Transportation Research Part E 130 (2019) 95–107

Contents lists available at ScienceDirect

Transportation Research Part E

journal homepage: www.elsevier.com/locate/tre

Optimal advertisement budget allocation and coordination in luxury fashion supply chains with multiple brand-tier products☆,☆☆

Tsan-Ming Choi[a], Na Liu[b,*]

[a] Business Division, Institute of Textiles and Clothing, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
[b] Business School, Beijing Institute of Fashion Technology, Beijing, China

ARTICLE INFO

Keywords:
Multi-product supply chain coordination
Advertisement
Luxury fashion products
Different brand tiers
Risk attitudes
Market share

ABSTRACT

Advertisement is critical in luxury fashion supply chains. In this paper, we analytically explore the optimal advertisement budget allocation strategy and coordination challenge when there are multiple brand-tier products in the market. In the basic model, we focus our analyses with respect to different risk attitudes that the luxury fashion brand takes and the coordination mechanisms. We show that irrespective of risk attitudes of the luxury fashion brand, the optimal advertisement strategy is a polarized strategy. We derive the coordination mechanisms to overcome the double marginalization effect for each risk attitude case. In the extended model, with the market share considerations, the optimal advertisement strategy is derived and is shown to be no longer polarized in general.

1. Introduction

1.1. Background and motivation

The luxury fashion industry is a very big global business. LVMH, which is commonly deemed as one of the world's most leading enterprises for luxury fashion brands (including the famous brand Louis Vuitton), achieved a total revenue of €46.8 billion in 2018. Table 1 shows the geographic business "footprint" of the LVMH group as of December 2018 (data extracted from the 2018 LVMH annual report). As we can see from Table 1, the business of LVMH has expanded all over the world by the end of 2018, and the total number of stores has exceeded 4000. In particular, LVMH's business not only focuses on the well-developed countries like the US, Europe and Japan, but also covers many Asian countries, including developing countries like China.

If we take a close look at different luxury fashion brands, we will see that they offer a lot of different products (or product lines) which target at different market segments. For example, in Louis Vuitton, the handbag of *Neverfull MM* is sold at a retail price from HKD10500 to HKD16300,[1] whereas the retail price of the handbag of *Petite Malle* is from HKD 42500 to HKD 179000.[2] It is evident

☆ Tsan-Ming Choi's research is partially supported by RGC(HK) under the GRF funding scheme (152294/16E). Na Liu's research is partially supported by Beijing Municipal Education Commission in China under project of CIT&TCD201804043.
☆☆ We sincerely thank the guest editor and three reviewers for their critical and helpful comments on this paper.
* Corresponding author.
E-mail address: sxyln@bift.edu.cn (N. Liu).
[1] https://hk.louisvuitton.com/eng-hk/search/neverful [Accessed on 12 Jul 2019].
[2] https://hk.louisvuitton.com/eng-hk/search/petite%20malle [Accessed on 12 Jul 2019].

https://doi.org/10.1016/j.tre.2019.08.009
Received 13 July 2019; Received in revised form 14 August 2019; Accepted 16 August 2019
Available online 06 September 2019
1366-5545/

Table 1
The geographic coverage and business scales of LVMH (as of December 2018).[1]

Location	Revenue	No. of Stores	No. of Employees
Asia (excluding Japan)	EURO 13,723 Million	1289	34,802
Japan	EURO 3351 Million	422	6905
Europe (excluding France)	EURO 8731 Million	1153	38,645
France	EURO 4491 Million	514	31,156
USA	EURO 11,207 Million	783	32,724
Other markets	EURO 5323 Million	431	11,856

[1] https://r.lvmh-static.com/uploads/2019/03/rapport-annuel-lvmh-2018_va.pdf [Accessed on 12 Jul 2019].

that the former product is a lower-tier product and the later one is a higher-tier product. They target at different market segments. This kind of multiple brand-tier product offering strategy is widely observed in the luxury fashion industry. This paper also examines this particular commonly observed industrial practice.

Note that the higher-tier (or "higher-end") luxury products are usually more expensive compared to the lower-tier (or "lower-end") luxury products counterpart. Thus, in this paper, the higher-tier and lower-tier products can be differentiated mainly with respect to the price levels.

It is also important to observe that the market demands for the higher-tier and the lower-tier products are related. To be specific, if a larger amount of the lower-tier product is expected to be sold in the market, the higher-tier product's demand will drop as the higher-tier consumers may feel the luxury fashion brand is less prestigious (i.e. a negative effect exists). However, on the contrary, if a larger amount of the higher-tier product is expected to be sold, the lower-tier product's demand will increase as the lower-tier consumers may perceive that the luxury fashion brand has a higher status with respect to the selling price of the lower-tier product. Thus, a positive influence exists.

The luxury fashion brands also face uncertainty when they offer the higher-tier and the lower-tier products. Given that the luxury fashion business is spending a huge amount of money for advertisement (Chiu et al., 2018), the corresponding budget allocation is a highly risky decision. It is commonly known that operations managers in practice can possess the risk averse, risk neutral or risk seeking attitude (Choi et al., 2008b, 2018c). As a result, it is important to incorporate the risk attitude of the operations manager into the decision making problem so that the optimal decision is made with respect to it.

In addition to product lines, in luxury fashion supply chains, a few critical operational issues are present. First, the way to increase demands for the luxury fashion products is crucial. Second, to enhance the efficiency of the supply chain with different brand-tier products is challenging and important. It is commonly known that for luxury fashion brands, pricing is more a strategic than operational issue. Many luxury fashion brands, like YSL, basically keep the price constant across seasons (see Yoganarasimhan, 2012; Chiu et al., 2018). Thus, in order to increase demand, advertisement becomes the most crucial tool. As reported, the famous luxury fashion groups like LVMH and Kering spent a huge amount of money every year solely on advertisement[3] (Chiu et al., 2018). Thus, how to spend the advertisement budget across different product lines is an important and practical operational issue that the luxury fashion brand has to decide season by season. Furthermore, for luxury fashion brands, market share is a critical concern and most luxury fashion brands would like to have a substantial demand for their products in order to show their presence in the luxury fashion business world. Last but not least, many luxury fashion brands have hired their own factories for production. In supply chain contracting, this is equivalent to the deployment of the commonly seen two-part-tariff contract, in which the retailer (i.e. the luxury fashion brand) will pay the manufacturer (i.e., the factory) a fixed lump sum of money just like the "salary" and also a per unit wholesale price of the product.

Table 2 shows the common features of a typical luxury fashion supply chain. To save space, we do not repeat the description here. In this paper, we will build formal analytical models to capture these features and theoretically explore the challenges in luxury fashion supply chain management.

Motivated by the importance of optimal advertisement budget allocation problem for the luxury fashion brand with different brand-tier products as well as the importance of achieving coordination for the luxury fashion supply chain, we build formal analytical models and conduct analyses with an aim to address the following three research questions:

1. What is the optimal advertisement strategy (in terms of advertisement budget allocation) for the risk sensitive luxury fashion brand with both a higher-tier (more expensive) and a lower-tier (less expensive) product lines?
2. What is the channel coordination scheme for the two-product-lines luxury fashion supply chain? If the luxury fashion brand is risk averse or risk seeking, what will be the impacts on the optimal advertisement strategy and the channel coordination schemes?
3. If the luxury fashion brand has the minimum market share target (i.e. expected demand target) in mind, how would it affect the optimal advertisement strategy and coordination scheme?

Addressing the above research questions yields various important insights. First, under the basic model in which we do not consider the market share issue, we prove that irrespective of the specific risk attitudes of the luxury fashion brand, the optimal

[3] http://www.thefashionlaw.com/home/lvmh-kering-and-chanel-slash-print-advertising-as-digital-reigns-supreme [Accessed on 12 Jul 2019].

Table 2
Features of luxury fashion supply chains.

	Details	Examples/References
Product lines	Different product lines. Usually a higher-tier and a lower-tier product lines	In Louis Vuitton, the *Neverfull MM* handbag is a lower-tier product whereas the *Petite Malle* handbag is a higher-tier product.
Increasing demand (operational wise)	By advertisement, but not by pricing because pricing is a strategic decision	This is known in the literature that huge budgets are allocated to advertisement in luxury fashion brands (see, e.g., Chiu et al. 2018), and also evidenced from real world practices that many luxury brands focus on advertisement to increase demand. For example, in 2016, Dior launched an advertisement and got a very effective result[a]
Advertising budget	Huge	LVMH spent $216.3 million for advertisement budget in print magazines in 2017, and Kering spent $97.2 million in 2017[b]
Market share consideration	Important	LVMH emphasizes on the importance of market share targets for most of its brands (e.g. Parfums Christian Dior, Guerlain, Bvlgari, Chaumet, TAG Heuer, etc) in the annual report[c]
Supply chain contracting	Many factories are hired by the luxury fashion brand (and hence the two-part-tariff contract is common)	Louis Vuitton has 16 French leather workshops by early 2019 to serve its rapidly growing demand[d] Burberry has two factories in UK and the third one in Italy is being developed[e] Prada owns 22 factories so far[f]

[a] http://www.bain.cn/pdfs/201801170740528448.pdf [Accessed on 12 Jul 2019].
[b] http://www.thefashionlaw.com/home/lvmh-kering-and-chanel-slash-print-advertising-as-digital-reigns-supreme [Accessed on 12 Jul 2019].
[c] https://r.lvmh-static.com/uploads/2019/03/rapport-annuel-lvmh-2018_va.pdf.
[d] https://fashionunited.uk/news/fashion/louis-vuitton-to-open-new-factories-in-france-to-meet-demand-for-leather-goods/2018032828870 [Accessed on 12 Jul 2019].
[e] https://www.burberryplc.com/content/dam/burberry/corporate/oar/documents/annual-report.pdf [Accessed on 31 Jul 2019].
[f] https://www.pradagroup.com/en/group/group-profile.html [Accessed on 31 Jul 2019].

advertisement strategy is a polarized strategy. Second, we derive the two-part-tariff based coordination mechanisms to overcome the double marginalization effect in the luxury fashion supply chain with both a higher-tier and a lower-tier products. We show analytically how to achieve coordination for each risk attitude case, and a more sophisticated contract is needed for the case when the luxury fashion brand is risk averse or risk seeking. Third, in the extended model, with the market share considerations, we find that the optimal advertisement strategy is in general un-polarized. Managerial implications are discussed.

As a remark, advertisement in the fast fashion companies is also an important topic. However, compared to luxury fashion, we believe that advertisement is relatively less important to fast fashion brands as they could easily match supply and demand by changing prices whereas luxury fashion cannot and seldom do so. Advertising is known to be critical in building the luxury brand, and helps develop a closer relationship between the luxury brand and consumers (Brioschi, 2006). It is confirmed in Freire (2014) that advertisements, including other marketing strategies, result in more and more loyal consumers because the brand image, which is especially crucial in luxury fashion, can be enhanced accordingly. As a real world evidence, fast fashion brands like Zara are also known to intentionally put very limited resources on advertisement (Ghemawat et al., 2003). In 2018, Zara only spent €352 million, including communications and other operating expenses, compared to the total sales revenue of €26,145 million.[4]

1.2. Contribution statements and organization

To the best of our knowledge, this is the first paper which analytically explores the optimal advertisement budget allocation and coordination challenge in the luxury fashion supply chain in the presence of two different brand-tier products. The paper also considers the presence of a luxury fashion brand with different risk attitudes as well as explores the impacts of market share constraints. Many novel managerial insights are developed which not only contribute to the supply chain management literature but also provide important managerial guidance to the managers of luxury fashion brands.

The remaining parts of this paper are organized as follows. We review the related literature in Section 2. We build the basic model for the luxury fashion supply chain in Section 3. We derive the optimal advertisement budget allocation decision and develop the channel coordination mechanisms in Section 4. We extend the analysis to the case when the luxury fashion brand considers the market share constraints in Section 5. We conclude this paper with a summary of findings, a discussion of managerial insights as well as a future research agenda in Section 6. To enhance exposition and readability, all technical proofs and detailed analytical derivations are all placed in Appendix (A), and a list of notation is shown in Table 3.

2. Related literature

This paper relates to three streams of research, namely advertisement in supply chains, luxury supply chain operations, and supply chain management with risk sensitive agents. We review some related studies as follows.

In operations management, optimal advertisement strategies in supply chains have been explored by numerous studies (Li et al.,

[4] https://static.inditex.com/annual_report_2018/en/2018-data.html [Accessed on 31 Jul 2019].

Table 3
The list of notation.

Notation	Meaning
H	Higher-tier product
L	Lower-tier product
RA	Risk averse
RN	Risk neutral
RS	Risk seeking
$\widetilde{D}_H$	Demand function of the higher-tier product
$\widetilde{D}_L$	Demand function of the lower-tier product
$\widetilde{\varepsilon}_H$	Independent random variables with mean 1 and standard deviation of σ_H
$\widetilde{\varepsilon}_L$	Independent random variables with mean 1 and standard deviation of σ_L
m_i	Unit production cost for product $i \in (H, L)$
c_i	Unit wholesale price for product $i \in (H, L)$
p_i	Unit retail selling price for product $i \in (H, L)$
λ	The proportion of advertisement budget spent on the higher-tier product
$\widetilde{\Pi}_i$	The profit function of product $i \in (H, L)$
$E[\widetilde{\Pi}_i]$	The expected profit function of product $i \in (H, L)$
$SD[\widetilde{\Pi}_i]$	The standard deviation of profit function of product $i \in (H, L)$
O_i	The mean-standard-deviation (MSD) objective function of the product $i \in (H, L)$

2019). For example, Sethi (1977) proposes the pioneering optimal advertising model with the consideration of budget constraints. He et al. (2009) apply the feedback control theory to study the optimal dynamic cooperative policy in the supply chain. Wu et al. (2011) explore the classic newsvendor inventory problem with the considerations of advertising decisions. Zheng et al. (2012) analytically investigate the optimal advertisement decisions in luxury fashion brands with the considerations of social influence effects. Chiu et al. (2018) examine the single-product optimal advertisement budget allocation for two market segments. The authors employ the mean-variance theory to explore the efficient frontiers for the optimization problem in the mean-variance space. Wang et al. (2018) study the advertisement policies via the use of "mobile apps". Zhang et al. (2018) explore the advertising decision in the presence of a competitive market environment. Most recently, Niu et al. (2019) investigate the effect of free-trial and advertisement for newly launched products. The authors interestingly show that the presence of competition may lead to a price war if the companies offer free-trial. Similar to the above studies, this paper also focuses on examining the optimal advertisement decision. However, different from all of them in terms of the perspective and model setting, this paper focuses on the case when there are two different brand-tier products and the luxury fashion brand can take up any risk attitude.

In luxury supply chain management, a lot of studies examine the conspicuous consumer behaviours (Choi and Shen, 2017). For example, Amaldoss and Jain (2005) conduct a scientific study on conspicuous consumer behaviours. Amaldoss and Jain (2015) analytically investigate the effects of market competition and social mutual influences for the optimal branding decision for conspicuous products. Straker and Wrigley (2016) conduct a case study on Burberry and report how to get the consumers involved via digital means. Amatulli et al. (2018) carry out a consumer study to see how consumers respond to the corporate social responsibility programs offered by luxury brands. For supply chain strategies, Moore and Birtwistle (2004) conduct a case study of Burberry's supply chain development. Phan et al. (2011) report a study on how social media operations are related to the management of luxury brands. Castelli and Sianesi (2015) propose methods to establish the list of critical factors which can ensure successful alignment of incentives in luxury supply chains. Brun et al. (2017) propose a strategy to manage luxury fashion supply chains. Shen et al. (2017a) explore the co-alliance strategies for luxury fashion brands. Shen et al. (2017b) investigate the social influence effects on luxury fashion operations. Choi (2019) proposes the use of blockchain technology to enhance diamond authentication in luxury supply chains. Some studies are devoted to the production side of luxury supply chains. For instance, Gregori and Marcone (2018) explore the production and operations management activities in luxury knit-wear companies. Similar to the above reviewed studies on luxury business operations, this paper also explores the luxury sector. However, different from all of them, this paper focuses on (i) highlighting the risk attitudes of the luxury fashion brand would affect the respective optimal decisions, and (ii) uncovering the channel coordination mechanisms, with reference to the observed industrial practices.

In recent years, risk analytics (Wu et al., 2017) and supply chain management with risk sensitive agents are very popular. In the literature, earlier studies such as Lau and Lau (1999), Gan et al. (2005), Choi et al. (2008a), and Choi et al. (2008b) all explore the channel coordination issue in two-echelon supply chains with risk sensitive agents. More recently, Tekin and Ozekici (2015) investigate the newsvendor supply chain with financial hedging under the mean-variance framework. Zhang et al. (2018) explore the quick response supply chain when the retailer is stochastically risk sensitive. Zhang et al. (2019) extend the prior studies on risk averse newsvendors under the mean-variance domain to the case with mean-variance-skewness-kurtosis domain. This paper follows this stream of research and also models the risk attitudes of the luxury fashion brand by using the mean-risk approach. See Chiu and Choi (2016) and Choi et al. (2019) for the reviews of some more related studies in the literature.

3. Basic model: luxury fashion supply chains

A luxury fashion brand offers two different tiers products, namely a higher-tier product and a lower-tier product. The luxury fashion brand needs to decide the optimal allocation of advertisement budget between the two products. We represent the amount of advertisement budget, given as a proportion, to be spent on the higher-tier product by λ, where $0 \leqslant \lambda \leqslant 1$. The remaining amount of advertisement budget, i.e. $(1-\lambda)$, is spent on the lower-tier product. Observe that for this problem, the advertisement cost is not explicitly included in the model because we consider the case when the advertisement budget is given and fixed (P.S.: Just like for many luxury fashion brands, there is always an annual budget for advertisement). The optimal decision is hence made with respect to how to divide the budget between the two product lines. Table 3 concisely shows a list of notation which would be used throughout this paper.

Demands of the two products ($\widetilde{D}_H$ denotes demand of the higher-tier product, and $\widetilde{D}_L$ denotes demand of the lower-tier product, where the subscripts H and L represent "higher" and "lower" tier products, respectively) are inter-related which exhibit a form similar to the model considerations in Chiu et al. (2018)[5]:

$$\widetilde{D}_H = (a + b\lambda - \gamma E[\widetilde{D}_L])\widetilde{\varepsilon}_H, \tag{3.1}$$

$$\widetilde{D}_L = (\alpha + \beta(1-\lambda) + \xi E[\widetilde{D}_H])\widetilde{\varepsilon}_L, \tag{3.2}$$

where $a,\ \alpha,\ b,\ \beta,\ \gamma,\ \xi > 0$; $\widetilde{\varepsilon}_H$ and $\widetilde{\varepsilon}_L$ are independent random variables with mean = 1 and standard deviation of σ_H and σ_L, respectively. In this paper, we use the notation with a tilde $\sim$ to represent a random variable, $E[\cdot]$ to denote expectation, and $SD[\cdot]$ to represent standard deviation.

Observe that the demand functions in (3.1) and (3.2) include the coefficients which represent the inter-relationships between demands of the higher-tier and the lower-tier products. In reality, they can be estimated by using the big data approach (Choi et al., 2018a), such as the use of machine intelligence (Sun et al., 2008) or multi-dimensional statistical methods (Ren et al., 2015).

It is straightforward to derive Lemma 3.1 which analytically shows the structural properties of the demand functions, including the expected demand and standard deviation of demand of each product.

Lemma 3.1. *(a)* $E[\widetilde{D}_L] = \frac{\alpha+\beta(1-\lambda)}{1+\gamma\xi} + \frac{\xi(a+b\lambda)}{1+\gamma\xi}$, $E[\widetilde{D}_H] = \frac{a+b\lambda}{1+\gamma\xi} - \frac{\gamma(\alpha+\beta(1-\lambda))}{1+\gamma\xi}$. *(b)* $SD[\widetilde{D}_L] = \left(\frac{\alpha+\beta(1-\lambda)}{1+\gamma\xi} + \frac{\xi(a+b\lambda)}{1+\gamma\xi}\right)\sigma_L$, $SD[\widetilde{D}_H] = \left(\frac{a+b\lambda}{1+\gamma\xi} - \frac{\gamma(\alpha+\beta(1-\lambda))}{1+\gamma\xi}\right)\sigma_H$.

Lemma 3.1 indicates that both the expected demand and standard deviation of demand can be expressed as a linear function of λ. This feature is critical in deriving the optimal advertisement budget allocation strategy.

For the higher-tier and lower-tier products, the unit production costs at the factory are given to be m_H and m_L, respectively. The unit wholesale prices offered by the factory to the luxury fashion brand are respectively c_H and c_L, and the luxury fashion brand sets the corresponding unit retail prices to be p_H and p_L. For each product $i \in (H, L)$: The profit function, expected profit function, standard deviation of profit function, and the mean-standard-deviation (MSD) objective function of the luxury fashion brand which sells the two products are given as follows:

$$\widetilde{\Pi}_i = (p_i - c_i)\widetilde{D}_i, \tag{3.3}$$

$$E[\widetilde{\Pi}_i] = (p_i - c_i)E[\widetilde{D}_i], \tag{3.4}$$

$$SD[\widetilde{\Pi}_i] = (p_i - c_i)SD[\widetilde{D}_i], \tag{3.5}$$

$$O_i = E[\widetilde{\Pi}_i] - kSD[\widetilde{\Pi}_i]. \tag{3.6}$$

Note that the profit function (3.3) is closely related to and "follows" the demand function. The demand function is derived from the original demand functions given by (3.1) and (3.2), which is also based on Chiu et al. (2018). The mutual inter-dependent demand relationships would be the key to explain the findings in the basic model. Moreover, in (3.6), k captures the risk attitude of the luxury fashion brand. When k is positive, the luxury fashion brand is risk averse as a higher standard deviation of profit yields a smaller objective function value O_i.[6] When k is negative, the luxury fashion brand is risk seeking. When k is 0, the luxury fashion brand is risk neutral and O_i is equal to $E[\widetilde{\Pi}_i]$. In this paper, we denote risk averse, risk neutral, and risk seeking by RA, RN, and RS, respectively.

[5] Despite being similar, the scenarios captured by the demand model in this paper and Chiu et al. (2018) are totally different. In this paper, we consider the scenario with two products (and hence two profit margins, etc.) whereas Chiu et al. (2018) consider one product with two different market segments. Furthermore, Chiu et al. (2018) focus on exploring the boundary cases which are not considered in this paper as we examine the more common case in which product demands are positive. In addition, this paper investigates supply chain coordination challenges and considers the impact of market share, which are not examined in Chiu et al. (2018).

[6] Note that the way we define the MSD objective function mainly considers the analytical tractability issue. We may also try other definition for it but we may not be able to derive closed-form analytical results.

Define:

$$O_{MSD}(\lambda) = \sum_{i\in(H,L)} E[\widetilde{\Pi}_i] - kSD[\widetilde{\Pi}_i]. \tag{3.7}$$

Observe that (3.7) defines the objective function for the luxury fashion brand with the considerations of both the expected profit and standard deviation of profit for each product (Tsiang, 1972). By maximizing $O_{MSD}(\lambda)$ with respect to λ, the optimal advertisement budget allocation problem can be found. As a remark, there are different approaches in modeling decision making under different risk attitudes. We adopt this objective function as it is intuitive, implementable, analytically tractable and follows the classical MSD framework (Tsiang, 1972; Lau, 1980). It is easy to find that $O_{MSD}(\lambda)$ can be rewritten as follows:

$$O_{MSD}(\lambda) = r_H\left(\frac{a+b\lambda}{1+\gamma\xi} - \frac{\gamma(\alpha+\beta(1-\lambda))}{1+\gamma\xi}\right) + r_L\left(\frac{\alpha+\beta(1-\lambda)}{1+\gamma\xi} + \frac{\xi(a+b\lambda)}{1+\gamma\xi}\right), \tag{3.8}$$

where

$$r_i = (p_i - c_i)(1 - k\sigma_i), \quad \text{for} \quad i \in (H, L). \tag{3.9}$$

Note that in (3.8) and (3.9), k reflects the risk attitude of the luxury fashion brand, $(p_i - c_i)$ is the profit margin for product $i \in (H, L)$, and σ_i captures demand uncertainty of product $i \in (H, L)$. In this paper, for the case when the luxury fashion brand is risk averse, we assume that $1 - k\sigma_i > 0$ which means the degree of risk aversion will not affect the brand's final product offering decision (P.S.: If $1 - k\sigma_i \leqslant 0$, then $r_i \leqslant 0$ which means offering product $i \in (H, L)$ is not beneficial).

4. Optimal advertisement strategies and coordination

4.1. Optimal advertisement budget allocation strategy

With the luxury fashion supply chain model developed in Section 3, we now proceed to derive the optimal advertisement strategy and explore the coordination challenge. Define the following notation:

$$\hat{\gamma} = \frac{r_L}{r_H}\left(1 - \frac{b\xi}{\beta}\right) - \frac{b}{\beta}, \tag{4.1}$$

$$\hat{\xi} = \frac{\beta}{b} - \frac{r_H}{r_L}\left(1 + \frac{\beta\gamma}{b}\right). \tag{4.2}$$

For (4.1) and (4.2), they are the critical thresholds which are important to help define the optimal advertisement budget allocation strategy as shown in Proposition 4.1.

Proposition 4.1.. *In the basic model, irrespective of the luxury fashion brand's risk attitude, under the MSD objective, the optimal advertisement budget allocation strategy is a polarized strategy (i.e., either $\lambda = 0$ or $\lambda = 1$). (a) If $\gamma \geqslant \hat{\gamma}$ or $\xi \geqslant \hat{\xi}$, then the optimal advertisement budget allocation strategy is $\lambda = 1$. (b) If $\gamma < \hat{\gamma}$ or $\xi < \hat{\xi}$, then the optimal advertisement strategy is $\lambda = 0$.*

Proposition 4.1 indicates that under the basic model, no matter whether the luxury fashion brand is risk averse, risk neutral or risk seeking, the optimal advertisement budget allocation decision is always a polarized strategy. To be specific, the decision rule can be interpreted in two ways, one with respect to γ and one with respect to ξ. If we take a look at (3.1) and (3.2), we will find that γ and ξ are both important and they capture the dependence of the two products' demands. To be specific, γ measures how much the demand of the lower-tier product negatively affects the demand of the higher-tier product. ξ captures how much the demand of the higher-tier product positively influences the demand of the lower-tier product. Proposition 4.1 shows that if either γ or ξ is sufficiently large (with respect to the corresponding critical thresholds $\hat{\gamma}$ and $\hat{\xi}$), then the optimal strategy is to allocate all advertisement budget to the higher-tier product. If either γ or ξ is sufficiently small (with respect to the corresponding critical thresholds $\hat{\gamma}$ and $\hat{\xi}$), then the optimal strategy is to allocate all advertisement budget to the lower-tier product. The result is in fact rather intuitive and logical because if the influence between the two products' demands is big enough (i.e. either γ or ξ is sufficiently large), advertising solely towards the higher-tier product can serve two purposes: (i) directly increasing the demand for the higher-tier product, and (ii) indirectly increasing the demand for the lower-tier product in a significant way (as either γ or ξ is sufficiently large). This is in fact optimal. One important point, which is less intuitive, is that the polarized strategy is independent of the risk attitude of the luxury fashion brand under the MSD framework. This result is different from the diversification concept in finance as well as in market segmentation risk analysis (see Chiu et al., 2018). Exploring further on $\hat{\gamma}$ and $\hat{\xi}$, we have Table 4.

Table 4
Sensitivity analyses on $\hat{\gamma}$ and $\hat{\xi}$ (↑ = increase; ↓ = decrease; − = no change).

	$\hat{\gamma}$	$\hat{\xi}$
r_L↑	↑if$\frac{b\xi}{\beta} < 1$	↑
	−if$\frac{b\xi}{\beta} = 1$	
	↓if$\frac{b\xi}{\beta} > 1$	
r_H↑	↓if$\frac{b\xi}{\beta} < 1$	↓
	−if$\frac{b\xi}{\beta} = 1$	
	↑if$\frac{b\xi}{\beta} > 1$	
b↑	↓	↓if$\frac{r_L}{r_H} \geqslant \gamma$
		↑if$\frac{r_L}{r_H} < \gamma$
β↑	↑	↑if$\frac{r_L}{r_H} \geqslant \gamma$
		↓if$\frac{r_L}{r_H} < \gamma$

Table 4 shows various interesting findings. First of all, how the critical thresholds are being affected by the changes of different parameters is not necessarily monotonic. In many cases, there are other mediating factors such as $\frac{b\xi}{\beta}$ for the case with $\hat{\gamma}$, and "$\frac{r_L}{r_H}$ versus γ" for the case with $\hat{\xi}$. As $\frac{r_L}{r_H}$ depends on the risk attitude of the luxury fashion brand, how $\hat{\xi}$ is being affected when b and β vary hence also relate to the risk attitude of the luxury fashion brand. However, the result not only depends the risk attitude type (i.e., risk averse, risk neutral, or risk seeking) but also the magnitude of the respective risk attitude (i.e., the value of k). The result is hence a mix of these factors together, which is complicated in general.

4.2. Channel coordination

Channel coordination in a supply chain (Moon and Feng, 2017; Choi et al., 2018b; Shen et al., 2019), also known as "supply chain coordination", commonly refers to the case in which individual supply chain members make a decision which is equal to the "optimal decision in the centralized supply chain" when the supply chain (expected) profitability is maximized (Choi et al., 2013; Chan et al., 2018b; Choi and Guo, 2018).[7] To develop the channel coordination mechanisms, we define the following:

$$r_{SC,i} = (p_i - m_i), \quad \text{for} \quad i \in (H, L), \tag{4.3}$$

$$\hat{\gamma}_{SC} = \frac{r_{SC,L}}{r_{SC,H}}\left(1 - \frac{b\xi}{\beta}\right) - \frac{b}{\beta}, \tag{4.4}$$

$$\hat{\xi}_{SC} = \frac{\beta}{b} - \frac{r_{SC,H}}{r_{SC,L}}\left(1 + \frac{\beta\gamma}{b}\right). \tag{4.5}$$

In the supply chain we considered, there is an upstream factory which produces for the luxury fashion brand. Owing to the double marginalization problem, the optimal advertisement budget allocation strategies for the supply chain and the luxury fashion brand are in general not the same, as shown in Lemma 4.1.

Lemma 4.1.. *For the supply chain system, denote the optimal λ by λ^*_{SC}: (a) If $\gamma \geqslant \hat{\gamma}_{SC}$ or $\xi \geqslant \hat{\xi}_{SC}$, then the optimal advertisement budget allocation strategy is $\lambda^*_{SC} = 1$. If $\gamma < \hat{\gamma}_{SC}$ or $\xi < \hat{\xi}_{SC}$, then the optimal advertisement budget allocation strategy is $\lambda^*_{SC} = 0$. (b) In general, $r_{SC,i} \neq r_i$ for $i \in (H, L)$, which implies $\hat{\gamma}_{SC} \neq \hat{\gamma}$ and $\hat{\xi}_{SC} \neq \hat{\xi}$, and hence the supply chain will not be coordinated by itself automatically.*

Lemma 4.1 indicates that in general, the luxury fashion supply chain is not coordinated in a decentralized setting. In order to achieve channel coordination, we need to consider separate cases. First of all, we need to define the participation constraints. To be specific, in the luxury fashion supply chain, the factory is willing to contract with and produce for the luxury fashion brand if it can at least achieve an expected profit of J_M (across the two products). Similarly, the luxury fashion brand has a minimum expected profit

[7] For supply chain coordination, in this paper, we focus on the classic definition which refers to the achievability of supply chain optimal expected profit. There are two other measures, namely the Pareto optimality concept (Chiu and Choi, 2016), and the proposal to achieve the goal of the supply chain central coordinator who has its own risk preference (Choi et al., 2008a). In this paper, we select the one for supply chain's expected profit maximization because: (i) It is really optimal in the sense that if the supply chain operates for a long time, the expected profit will become the real average profit. (ii) Considering the other optimization objectives will complicate the analyses and we will have lots of other cases. As coordination is just a part of this paper, we would vote for the expected profit maximization case for neatness of results and analyses.

target of J_{LFB} (across the two products). When the luxury fashion supply chain's expected profit is maximized, the respective expected profit is denoted by Π_{SC*}. In this paper, channel coordination (or "coordination" in short) refers to the case when the luxury fashion supply chain's expected profit is maximized while the factory and the luxury fashion brand have their minimum expected profit targets being met.

For a notational purpose, we define the following:

$$\bar{D}^*_{SC,i} = E[\widetilde{D}_i(\lambda^*_{SC})], \tag{4.6}$$

$$G_{S*} = \sum_{i\in(H,L)} (-p_i\delta_i + c_i - m_i)\bar{D}^*_{SC,i}, \tag{4.7}$$

$$G_{R*} = \sum_{i\in(H,L)} (p_i\delta_i + c_i - m_i)\bar{D}^*_{SC,i}. \tag{4.8}$$

It is known that the two-part-tariff (TPT) contract (Choi, 2018) is commonly used in the luxury fashion industry (see Table 2). Thus, in developing the channel coordination mechanisms, we try to explore how efficient supply chain contracting mechanisms can be developed on top of the TPT contract.

As we will show later on, if the luxury fashion brand is risk neutral, coordination can be achieved easily by the TPT contract alone. However, if the luxury fashion brand is risk averse or risk seeking, we need to employ the more sophisticated hybrid TPT contracts. We describe three kinds of related supply chain contracts in the context of luxury fashion supply chain as follows.

The two-part-tariff (TPT) contract: Under the TPT contract (Chen et al., 2017; Biswas et al., 2018; Choi, 2018), the factory produces the products and supplies them to the luxury fashion brand at a certain unit wholesale price for each product, and charge a lump sum of fixed payment. This is similar to the case when the factory is hired/owned by the luxury fashion brand and the fixed lump sum payment is the "salary".

The sales-rebate two-part-tariff (STPT) contract: Under the STPT contract, on top of the TPT contract, the factory offers a sales incentive called "sales rebate" (Chiu et al,. 2011; Heydari and Asl-Najafi, 2018; Chiu et al., 2019) to the luxury fashion brand. With the (positive) sales rebate, the luxury fashion brand can make more money than the product's selling price for each unit of product sold.

The revenue-sharing two-part-tariff (RTPT) contract: Under the RTPT contract, on top of the TPT contract, the factory charges a share of the luxury fashion brand's revenue per unit product sold (i.e., "revenue sharing") (Chan et al., in press-a; Fatehi and Wagner, 2019). However, if the rate of "revenue sharing" is negative, the factory actually sponsors the luxury fashion brand and the "revenue share" would become a "sales rebate".

Table 5 summarizes the key parameters of the three types of TPT based contracts and Proposition 4.2 shows the results and the contracting details are shown in the proof. As we mentioned in Section 3, we employ the subscripts RA, RN and RS to denote risk averse, risk neutral and risk seeking, respectively.

Proposition 4.2.. *(a) If the luxury fashion brand is risk neutral, the supply chain can be coordinated by setting the unit wholesale price* $c_i = m_i$, *for* $i \in (H, L)$, *and the luxury fashion brand transfers a fixed fee* T_{RN} *(under the TPT contract* $\Theta_{TPT}(c_i; T_{RN})$*) to the factory where* $J_M \leqslant T_{RN} \leqslant \Pi_{SC*} - J_M$. *(b) If the luxury fashion brand is risk averse, the supply chain can be coordinated by using a sales-rebate two-part-tariff (STPT) contract* $\Theta_{STPT}(c_i; \delta_i; T_{RA})$, *in which the sales rebate of* $\delta_i = \frac{1}{p_i}\left(c_i + \frac{p_i - m_i}{1 - k\sigma_i}\right) - 1$ *is set for* $i \in (H, L)$,*where* $\delta_i > 0$, *and the luxury fashion brand transfers a fixed fee* T_{RA} *to the factory where* $J_M - G_{S*} \leqslant T_{RA} \leqslant \Pi_{SC*} - J_{LFB}$. *(c) If the luxury fashion brand is risk seeking, the supply chain can be coordinated by a revenue-sharing two-part-tariff (RTPT) contract* $\Theta_{RTPT}(c_i; \eta_i; T_{RS})$, *in which the revenue share of* $\eta_i = 1 - \frac{1}{p_i}\left(c_i + \frac{p_i - m_i}{1 - k\sigma_i}\right)$*is set for* $i \in (H, L)$, *and the luxury fashion brand transfers a fixed fee* T_{RS} *to the factory where* $J_M + G_{R*} \leqslant T_{RS} \leqslant \Pi_{SC*} - J_{LFB}$.

Proposition 4.2 indicates that to coordinate the luxury fashion supply chain with both the higher-tier and lower-tier products, the luxury fashion brand's risk attitude is a critical factor. When the luxury fashion brand is risk neutral, the TPT contract which adopts the classic "supply at cost with credit transfer" approach can perfectly coordinate the channel. This is the simplest scenario. However, once the luxury fashion brand is risk averse or risk seeking, the situation is much more complicated and we need additional contract parameters to help. In particular, when the luxury fashion brand is risk averse, the sales rebate contract can help entice the luxury fashion brand to decide the optimal advertisement budget allocation to be the same as the supply chain system's optimal decision. When the luxury fashion brand is risk seeking, the case is tricky as a revenue sharing contract can be issued to help coordinate the decision. However, the revenue share η_i can be negative or positive. When η_i is positive, it is a standard revenue share. When η_i is negative, the contract is in fact a sales rebate. For all risk attitude cases, the presence of the credit transfer T_l, where $l \in (RA, RN, RS)$,

Table 5
The three types of TPT based supply chain contracts.

Contracts	Notation	Parameters
Two-part-tariff	$\Theta_{TPT}(c_i; T_{RN})$	c_i, T_{RN}
Sales-rebate two-part-tariff	$\Theta_{STPT}(c_i; \delta_i; T_{RA})$	c_i, δ_i, T_{RA}
Revenue-share two-part-tariff	$\Theta_{RTPT}(c_i; \eta_i; T_{RS})$	c_i, η_i, T_{RS}

helps to ensure the minimum profit requirements (i.e. the participation constraints) of the factory and the luxury fashion brand will definitely be fulfilled by properly setting the credit transfer T_i within the corresponding range as defined in Proposition 4.2. Note that the STPT and RTPT contracts have three contract parameters. However, to coordinate the supply chain with the corresponding risk sensitive luxury fashion brand, controlling the fixed payment (T_{RA} or T_{RS}) and an additional contract parameter will be sufficient already.

5. Extended analysis: market share considerations

For many luxury fashion brands, when they plan their advertisement budget allocation strategy, one key aspect is the market share. In other words, they want to ensure the expected market demand of each product hits a certain target. This is critical to ensure that the product's presence in the market is substantial. In this extended analysis, we examine the case with this market share consideration. As we will see, considering the market share of each product changes the optimal advertisement budget allocation strategy and the polarized strategy is no longer optimal.

Suppose that the luxury fashion brand has set the minimum expected demand target for product $i \in (H, L)$ as Δ_i. To avoid trivial cases, we consider the setting of Δ_i is feasible which means that the optimal λ exists. For a notational purpose, we define the following critical parameters:

$$\hat{\lambda}_H = \frac{(1+\gamma\xi)\Delta_H}{b+\gamma\beta} + \frac{\xi(\alpha+\beta)-a}{b+\gamma\beta}, \tag{5.1}$$

$$\hat{\lambda}_L = \frac{(1+\gamma\xi)\Delta_L - (\alpha+\beta+a\xi)}{\xi b-\beta}, \tag{5.2}$$

$$\Omega = \xi b - \beta. \tag{5.3}$$

Lemma 5.1 shows the feasible regions of λ which helps define the optimal advertisement budget allocation strategy.

Lemma 5.1.. *With the market share consideration, the feasible region of λ is determined by the critical threshold $\Omega = \xi b - \beta$. (a) If $\Omega > 0$, then the feasible region is $\max(\hat{\lambda}_H, \hat{\lambda}_L) \leqslant \lambda \leqslant 1$. (b) If $\Omega = 0$, then the feasible region is $\hat{\lambda}_H \leqslant \lambda \leqslant 1$. (c) If $\Omega < 0$, then the feasible region is $\hat{\lambda}_H \leqslant \lambda \leqslant \hat{\lambda}_L$.*

Lemma 5.1 indicates that the polarized decisions, i.e., $\lambda = 0$ and $\lambda = 1$ may no longer be feasible when the market share consideration is incorporated into the optimization problem. This has a very important implication regarding the optimal advertisement budget allocation decision, as shown in Proposition 5.1.

Proposition 5.1.. *With the market share consideration, irrespective of the luxury fashion brand's risk attitude, under the MSD objective, the optimal advertisement budget allocation strategy is characterized as follows: (a) If $\gamma \geqslant \hat{\gamma}$ or $\xi \geqslant \hat{\xi}$, then the optimal advertisement budget allocation strategy is (i) $\lambda = 1$ if $\Omega \geqslant 0$, and (ii) $\lambda = \hat{\lambda}_L$ if $\Omega < 0$. (b) If $\gamma < \hat{\gamma}$ or $\xi < \hat{\xi}$, then the optimal advertisement budget allocation strategy is (i) $\lambda = \max(\hat{\lambda}_H, \hat{\lambda}_L)$ if $\Omega > 0$, and (ii) $\lambda = \hat{\lambda}_H$ if $\Omega \leqslant 0$.*

Proposition 5.1 concisely shows the optimal advertisement budget allocation solution for the luxury fashion brand when the market share is considered (as reflected by the minimum expected demand requirement for each product). The findings clearly indicate that the polarized strategy need not be optimal anymore. This is a very important finding as it helps to explain why in the real world, some luxury fashion brands may vote for dividing the advertisement budget in some proportion (say, "50%:50%", "60%:40%", etc.) between the two tiers of products. Thus, we argue that the presence of market share consideration is critical.

As a remark, for the coordination mechanism, the incentive alignment schemes proposed in Proposition 4.2 basically can all be applied to help achieve channel coordination for the luxury supply chain when the luxury fashion brand has the market share considerations. Of course, owing to the change of the feasible region for λ, the optimal expected profit of the luxury fashion supply chain with market share consideration is in general different from the case without market share consideration, and we define it as Π^{MS}_{SC*}, where the superscript MS represents the "market share" scenario[8]. We have Proposition 5.2.

Proposition 5.2.. *In the extended model with the market share consideration, the luxury fashion supply chain can basically be coordinated by the same contractual arrangements as shown in Proposition 4.2, with the change that Π_{SC*} is replaced by Π^{MS}_{SC*}.*

Proposition 5.2 shows that the channel coordination mechanisms proposed in Proposition 4.2 are basically robust and they can achieve channel coordination irrespective whether the market share targets are considered or not.

[8] In this paper, when there are market share considerations by the luxury fashion brand, we will impose the same market share constraints on the luxury fashion supply chain.

6. Conclusion

6.1. Concluding remarks and major findings

Motivated by the importance of optimal advertisement budget allocation and channel coordination for luxury fashion supply chains with different multiple brand-tier products, we have conducted an analytical study in this paper. Some important findings and managerial insights are summarized below.

Optimal advertisement budget allocation strategies: We have analytically proven that in the basic model when there is no consideration of market share, the optimal advertisement budget allocation strategies are always polarized for all kinds of risk attitudes under the MSD objective (see Proposition 4.1). This is a very interesting result as it shows something different from the literature. To be specific, Chiu et al. (2018) show that the risk averse luxury fashion brand may diversify risk by allocating the advertisement budget between two groups of consumers and only the risk neutral luxury fashion brand will adopt the polarized strategy. Our finding shows a different result when we consider the presence of two different brand-tier products, instead of two groups of consumers. This hence calls for a more careful planning of the optimal advertisement budget allocation between different situations. Special attention should be paid to the specific problem domain and scope of the problem.

Coordination mechanism: It is commonly known that the TPT contract is widely used in the luxury fashion industry. Thus, we have explored how versatile supply chain contracting mechanisms can be developed on top of it. As we have proven in Proposition 4.2, to coordinate the luxury fashion supply chain with both the higher-tier and lower-tier products, the luxury fashion brand's risk attitude is a critical factor. To be specific, we have found that when the luxury fashion brand is risk neutral, the TPT contract which adopts the classic "supply at cost with credit transfer" approach is good enough to coordinate the channel. However, once the luxury fashion brand is risk averse or risk seeking, we need to apply a more sophisticated contract with more parameters in order to achieve channel coordination. In particular, when the luxury fashion brand is risk averse, the sales rebate TPT contract can help entice the luxury fashion brand to set the optimal advertisement budget allocation the same as the supply chain system's optimal decision. When the luxury fashion brand is risk seeking, the case is trickier as we need to apply a revenue sharing TPT contract to help coordinate the channel. Although we call it the revenue sharing TPT contract, the "revenue share" may become a "sales rebate" in some cases.

Impacts of risk attitudes and sensitivity: Comparing the cases with different risk attitudes, we have found that the specific risk attitude does not affect the polarized nature of the optimal advertisement budget allocation problem under the basic model. This point is different from some reported studies in the literature (e.g., Chiu et al., 2018). However, different risk attitudes would affect the respective required coordination mechanisms as what we have discussed above. As a result, when the luxury fashion brand and its factory partner develop supply chain contracts, special attention should be paid to the risk attitude of the luxury fashion brand.

Market share target: In the extended analysis, we have considered the scenario when the luxury fashion brand has set a minimum expected demand target for each brand-tier product. This is known as the market share consideration. We have proven that the expected minimum target demand for each product affects the optimal advertisement budget allocation significantly and makes it un-polarized in general. On the contrary, the supply chain coordination contractual mechanism is not affected by the presence of the market share considerations and constraints. In other words, the same channel coordination contracting mechanisms (with the only change in terms of the "optimal supply chain expected profit" as it is different between the cases with and without the market share constraint) can be applied to coordinate the supply chain with market share considerations and hence the coordination schemes are robust.

Table 6 summarizes some important analytically proven core findings regarding the impacts of risk attitudes and sensitivity, the impacts of market share considerations, as well as the channel coordination mechanisms in luxury fashion supply chains with two different brand-tier products.

Table 6
Core findings.

	Details	References
Impacts of risk attitudes	Different risk attitudes affect the respective required coordination mechanisms. But the optimal advertisement budget allocation mechanism is still a polarized mechanism irrespective of whether the luxury fashion brand is risk averse, risk neutral or risk seeking	Proposition 4.1, and Proposition 4.2
Impacts of market share considerations	The market share considerations, modelled by the expected minimum target demand for each product, affects the optimal advertisement budget allocation and makes it un-polarized in general. However, the format of supply chain coordination mechanism is not affected by the presence of this market share considerations	Proposition 4.2, Proposition 5.1, and Proposition 5.2
Supply chain coordination with higher-tier and lower-tier branded products	To coordinate the luxury fashion supply chain with both the higher-tier and lower-tier products, the luxury fashion brand's risk attitude is a critical factor. When the luxury fashion brand is risk neutral, the TPT contract which adopts the classic "supply at cost with credit transfer" approach can perfectly coordinate the channel. However, once the luxury fashion brand is risk averse or risk seeking, we need to apply a more sophisticated contract in order to achieve channel coordination. In particular, when the luxury fashion brand is risk averse, the sales rebate TPT contract can help. When the luxury fashion brand is risk seeking, the case is trickier as we need to apply a "revenue sharing TPT" contract to help coordinate the channel	Proposition 4.2, and Proposition 5.2

6.2. Future studies

For future research, we can explore the situation when the luxury fashion brand sets its multiple brand-tier product lines strategy to cope with the copycat phenomenon commonly observed in the fashion industry. The optimal advertisement budget allocation for the luxury fashion brand under the case with a copycat product in the market can be examined. To be specific, whether and how the advertisement budget allocation towards the lower-tier brand product would affect the copycat's threat can be examined. Moreover, the optimal advertisement budget allocation in other types of fashion brands, such as fast fashion brands, could also be an interesting topic for further exploration in the future. However, as the operations models are different, other decisions such as pricing may also need to be considered and the analytical model may become more complicated. Last but not least, it will also be interesting to examine the problem by using the multi-methodological approach (Choi, 2016; Choi et al., 2016; Sheu and Choi, 2019) which may yield some more interesting results.

Appendix (A). : All proofs

Proof of Lemma 3.1. *From (3.1) and (3.2), we have the following*:

$$\widetilde{D}_H = (a + b\lambda - \gamma E[\widetilde{D}_L])\widetilde{\varepsilon}_H,$$

$$\widetilde{D}_L = (\alpha + \beta(1-\lambda) + \xi E[\widetilde{D}_H])\widetilde{\varepsilon}_L.$$

Obviously, we have the following:

$$E[\widetilde{D}_H] = (a + b\lambda - \gamma E[\widetilde{D}_L]), \tag{A1}$$

$$E[\widetilde{D}_L] = (\alpha + \beta(1-\lambda) + \xi E[\widetilde{D}_H]). \tag{A2}$$

From (A1) and (A2), we have $E[\widetilde{D}_L] = \frac{\alpha+\beta(1-\lambda)}{1+\gamma\xi} + \frac{\xi(a+b\lambda)}{1+\gamma\xi}$ *and* $E[\widetilde{D}_H] = \frac{a+b\lambda}{1+\gamma\xi} - \frac{\gamma(\alpha+\beta(1-\lambda))}{1+\gamma\xi}$.
Similarly, we can find that:
$SD[\widetilde{D}_L] = \left(\frac{\alpha+\beta(1-\lambda)}{1+\gamma\xi} + \frac{\xi(a+b\lambda)}{1+\gamma\xi}\right)\sigma_L$, $SD[\widetilde{D}_H] = \left(\frac{a+b\lambda}{1+\gamma\xi} - \frac{\gamma(\alpha+\beta(1-\lambda))}{1+\gamma\xi}\right)\sigma_H$. *(Q.E.D.)*

Proof of Proposition 4.1:. *From (3.8), we have*: $O_{MSD}(\lambda) = r_H\left(\frac{a+b\lambda}{1+\gamma\xi} - \frac{\gamma(\alpha+\beta(1-\lambda))}{1+\gamma\xi}\right) + r_L\left(\frac{\alpha+\beta(1-\lambda)}{1+\gamma\xi} + \frac{\xi(a+b\lambda)}{1+\gamma\xi}\right)$. *Taking the first order derivative, we have*:

$$\frac{dO_{MSD}(\lambda)}{d\lambda} = \frac{r_L(b\xi-\beta) + r_H(b+\gamma\beta)}{1+\gamma\xi}. \tag{A3}$$

Solving $\frac{dO_{MSD}(\lambda)}{d\lambda} = 0$ *for* γ *yields* $\hat{\gamma} = \frac{r_L}{r_H}\left(1 - \frac{b\xi}{\beta}\right) - \frac{b}{\beta}$.
Solving $\frac{dO_{MSD}(\lambda)}{d\lambda} = 0$ *for* ξ *yields* $\hat{\xi} = \frac{\beta}{b} - \frac{r_H}{r_L}\left(1 + \frac{\beta\gamma}{b}\right)$.
Thus, irrespective of the luxury fashion brand's risk attitude, under the MSD objective, the optimal advertisement strategy is a polarized strategy (i.e., either $\lambda = 0$ *or* $\lambda = 1$*) as* $\frac{dO_{MSD}(\lambda)}{d\lambda}$ *is a monotonic function. If* $\gamma \geqslant \hat{\gamma}$ *or* $\xi \geqslant \hat{\xi}$*, then* $\frac{dO_{MSD}(\lambda)}{d\lambda} > 0$ *and hence the optimal advertisement budget allocation strategy is* $\lambda = 1$*. If* $\gamma < \hat{\gamma}$ *or* $\xi < \hat{\xi}$*, then* $\frac{dO_{MSD}(\lambda)}{d\lambda} < 0$ *and hence the optimal advertisement budget allocation strategy is* $\lambda = 0$. *(Q.E.D.)*

Proof of Lemma 4.1:. *For the supply chain system, the expected supply chain profit is given as follows*:

$$EP_{SC}(\lambda) = \sum_{i\in(H,L)} E[\widetilde{\Pi}_{SC,i}], \tag{A4}$$

where

$$E[\widetilde{\Pi}_{SC,i}] = (p_i - m_i)E[\widetilde{D}_i], \quad \text{for} \quad i \in (H, L). \tag{A5}$$

It is easy to show that $\frac{dEP_{SC}(\lambda)}{d\lambda}$ *is a monotonic function. Thus, using the same argument as in the proof of Proposition 4.1, we can find the thresholds* $\hat{\gamma}_{SC} = \frac{r_{SC,L}}{r_{SC,H}}\left(1 - \frac{b\xi}{\beta}\right) - \frac{b}{\beta}$, *and* $\hat{\xi}_{SC} = \frac{\beta}{b} - \frac{r_{SC,H}}{r_{SC,L}}\left(1 + \frac{\beta\gamma}{b}\right)$.

Then, we have: (a) *If* $\gamma \geqslant \hat{\gamma}_{SC}$ *or* $\xi \geqslant \hat{\xi}_{SC}$*, then the optimal advertisement budget allocation strategy is* $\lambda^*_{SC} = 1$*. If* $\gamma < \hat{\gamma}_{SC}$ *or* $\xi < \hat{\xi}_{SC}$*, then the optimal advertisement budget allocation strategy is* $\lambda^*_{SC} = 0$. (b) *In general,* $r_{SC,i} \neq r_i$*, for* $i \in (H, L)$ *and hence the supply chain will not be coordinated naturally as* $\hat{\gamma} \neq \hat{\gamma}_{SC}$ *and* $\hat{\xi} \neq \hat{\xi}_{SC}$. *(Q.E.D.)*

Proof of Proposition 4.2:.

(a) *When the luxury fashion brand is risk neutral, under the TPT contract: The expected profit of the factory is given to be:* $E[\widetilde{\Pi}_F] = \sum_{i\in(H,L)} (c_i - m_i)E[\widetilde{D}_i] + T_{RN}$. *When* $c_i = m_i$*, for* $i \in (H, L)$*, we have:* $\hat{\gamma} = \hat{\gamma}_{SC}$ *and* $\hat{\xi} = \hat{\xi}_{SC}$*, which implies that the luxury fashion brand's optimal decision is the same as the centralized supply chain's optimal decision. If the luxury fashion brand transfers a fixed*

fee T_{RN} (under the TPT contract) to the factory where $J_M \leqslant T_{RN} \leqslant \Pi_{SC*} - J_M$, then both the factory and the luxury fashion brand will have their minimum expected profit requirements being fulfilled. Coordination is hence achieved.

(b) *When the luxury fashion brand is risk averse: Under the sales-rebate two-part-tariff (STPT) contract $\Theta_{STPT}(c_i; \delta_i; T_{RA})$, setting the sales rebate of $\delta_i = \frac{1}{p_i}\left(c_i + \frac{p_i - m_i}{1 - k\sigma_i}\right) - 1$ for $i \in (H, L)$, where $\delta_i > 0$, we have: $\hat{\gamma} = \hat{\gamma}_{SC}$ and $\hat{\xi} = \hat{\xi}_{SC}$.*

Under $\Theta_{STPT}(c_i; \delta_i; T_{RA})$, the expected profit of the factory becomes:

$$E[\widetilde{\Pi}_F|\Theta_{STPT}(c_i; \delta_i; T_{RA})] = \sum_{i\in(H,L)} (c_i - m_i - p_i\delta_i)E[\widetilde{D}_i] + T_{RA}.$$

Thus, when the luxury fashion brand transfers a fixed fee T_{RA} to the factory where $J_M - G_{S*} \leqslant T_{RA} \leqslant \Pi_{SC*} - J_{LFB}$, then both the factory and the luxury fashion brand will have their minimum expected profit requirements being fulfilled. Channel coordination is achieved.

(c) *When the luxury fashion brand is risk seeking: Under the revenue-sharing two-part-tariff (RSTPT) contract $\Theta_{RTPT}(c_i; \eta_i; T_{RS})$, setting the revenue share of $\eta_i = 1 - \frac{1}{p_i}\left(c_i + \frac{p_i - m_i}{1 - k\sigma_i}\right)$ for $i \in (H, L)$, we have: $\hat{\gamma} = \hat{\gamma}_{SC}$ and $\hat{\xi} = \hat{\xi}_{SC}$.*

Under $\Theta_{RTPT}(c_i; \eta_i; T_{RS})$, the expected profit of the factory becomes:

$$E[\widetilde{\Pi}_F|\Theta_{RTPT}(c_i; \delta_i; T_{RS})] = \sum_{i\in(H,L)} (c_i - m_i + p_i\eta_i)E[\widetilde{D}_i] + T_{RS}.$$

Thus, when the luxury fashion brand transfers a fixed fee T_{RS} to the factory where $J_M + G_{R*} \leqslant T_{RS} \leqslant \Pi_{SC*} - J_{LFB}$, then both the factory and the luxury fashion brand will have their minimum expected profit requirements being fulfilled. Channel coordination is achieved.(*Q.E.D.*)

Proof of Lemma 5.1:. *From* Lemma 3.1, *we have: $E[\widetilde{D}_H] = \frac{a + b\lambda}{1 + \gamma\xi} - \frac{\gamma(\alpha + \beta(1 - \lambda))}{1 + \gamma\xi}$. Thus, $E[\widetilde{D}_H] \geqslant \Delta_H$ is equivalent to the following:*

$$E[\widetilde{D}_H] \geqslant \Delta_H \Leftrightarrow \lambda \geqslant \hat{\lambda}_H = \frac{(1 + \gamma\xi)\Delta_H}{b + \gamma\beta} + \frac{\xi(\alpha + \beta) - a}{b + \gamma\beta}. \tag{A6}$$

From Lemma 3.1, *we have: $E[\widetilde{D}_L] = \frac{\alpha + \beta(1 - \lambda)}{1 + \gamma\xi} + \frac{\xi(a + b\lambda)}{1 + \gamma\xi}$. Thus, $E[\widetilde{D}_L] \geqslant \Delta_L$ is equivalent to the following cases with $\Omega = \xi b - \beta$:*

Case 1: If $\Omega > 0$, then $E[\widetilde{D}_L] \geqslant \Delta_L \Leftrightarrow \lambda \geqslant \hat{\lambda}_L$. (A7)

Case 2: If $\Omega = 0$, since$E[\widetilde{D}_L(\lambda = 0)] > 0 \Leftrightarrow \frac{a\xi + \alpha + \beta}{1 + \gamma\xi} > 0$ and $E[\widetilde{D}_L(\lambda = 0)] > \Delta_L$ (as Δ_L must be less than the expected demand for the lower-tier product when all advertisement budget is allowed to it (i.e., $E[\widetilde{D}_L(\lambda = 0)]$)), then we have:

$$E[\widetilde{D}_L] \geqslant \Delta_L. \tag{A8}$$

Case 3: If $\Omega < 0$, then $E[\widetilde{D}_L] \geqslant \Delta_L \Leftrightarrow \lambda \leqslant \hat{\lambda}_L$. (A9)

With the market share consideration, the feasible region of λ is determined by combining (A6) with the cases defined in (A7), (A8) and (A9): (a) If $\Omega > 0$, then we have $\max(\hat{\lambda}_H, \hat{\lambda}_L) \leqslant \lambda \leqslant 1$. (b) If $\Omega = 0$, then we have $\hat{\lambda}_H \leqslant \lambda \leqslant 1$. (c) If $\Omega < 0$, then we have $\hat{\lambda}_H \leqslant \lambda \leqslant \hat{\lambda}_L$. (*Q.E.D.*)

Proof of Proposition 5.1. Proposition 5.1 *is a direct result from* Lemma 5.1 (*the feasible set for the optimal λ*) *with market share considerations, as well as* Proposition 4.1 (*the optimal advertisement budget allocation strategy*). *To be specific, with the market share consideration, irrespective of the luxury fashion brand's risk attitude, under the MSD objective, the optimal advertisement budget allocation strategy is characterized as follows:*

(a) *If $\gamma \geqslant \hat{\gamma}$ or $\xi \geqslant \hat{\xi}$, then the optimal advertisement budget allocation strategy is (i) $\lambda = 1$ if $\Omega > 0$ and $\Omega = 0$; (ii) $\lambda = \hat{\lambda}_L$ if $\Omega < 0$.*

(b) *If $\gamma < \hat{\gamma}$ or $\xi < \hat{\xi}$, then the optimal advertisement budget allocation strategy is (i) $\lambda = \max(\hat{\lambda}_H, \hat{\lambda}_L)$ if $\Omega > 0$; (ii) $\lambda = \hat{\lambda}_H$ if $\Omega < 0$ and $\Omega = 0$.* (*Q.E.D.*)

Proof of Proposition 5.2:. *Similar to the proof of* Proposition 4.2, *with the only change of the optimal expected profit of the supply chain.* (*Q.E.D.*)

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