_0) = 0.7$ and $E(\mu_{-1}) = 0$ (i.e., $\mu_{-1}(u)$ is absolutely low or 0), $[\mu_{s^1_0}(u)]$ with $E(\mu_{s^1_0}) = (1 - 0.25)E(\mu_0) + 0.25\, E(\mu_{-1}) = 0.525$.

According to Eq.(14), generated formal linguistic concept $([\mu_{s^1_2}(u)], \{h_1s_0\})$ has formal 2-tuple linguistic concept representation $([\mu_{s^1_2}(u)], \{(s_0, \alpha_1)\})$, in which

$$\alpha_1 = 0.5 - \frac{1}{2(1+1)} = 0.25.$$

By $E(\mu_0) = 0.7$ and $E(\mu_1) = 1.65$, $[\mu_{s^1_2}(u)]$ with $E(\mu_{s^1_2}) = (1 - 0.25)E(\mu_0) + 0.25\, E(\mu_1) = 0.9375$.

According to Eq.(15), generated formal linguistic concept $([\mu_{s^1_6}(u)], \{h_2s_2\})$ has formal 2-tuple linguistic concept representation $([\mu_{s^1_6}(u)], \{(s_2, \alpha_3)\})$, in which

$$\alpha_3 = \frac{1}{2(1+1)} = 0.25.$$

By $E(\mu_2) = 2.2$ and $E(\mu_3) = 2.5$ (i.e., $\mu_{-1}(u)$ is absolutely high or 2.5), $[\mu_{s^1_6}(u)]$ with $E(\mu_{s^1_6}) = (1 - 0.25)\, E(\mu_2) + 0.25\, E(\mu_3) = 2.275$.

According to Eq.(16), generated formal linguistic concept $([\mu_{s^1_4}(u)], \{h_1s_2\})$ has formal 2-tuple linguistic concept representation $([\mu_{s^1_4}(u)], \{(s_2, \alpha_4)\})$, in which $\alpha_4 = \frac{1}{2(1+1)} - 0.5 = -0.25$. By $E(\mu_1) = 1.65$ and $E(\mu_2) = 2.2$, $[\mu_{s^1_4}(u)]$ with $E(\mu_{s^1_4}) = (1 - 0.25)E(\mu_2) + 0.25\, E(\mu_1) = 2.0625$.

Generated formal linguistic concepts

$$\begin{aligned} DH = \{&([\mu_{s^1_0}(u)], \{s^1_0\}), ([\mu_0(u)], \{s^1_1 = s_0\}), ([\mu_{s^1_2}(u)], \\ &\{s^1_2\}), ([\mu_1(u)], \{s^1_3 = s_1\}), ([\mu_{s^1_4}(u)], \{s^1_4\}), \\ &([\mu_2(u)], \{s^1_5 = s_2\}), ([\mu_{s^1_6}(u)], \{s^1_6\})\}. \end{aligned}$$

The generated linguistic term $h_2s^1_4$ is very slightly high, $([\mu_{h_2s^1_4}(u)], \{h_2s^1_4\})$ has formal 2-tuple linguistic concept representation $([\mu_{h_2s^1_4}(u)], \{(s^1_4, \alpha)\})$, in which $\alpha = \frac{1}{2(1+1)} = 0.25$. By $E(\mu^1_5) = E(\mu_2) = 2.2$ and $E(\mu^1_4) = E(\mu_{h_1s_2}) = 2.0625$, $[\mu_{h_2s^1_4}(u)]$ with $E(\mu_{h_2s^1_4}) = (1 - 0.25)E(\mu^1_4) + 0.25\, E(\mu^1_5) = 2.096875$.

Suppose 2-tuple linguistic term $(s_2, -0.1875)$ on primary linguistic terms $H = \{s_0, s_1, s_2\}$ with fuzzy sets $\mu_0(u) = (0, 0, 1.4, 1.6)$, $\mu_1(u) = (1.4, 1.65, 1.9)$ and $\mu_2(u) = (1.7, 1.9, 2.5, 2.5)$ on $(0, 2.5m]$. According to Property 3, fuzzy sets $[\mu(u)]$ with

$$\begin{aligned} E(\mu) &= (1 - 0.1875)E(\mu_2) + 0.1875\, E(\mu_1) \\ &= 0.8125 \times 2.2 + 0.1875 \times 1.65 = 2.096875 \end{aligned}$$

is described by $(s_2, -0.1875)$, *i.e.*, $([\mu(u)], \{(s_2, -0.1875)\})$ is a formal 2-tuple linguistic concept, according to Corollary 2, $([\mu_{h_2s^1_4}(u)], \{h_2s^1_4\}) = ([\mu_{h_2s^1_4}(u)], \{h_2h_1s_2\})$ is the formal 2-tuple linguistic concept representation $([\mu(u)], \{(s_2, -0.1875)\})$, in other words, 2-tuple linguistic term $(s_2, -0.1875)$ is natural language "very slightly high". Fig.(9) shows linguistic terms H, DH, $h_2s^1_4$ and $(s_2, -0.1875)$ with their fuzzy sets.

Example 5: A company is to plan the development of large projects for the following five years, there are three possible projects a_j $(j = 1, 2, 3)$ to be assessed by c_1 (financial perspective), c_2 (the customer satisfaction), c_3 (internal business process perspective) and c_4 (learning and growth perspective), where their weights are 0.2, 0.3, 0.1 and 0.4, respectively. Fuzzy sets on [0, 1] or linguistic terms $H = \{\text{nothing } (s_0), \text{very bad } (s_1), \text{bad } (s_2), \text{medium } (s_3), \text{good } (s_4), \text{very good } (s_5), \text{perfect } (s_6)\}$ with fuzzy sets on [0, 1] are utilized to evaluate projects according to c_1, c_2, c_3 and c_4, they are shown in Fig.(10). Evaluation information is shown in Table 1, in which for $a, b, c \in [0, 1]$, (a, b, c) is a triangular fuzzy set. It is a mixed decision making problem so as to select the most important of project.

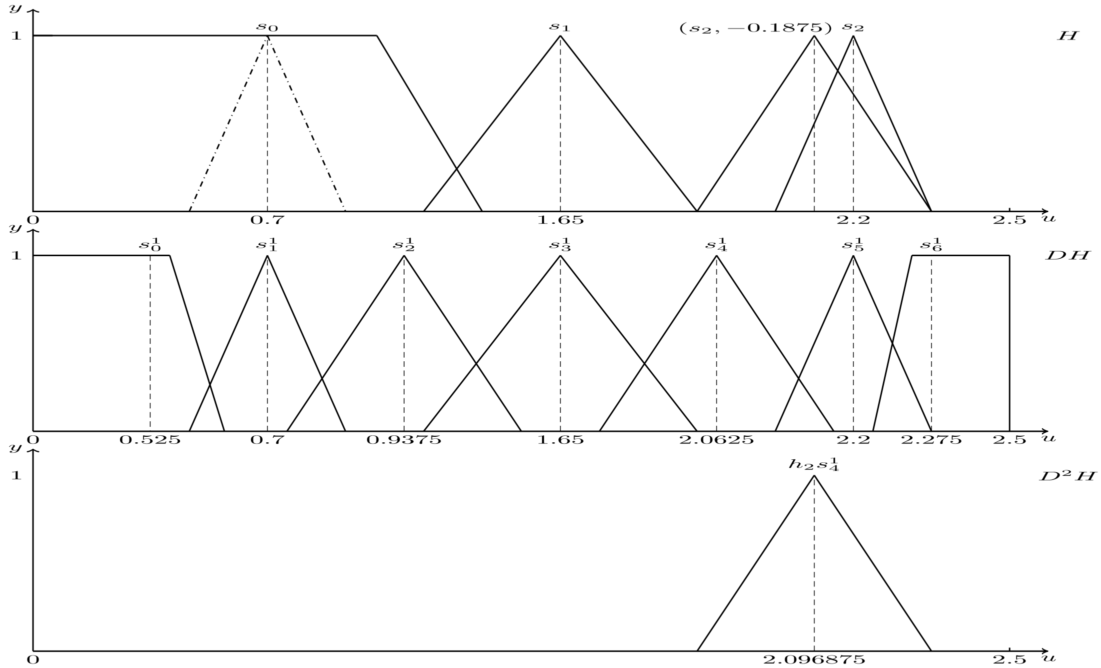

FIGURE 9. Linguistic terms H, DH, $h_2 s_4^1$ and $(s_2, -0.1875)$ with fuzzy sets.

TABLE 1. The evaluation information of projects by c_1, c_2, c_3 and c_4.

	$c_1(0.2)$	$c_2(0.3)$	$c_3(0.1)$	$c_4(0.4)$
a_1	$(0.2, 0.3, 0.4)$	s_5	$(0.3, 0.45, 0.6)$	s_3
a_2	s_6	$(0.4, 0.55, 0.7)$	s_3	s_4
a_3	$(0.1, 0.3, 0.5)$	$(0.4, 0.6, 0.8)$	$(0.7, 0.85, 1)$	s_5

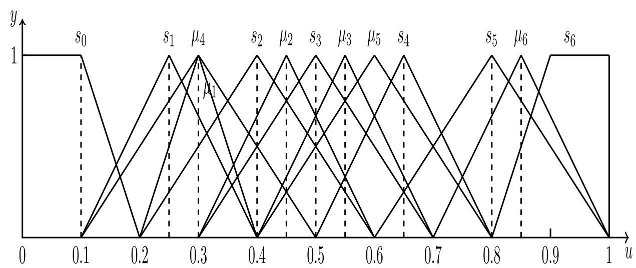

FIGURE 10. Fuzzy sets or linguistic terms on [0, 1].

Because the problem is a mixed decision making problem, it is necessary that evaluation information is unified by linguistic terms or fuzzy sets. "Unified by linguistic terms" means that linguistic decision making methods can be utilized to solve the problem. "Unified by fuzzy sets" means that decision making methods based on fuzzy sets can be utilized to solve the problem.

In the framework of 2-tuple fuzzy linguistic representation model, fuzzy set $\mu(u)$ can be transformed into 2-tuple linguistic terms on H by the transformation function [38], i.e., $T(\mu) = \{(s_i, \beta_i)|s_i \in H, \beta_i = max\{min\{\mu_{s_i}(u), \mu(u)\}|u \in U\}\}$ and $L(\mu) = \Delta(\frac{\Sigma_{i=0}^{g} i\beta_i}{\Sigma_{i=0}^{g}\beta_i}) = (s_j, \alpha)$, fuzzy sets of Table 1 transformed into 2-tuple linguistic terms on H are shown in Table 2. Then 2-tuple linguistic weighted aggregation operator is used to combine multiple criteria evaluation information and 2-tuple linguistic term result can be obtained, i.e., evaluation result of a_1 is $0.2 \times (s_1, 0.45) + 0.3 \times s_5 + 0.1 \times (s_3, -0.5) + 0.4 \times s_3 = (s_3, 0.24)$, evaluation results of a_2 and a_3 are $(s_4, 0.087)$ and $(s_4, -0.054)$. The most important project is a_2 due to $(s_4, 0.087) > (s_4, -0.054) > (s_3, 0.24)$.

In linguistic information processing via fuzzy sets of linguistic values, the aggregation operators of fuzzy sets is utilized to combine multiple criteria evaluation information, in the paper, the weighted aggregation operator of triangular fuzzy sets is

$$G(\{(\mu_i(u) = (a_i, b_i, c_i), w_i)|i = 1, \cdots, n\}) = (a, b, c),$$

where w_i is weight of triangular fuzzy set $\mu_i(u)$, $a = \sum_{i=1}^{n} w_i a_i$, $b = \sum_{i=1}^{n} w_i b_i$ and $c = \sum_{i=1}^{n} w_i c_i$, i.e., evaluation result of a_1 is $\mu_7(u) = (0.37, 0.545, 0.72)$ due to $0.37 = 0.2 \times 0.2 + 0.3 \times 0.6 + 0.1 \times 0.3 + 0.4 \times 0.3$, $0.545 = 0.2 \times 0.3 + 0.3 \times 0.8 + 0.1 \times 0.45 + 0.4 \times 0.5$ and $0.72 = 0.2 \times 0.4 + 0.3 \times 1 + 0.1 \times 0.6 + 0.4 \times 0.7$, evaluation results of a_2 and a_3 are $\mu_8(u) = (0.51, 0.655, 0.8)$ and $\mu_9(u) = (0.45, 0.645, 0.84)$. Then the transformation function [38] is utilized to transform triangular fuzzy sets results into 2-tuple linguistic terms, i.e., $T(\mu_7) = \{(s_1, 0.09), (s_2, 0.61), (s_3, 0.88), (s_4, 0.68), (s_5, 0.32)\}$ and $L(\mu_7) = (s_3, 0.26)$, $T(\mu_8) = \{(s_2, 0.05), (s_3, 0.12), (s_4, 0.19), (s_5, 0.55)\}$ and $L(\mu_8) = (s_4, 0.36)$, $T(\mu_9) = \{(s_2, 0.07), (s_3, 0.12), (s_4, 0.17), (s_5, 0.11), (s_6, 0.02)\}$ and $L(\mu_9) = (s_4, -0.22)$. The most important project is a_2 due to $(s_4, 0.36) > (s_4, -0.22) > (s_3, 0.26)$.

TABLE 2. Formal 2-tuple linguistic concepts of fuzzy sets.

Fuzzy set	$T(\mu)$	2-tuple linguistic term	$E(\mu)$	formal 2-tuple linguistic concept
$\mu_1(u)$	$\{(s_1, 0.8), (s_2, 0.66)\}$	$(s_1, 0.45)$	0.3	$([\mu_1(u)], \{(s_1, \frac{1}{3})\})$
$\mu_2(u)$	$\{(s_1, 0.33), (s_2, 0.86), (s_3, 0.86), (s_4, 0.33)\}$	$(s_3, -0.5)$	0.45	$([\mu_2(u)], \{(s_3, -0.5)\})$
$\mu_3(u)$	$\{(s_2, 0.57), (s_3, 0.85), (s_4, 0.67), (s_5, 0.29)\}$	$(s_3, 0.29)$	0.55	$([\mu_3(u)], \{(s_3, \frac{1}{3})\})$
$\mu_4(u)$	$\{(s_0, 0.08), (s_1, 0.85), (s_2, 0.75), (s_3, 0.5)\}$	$(s_2, -0.17)$	0.3	$([\mu_4(u)], \{(s_1, \frac{1}{3})\})$
$\mu_5(u)$	$\{(s_2, 0.5), (s_3, 0.75), (s_4, 0.86), (s_5, 0.5)\}$	$(s_4, -0.48)$	0.6	$([\mu_5(u)], \{(s_4, -0.5)\})$
$\mu_6(u)$	$\{(s_4, 0.33), (s_5, 0.86), (s_6, 0.8)\}$	$(s_5, 0.24)$	0.85	$([\mu_6(u)], \{(s_6, -0.5)\})$

In formal linguistic concepts, according to Eqs.(3)-(7) and Definition 4, triangular fuzzy sets or linguistic terms of Table 1 are unified by formal 2-tuple linguistic concepts shown in Table 2. Then 2-tuple linguistic weighted aggregation operator is used to combine intensions of formal 2-tuple linguistic concepts, i.e., evaluation result of a_1 is $0.2 \times (s_1, \frac{1}{3}) + 0.3 \times s_5 + 0.1 \times (s_3, -0.5) + 0.4 \times s_3 \doteq (s_3, 0.22)$, evaluation results of a_2 and a_3 are $(s_4, 0.10)$ and $(s_4, -0.15)$. The most important project is a_2 due to $(s_4, 0.10) > (s_4, -0.15) > (s_3, 0.22)$. If linguistic hedges $D_2 = \{$rather, much, extremely, significantly$\}$ are utilized to construct and manage multi-granular linguistic values on H, 2-tuple linguistic term $(s_4, 0.10)$ is "rather good", according to Eq.(15), formal 2-tuple linguistic concept $([\mu(u)], \{(s_4, 0.10)\})$ is the generated formal linguistic concept $([\mu(u)], \{$rather good$\})$ in D_2H, $[\mu(u)]$ with $E(\mu) = (1 - 0.10) \times 0.65 + 0.10 \times 0.8 = 0.665$.

VII. CONCLUSIONS

Constructing and managing multi-granular linguistic values via formal linguistic concepts have been proposed in the paper, a formal linguistic concept is a linguistic value, an equivalence class of fuzzy sets on the universe is used to represent extension and linguistic term is used to describe intension of the linguistic value. Then new symbolic translation and transformation function have been provided to construct formal 2-tuple linguistic concepts and manage multi-granular formal linguistic concepts in the hierarchy of generated formal linguistic concepts. Compared with symbolic approaches or fuzzy set methods in processing linguistic information, advantages of symbolic approaches and fuzzy set methods in linguistic information processing are combined in the method based on formal linguistic concept, *i.e.*, fuzzy sets and linguistic term of linguistic value is entity, continuous linguistic domain, easy computation, without loss of information on symbolic linguistic terms can be finished by intensions of linguistic values and meaning of symbolic result can be represented by extension and generated natural or artificial linguistic term, in addition, symbolic translation and transformation function based on $E(\mu)$ of fuzzy set $\mu(u)$ are easier than in symbolic or fuzzy set methods, which needs complex calculation between fuzzy sets $\mu(u)$ and $\{\mu_0(u), \cdots, \mu_g(u)\}$, these can be seen from Examples 4 and 5. It seems that constructing and managing multi-granular linguistic values via formal linguistic concepts can overcome drawbacks of symbolic approaches and fuzzy set methods in linguistic information processing.

Up to now, there are many equivalence relations on fuzzy sets, combining them with linguistic hedges and linguistic terms to construct and manage multi-granular linguistic values are our future works.

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