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Fractional logistic models in the frame of fractional operators generated by conformable derivatives

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ABSTRACT

In this article, we study different types of fractional-order logistic models in the frame of Caputo type fractional operators generated by conformable derivatives (Caputo CFDs). We present the existence and uniqueness theorems to solutions of these models and discuss their stability by perturbing the equilibrium points. Finally, we furniture our results by illustrative numerical examples for the studied models.

1. Introduction

Fractional calculus is a branch of mathematical analysis that takes into consideration the integration and differentiation of real or complex order. In spite of the fact that this kind of calculus is old, it gained popularity and started to catch the interest of scientists only in the last 20 or 30 years because important results were reported when fractional derivatives and integrals were applied to describe many real world phenomena [1–10]. A big virtue of the fractional calculus is that there are many different fractional derivatives or integrals. This virtue gives the opportunity to choose the most appropriate derivative or integral in order to describe complex systems of real world problems eligibly. Nevertheless, in order to have better mathematical models of real world problems, scientists started to disclose some new types of fractional integrals and consequently fractional derivatives using two main methods. The first method is the traditional method based on iterating to find the nth order integral and then replacing n by any number α. Hadamard, generalized fractional operators and the fractional operators generated from conformable derivatives, can be considered as examples of fractional operators obtained using this approach [11–17]. These operators usually have singular kernels. The second method is subject to the limiting process and using some properties of the Dirac-Delta functions. Among these operators we advert Caputo-Fabrizio and Atangana-Baleanu fractional derivatives [18,19]. These operators embodies nonsingular kernels. For more details on such operators and their applications we refer to [20–22]

The literature reveals that the logistic equation in the frame of fractional derivatives were tackled by many scientists (see [23–25] and the references therein). In this article we discuss the fractional-order modified quadratic and cubic logistic models given, respectively, by

$$({}_{t_0}D^{\alpha,\rho}x)(t) = rx(t)(1-x(t)), \quad t > t_0, \; x(t_0) = x_0, \tag{1}$$

and

$$({}_{t_0}D^{\alpha,\rho}x)(t) = rx(t)\left(1-\frac{(x(t))}{k}\right)(x(t)-m), \quad t > t_0, \; x(t_0) = x_0, \tag{2}$$

$\alpha \in (0, 1]$, $\rho > 0$, and $r, m, k > 0$. Here, ${}_{t_0}D^{\alpha,\rho}$ represents the left-Caputo conformable fractional derivatives (Caputo CFDs) generated in [17] depending on an open problem raised in [26]. It is worth mentioning that physical reason for using conformable derivative in the logistic models can be connected to the recent works of [27–30].

In this article, we present the existence and uniqueness theorems for Eqs. (1) and (2), followed by a detailed discussion of the stability of these equations. Then, we furnish our results with some numerical illustrative examples.

This article is organized as follows: In Section 2, some preliminary results are presented. The existence and uniqueness of a

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certain initial value problems that embodies a fractional conformable derivative are presented in Section 3. In Section 4, the stability of models (1) and (2) are discussed. In Section 5, numerical discussion is presented. The last section is committed to the conclusion.

2. Preliminary results

Prior to presenting the main results, we recall and present some definitions and theorems which will be used intensively in our study.

The original definition of the conformable derivatives [26] is defined by

$$({}_{t_0}T^{\alpha}f)(t)=\lim_{\epsilon\to 0}\frac{f(t+\epsilon(t-t_0)^{1-\alpha})-f(t)}{\epsilon},$$

where $f:[t_0,\infty)\to\mathbb{R}$ of order α ($\alpha\in(0,1]$). If $({}_{t_0}T^{\alpha}f)(t)$ exists on (t_0,b) then $({}_{t_0}T^{\alpha}f)(t_0)=\lim_{t\to t_0^+}({}_{t_0}T^{\alpha}f)(t)$.

If f is differentiable then one may deduce that

$$({}_{t_0}T^{\alpha}f)(t)=(t-t_0)^{1-\alpha}f'(t). \tag{3}$$

The corresponding conformable left integral is defined as [26]

$${}_{t_0}I^{\alpha}f(x)=\int_{t_0}^{x}f(t)\frac{dt}{(t-t_0)^{1-\alpha}},\quad 0<\alpha<1.$$

Definition 1 Jarad et al. [17]**.** The left-fractional conformable integral operator is defined by

$${}_{t_0}\mathfrak{I}^{\alpha,\rho}f(x)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{x}\left(\frac{(x-t_0)^{\rho}-(t-t_0)^{\rho}}{\rho}\right)^{\alpha-1}f(t)\frac{dt}{(t-t_0)^{1-\rho}}, \tag{4}$$

where $\alpha\in\mathbb{C}, Re(\alpha)\geq 0$.

Definition 2 Jarad et al. [17]**.** The left-fractional conformable derivative of order $\alpha\in\mathbb{C}, Re(\alpha)\geq 0$ in the Caputo setting is defined by

$$\begin{aligned}{}_{t_0}D^{\alpha,\rho}f(x)&=({}_{t_0}\mathfrak{I}^{n-\alpha,\rho})\,{}^{n}_{t_0}T^{\rho}f(x)\\&=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{x}\left(\frac{(x-t_0)^{\rho}-(t-t_0)^{\rho}}{\rho}\right)^{n-\alpha-1}\\&\quad\times{}^{n}_{t_0}T^{\rho}f(t)\frac{dt}{(t-t_0)^{1-\rho}},\end{aligned} \tag{5}$$

where $n=[Re(\alpha)]+1, {}^{n}_{t_0}T^{\rho}=\underbrace{{}_{t_0}T^{\rho}_{t_0}T^{\rho}\cdots{}_{t_0}T^{\rho}}_{n\,times}$ and ${}_{t_0}T^{\rho}$ is the left conformable differential operator presented in (3).

The following identity is essential to solve linear conformable fractional differential equations [17]

$$({}_{t_0}\mathfrak{I}^{\alpha,\rho}(t-t_0)^{\rho\nu-\rho})(x)=\frac{1}{\rho^{\alpha}}\frac{\Gamma(\nu)}{\Gamma(\alpha+\nu)}(x-t_0)^{\rho(\alpha+\nu-1)},\quad Re(\nu)>0. \tag{6}$$

The following theorem is the main tool to obtain the solution representation.

Theorem 1 Jarad et al. [17]**.** *Let* $f\in C^{n}_{\alpha,t_0}[t_0,b]=\{f:[t_0,b]\to\mathbb{R}:{}^{n-1}_{t_0}T^{\rho}f\in I_{\rho}([t_0,b])\}$, $n=[\alpha]+1$. *Then,*

$$({}_{t_0}\mathfrak{I}^{\alpha,\rho}_{t_0}D^{\alpha,\rho}f)(t)=f(t)-\sum_{k=0}^{n-1}\frac{{}^{k}_{t_0}T^{\rho}f(t_0)(t-t_0)^{\rho k}}{\rho^{k}k!}, \tag{7}$$

where $I_{\rho}([t_0,b])$ *is the space defined in Definition 3.1 in [26].*

More properties of the left-fractional conformable integrals and derivatives can be found in [17].

3. Existence and uniqueness theorems

Consider the system

$${}_{t_0}D^{\alpha,\rho}x(t)=f(t,x(t)),\quad x(t_0)=x_0,\ t\in(t_0,b], \tag{8}$$

where $\alpha\in(0,1)$, f: $[t_0,b)\times G$, G an open subset of $\mathbb{R}$ or more generally of $\mathbb{C}$, and

$$\begin{aligned}g(t)&=f(t,x(t))\in C_{\gamma,\rho}[t_0,b]\\&=\left\{y:(t_0,b]\to\mathbb{R}:\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\gamma}y(t)\in C[t_0,b]\right\},\\&0\leq\gamma<1,\ \rho>0.\end{aligned}$$

The space $C_{\gamma,\rho}[t_0,b]$ is a Banach space when it is endowed by the norm

$$\|y\|=\sup_{t}|e^{-N(t-t_0)^{\rho}}(\frac{t^{\rho}-t_0^{\rho}}{\rho})^{\gamma}y(t)|,\quad N>0, \tag{9}$$

which is equivalent to the norm $\|y\|_{\gamma,\rho}=\sup_t|(\frac{t^{\rho}-t_0^{\rho}}{\rho})^{\gamma}y(t)|$. When $\gamma=0$ we accept that $C_{\gamma,\rho}[t_0,b]=C[t_0,b]$ the space of continuous functions on $[t_0,b]$ and when $\rho=1$ we accept $C_{\gamma,\rho}[t_0,b]=C_{\gamma}[t_0,b]$ (see [3], page 4).

Definition 3. A function $x(t)$ is said to be a solution of the initial value problem (8) if

1. $(t,x(t))\in D,\quad D=[t_0,b]\times B,\quad B=\{x\in\mathbb{R}:|x|\leq L\}\subset G,\quad L>0$
2. $x(t)$ satisfies (8).

Theorem 2. *The conformable fractional initial value problem (8) has a unique solution in the space*

$$C^{\alpha,0}_{\gamma,\rho}[t_0,b]=\{y(t)\in C[t_0,b]:aD^{\alpha,\rho}y(t)\in C_{\gamma,\rho}[t_0,b]\},$$

with $0\leq\gamma<1$ *and* $\gamma\leq\alpha$*, provided that*

$$\frac{A}{\rho^{\alpha}N^{\alpha-1}}<1, \tag{10}$$

and f satisfies the Lipschitzian condition

$$|f(t,y_1)-f(t,y_2)|\leq A|y_1-y_2|,\quad A>0. \tag{11}$$

Proof. First let's prove the existence of a unique solution $y(t)$ in the space $C[t_0,b]$. Define the operator Ψ: $C[t_0,b]\to C[t_0,b]$ by

$$(\Psi x)(t)=x_0+{}_{t_0}\mathfrak{I}^{\alpha,\rho}f(t,x(t)). \tag{12}$$

where the space $C[t_0,b]$ is endowed with the norm $\|y\|_C=\sup_t|e^{-N(t-t_0)^{\rho}}y(t)|$, which is equivalent to the sup norm. For any $y_1,y_2\in B$, by the help of the Lipschitzian condition (11) we have

$$\begin{aligned}&|e^{-N(t-t_0)^{\rho}}(\Psi y_1(t)-\Psi y_2(t))|\\&\leq\frac{A\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_0}^{t}e^{-N.K(t,s)}K(t,s)^{\alpha-1}(s-t_0)^{\rho-1}ds\|y_1-y_2\|_C.\end{aligned} \tag{13}$$

Using the change of variable $u=K(t,s)=(t-t_0)^{\rho}-(s-t_0)^{\rho}$, it follows that

$$\|\Psi y_1-\Psi y_2\|\leq\frac{A\rho^{1-\alpha}}{\rho\Gamma(\alpha)}\int_0^{(t-t_0)^{\rho}}u^{\alpha-1}e^{-Nu}du\|y_1-y_2\|_C, \tag{14}$$

and, hence, by using the definition of Gamma function we have,

$$\|\Psi y_1-\Psi y_2\|_C\leq\frac{A}{\rho^{\alpha}N^{\alpha-1}}\|y_1-y_2\|_C. \tag{15}$$

By the assumption (10), the mapping Ψ is a contraction and hence by Banach fixed point theorem it has a unique fixed point $x\in C[t_0,b]$. Moreover,

$$\lim_{m\to\infty}\|T^{m}x_0-x\|_C=0. \tag{16}$$

From the definition of Ψ, x has the form

$$x(t)=x_0+{}_{t_0}\mathfrak{I}^{\alpha,\rho}f(t,x(t)). \tag{17}$$

From Theorem 1 with $n=1$, it is clear that if x satisfies the initial value problem (8), then it has the representation (17). Conversely, if x has the representation (17), then clearly $x(t_0)=x_0$ and by the help of Theorem 3.6 and (57) of Definition 4.1 with $n=1$ in [17], x will satisfy the Eq. (8). Hence, x has the representation (17) if and only if it satisfies the initial value problem (8). Finally, if $\|.\|$ denotes the norm defined in (9) then we have

$$\|aD^{\alpha,\rho}T^m x_0 - D^{\alpha,\rho}x\| \le A\|T^m x_0 - x\| \le A(b^\rho - t_0^\rho)^\gamma \|T^m x_0 - x\|_C.$$

From (16), we conclude that $\lim_{m\to\infty}\|aD^{\alpha,\rho}T^m x_0 - aD^{\alpha,\rho}x\| = 0$. That is ${}_aD^{\alpha,\rho}x \in C_{\gamma,\rho}[t_0,b]$ and $x\in C^{\alpha,0}_{\gamma,\rho}[t_0,b]$. □

Theorem 3. *The initial value problem (1) has a unique solution in the space $C^{\alpha,0}_{\gamma,\rho}[t_0,b]$ with $0\le\gamma<1$ and $\gamma\le\alpha$, provided that*

$$\frac{r(1+2L)}{\rho^\alpha N^{\alpha-1}} < 1. \tag{18}$$

Proof. The proof follows by using Theorem 2 by taking $f(t,x(t)) = rx(t)(1-x(t))$ and noting that

$$|f(t,y_1)-f(t,y_2)| = r|(y_1-y_2)(1+y_1+y_2)| \le r(1+2L)|y_1-y_2|.$$

That is with the Lipschitz constant $A=r(1+2L)$. □

Theorem 4. *The initial value problem (2) has a unique solution in the space $C^{\alpha,0}_{\gamma,\rho}[t_0,b]$ with $0\le\gamma<1$ and $\gamma\le\alpha$, provided that*

$$\frac{r\left(-m+(1+\frac{m}{k})2L+\frac{L^2}{k}\right)}{\rho^\alpha N^{\alpha-1}} < 1. \tag{19}$$

Proof. The proof follows by using Theorem 2 by taking $f(t,x(t)) = rx(t)(1-\frac{x(t)}{k})(x(t)-m)$ and noting that

$$|f(t,y_1)-f(t,y_2)| = r\left|(y_1-y_2)\left(-m+\left(1+\frac{m}{k}\right)(y_1+y_2) - \frac{1}{k}[(y_1-y_2)^2 - y_1y_2]\right)\right|,$$

and hence, since $y_1, y_2\in B$, that

$$|f(t,y_1)-f(t,y_2)| \le r\left(-m+\left(1+\frac{m}{k}\right)2L+\frac{L^2}{k}\right)|y_1-y_2|.$$

That is with the Lipschitz constant $A=r(-m+(1+\frac{m}{k})2L+\frac{L^2}{k})$. □

Below we use successive approximation depending on Theorem 2 to find the solution of the Caputo conformable fractional linear differential equation with constant coefficient of order $\alpha\in(0,1)$. The obtained solution representation will be the key for proceeding in the next stability analysis section. Consider the initial value problem

$${}_{t_0}D^{\alpha,\rho}x(t) = \lambda x(t)+f(t), t>t_0, \quad x(t_0)=x_0, \tag{20}$$

where $\alpha\in(0,1)$, f real-valued function and $\rho>0$.

Theorem 5. *The solution of the Caputo initial value problem (20) is given by*

$$x(t) = E_\alpha\left(\frac{\lambda}{\rho^\alpha}(t-t_0)^{\rho\alpha}\right)x_0 + \int_{t_0}^t \left(\frac{(t-t_0)^\rho-(s-t_0)^\rho}{\rho}\right)^{\alpha-1} \times E_{\alpha,\alpha}\left(\lambda\left(\frac{(t-t_0)^\rho-(s-t_0)^\rho}{\rho}\right)^\alpha\right) f(s)(s-t_0)^{\rho-1}ds, \tag{21}$$

where $E_\alpha(.)$ and $E_{\alpha,\alpha}(.)$ are the Mittag-Leffler functions of 1 parameter and 2 parameters, respectively [3].

Proof. Upon Theorem 2, consider the successive approximation

$$x_m(t) = x_0 + \lambda {}_{t_0}\mathfrak{I}^{\alpha,\rho}x_{m-1}(t) + {}_{t_0}\mathfrak{I}^{\alpha,\rho}f(t), \quad m=1,2,\ldots, x_0(t)=x_0.$$

Then, by (6), by writing $x_0 = x_0(t-t_0)^{\rho(1)-\rho}$ we have

$$x_1(t) = x_0 + \lambda\frac{\Gamma(1)}{\rho^\alpha\Gamma(\alpha+1)}(t-t_0)^{\rho\alpha}x_0 + {}_{t_0}\mathfrak{I}^{\alpha,\rho}f(t).$$

Proceeding inductively and by making use of (6) and writing $\rho k\alpha = \rho(k\alpha+1-1)$, we have

$$x_m(t) = x_0\sum_{k=0}^m \frac{\lambda^k(t-t_0)^{k\rho\alpha}}{\rho^{k\alpha}\Gamma(k\alpha+1)} + \sum_{k=1}^m \lambda^{k-1}_{t_0}\mathfrak{I}^{k\alpha,\rho}f(t). \tag{22}$$

Then, we reach our claim by expanding ${}_{t_0}\mathfrak{I}^{k\alpha,\rho}$ in the second summation, shifting the index k, interchanging the order of the integral and summation, and letting $m\to\infty$. □

Remark 1. If $\rho=1$ in Theorem 5, then we obtain the results of Example 4.9 in [3].

4. Stability analysis for the logistic models

In the following two subsections we discuss the stability of the logistic models (1) and (2) by perturbing the equilibrium points. Assume $\alpha\in(0,1]$, $\rho>0$ and consider the fractional initial value problem

$${}_{t_0}D^{\alpha,\rho}x(t) = f(x), \quad x(t_0)=x_0, \quad t>t_0. \tag{23}$$

Since the Caputo CFD of the constant function is zero, then z is an equilibrium point of the system (23) if $f(z)=0$. Assume z is an equilibrium point and let $x(t)=z+\theta(t)$. Then, ${}_{t_0}D^{\alpha,\rho}(z+\theta) = f(z+\theta)$ and hence ${}_{t_0}D^{\alpha,\rho}\theta(t) = f(z+\theta)$. By expanding in Taylor series in powers of θ it follows that

$${}_{t_0}D^{\alpha,\rho}\theta(t) = f(z+\theta) \simeq f(z)+f'(z)\theta+\frac{f''(z)}{2}\theta^2+\ldots \simeq f'(z)\theta.$$

Hence, we get the perturbed system

$${}_{t_0}D^{\alpha,\rho}\theta(t) = f'(z)\theta(t), \quad t>t_0, \quad \theta(t_0)=x_0-z. \tag{24}$$

4.1. Analysis of the fractional-order modified quadratic logistic model (1)

We see that model (1) has the equilibrium points $z=0,1$. The corresponding right hand side function of model (1) is $f(x)=rx(1-x)$ and hence $f'(x)=r(1-2x)$ and $f'(0)=r$, $f'(1)=-r$.

The perturbed system associated to the equilibrium point $z=0$ is the fractional linear system

$${}_{t_0}D^{\alpha,\rho}\theta(t) = r\theta(t), \quad \theta(t_0)=x_0. \tag{25}$$

Using Theorem 5, the solution of system (25) is given by

$$\theta(t) = E_\alpha\left(\frac{r}{\rho^\alpha}(t-t_0)^{\rho\alpha}\right)x_0 = \sum_{k=0}^\infty\left(\frac{r}{\rho^\alpha}\right)^k\frac{(t-t_0)^{k\rho\alpha}}{\Gamma(k\alpha+1)}x_0,$$

and hence the equilibrium point $z=0$ is unstable.

In addition, the perturbed system associated to the equilibrium point $z=1$ is the fractional linear system

$${}_{t_0}D^{\alpha,\rho}\theta(t) = -r\theta(t), \quad \theta(t_0)=x_0-1. \tag{26}$$

The solution of system (26) is given by

$$\theta(t) = E_\alpha\left(\frac{-r}{\rho^\alpha}(t-t_0)^{\rho\alpha}\right)(x_0-1) = \sum_{k=0}^\infty\left(\frac{-r}{\rho^\alpha}\right)^k\frac{(t-t_0)^{k\rho\alpha}}{\Gamma(k\alpha+1)}(x_0-1),$$

and hence the equilibrium point $z=1$ is asymptotically stable.

4.2. Analysis of the fractional-order modified quadratic logistic model (2)

We see that the model (2) has the equilibrium points $z_1 = 0, z_2 = m$ and $z_3 = k$. The corresponding right hand side function of model (2) is $f(x) = rx(t)(1 - \frac{x(t)}{k})(x(t) - m)$ and hence $f'(x) = r(1 - \frac{(x(t))}{k})(x(t) - m) + rx(t)(1 - \frac{(x(t))}{k}) - \frac{r}{k}x(t)(x(t) - m)$ and $f'(0) = -rm, \quad f'(m) = rm(1 - \frac{m}{k})$, and $f'(k) = -r(k - m)$.

The perturbed system associated to the equilibrium point $z = 0$ is the fractional linear system

$$ {}_{t_0}D^{\alpha,\rho}\theta(t) = -rm\theta(t), \quad \theta(t_0) = x_0, \quad t > t_0. \tag{27}$$

Using Theorem 5, the solution of system (27) is given by

$$\theta(t) = E_\alpha\left(\frac{-rm}{\rho^\alpha}(t - t_0)^{\rho\alpha}\right)x_0 = \sum_{k=0}^{\infty}\left(\frac{-rm}{\rho^\alpha}\right)^k\frac{(t - t_0)^{k\rho\alpha}}{\Gamma(k\alpha + 1)}x_0.$$

Since $r, \ \rho > 0$ the equilibrium point $z_1 = 0$ is asymptotically stable.

Also the perturbed system associated to the equilibrium point $z_2 = m$ is the fractional linear system

$$ {}_{t_0}D^{\alpha,\rho}\theta(t) = rm\left(1 - \frac{m}{k}\right)\theta(t), \quad \theta(t_0) = x_0 - m. \tag{28}$$

The solution of system (28) is given by

$$\begin{aligned}\theta(t) &= E_\alpha\left(\frac{rm(1 - \frac{m}{k})}{\rho^\alpha}(t - t_0)^{\rho\alpha}\right)(x_0 - m)\\ &= \sum_{k=0}^{\infty}\left(\frac{rm(1 - \frac{m}{k})}{\rho^\alpha}\right)^k\frac{(t - t_0)^{k\rho\alpha}}{\Gamma(k\alpha + 1)}(x_0 - m).\end{aligned}$$

Since $r, \ m, \ k, \ \rho > 0$ and $m < k$, then the equilibrium point $z_2 = m$ is unstable.

Finally, the perturbed system associated to the equilibrium point $z_3 = k$ is the fractional linear system

$$ {}_{t_0}D^{\alpha,\rho}\theta(t) = -r(k - m)\theta(t), \quad \theta(t_0) = x_0 - k. \tag{29}$$

The solution of system (29) is given by

$$\begin{aligned}\theta(t) &= E_\alpha\left(\frac{-r(k - m)}{\rho^\alpha}(t - t_0)^{\rho\alpha}\right)(x_0 - k)\\ &= \sum_{k=0}^{\infty}\left(\frac{-r(k - m)}{\rho^\alpha}\right)^k\frac{(t - t_0)^{k\rho\alpha}}{\Gamma(k\alpha + 1)}(x_0 - k).\end{aligned}$$

Since $r, \ m, \ k, \ \rho > 0$ and $m < k$, then the equilibrium point $z_3 = k$ is asymptotically stable.

5. Numerical discussion

In this section we use the iterative power series method [31,32] for the numerical simulations of nonlinear problems (1) and (2). This method proves to be a very effective tool for solving nonlinear fractional differential equations. A brief description of this method is discussed below. We firstly subdivide the interval $[t_0, \ T]$ into N uniform subintervals $I_n = [t_n, t_{n+1}]$, for $n = 0, \cdots, N - 1$ with $t_n = t_0 + nh$ and $h = (T - t_0)/N$. Assume that the exact solution of Eq. (1) subject to condition (2) can be approximated by a function $X_n(t)$ on I_n represented by

$$X_n(t) = \sum_{j=0}^{n_{max}} c_{n,j}(t - t_n)^{kr\alpha}, \quad t \in I_n. \tag{30}$$

The coefficients $c_{n,\,j}$ can be obtained by minimizing the residual

$$R_n(t) = ({}_{t_0}D^{\alpha,\rho}X_n)(t) - rX_n(t)(1 - X_n(t)).$$

In the following examples, we used $h = 0.01$ and $n_{\max} = 3$. For more details, the reader is referred to [31,32].

Table 1
Absolute errors of solution trajectories for Example 1 at $\alpha = 1/2$ and $\rho = 1/2$ for different values of initial point, $x(0)$.

$x(0)$	$E(t)$
0.1	1.35582×10^{-14}
0.5	5.19429×10^{-9}
0.9	6.82237×10^{-13}
2.0	7.41072×10^{-7}

Example 1. Consider the following modified quadratic logistic model

$$({}_{t_0}D^{\alpha,\rho}x)(t) = rx(t)(1 - x(t)), \quad t > 0, \ x(0) = x_0, \tag{31}$$

where $r = 1/2$.

Obviously, this equation has two equilibria given by $x_1 = 0$ and $x_2 = 1$. Our aim in this example is to discuss the effect of α, ρ, and x_0 on the solution trajectories. Fig. 1 shows the solution trajectories as the initial point, at $t_0 = 0$, changes in the set $\{0.1, 0.5, 0.9, 2\}$ when $\alpha = 1/2$ and $\rho = 1/2$. One can clearly see that the solution trajectories converge to $x_2 = 1$ asymptotically for any x_0. Thus, we conclude that $x_2 = 1$ is asymptotically stable equilibrium solution whereas $x_1 = 0$ is unstable equilibrium solution. Notice that the rate of convergence of solution trajectories to the equilibrium solutions is strongly dependent on the initial points. Obviously, the rate of convergence to a steady state is higher as the initial point closer to the value of the steady state, $x_2 = 1$.

It should be noted that, since the exact solution of problem (31) is unknown, we measure the error bound using the residual to Eq. (31) as follows:

$$E(t) = \max_{t\in I}|Res(t)|, \tag{32}$$

where $I := [t_0, \ T]$ is the considered time domain and $Res(t)$ is the residual to Eq. (31) defined by

$$Res(t) = ({}_{t_0}D^{\alpha,\rho}x)(t) - rx(t)(1 - x(t)). \tag{33}$$

Table 1 displays the error bounds, $E(t)$, for different initial points of the solution trajectories which clearly indicates the accuracy of the present results.

The effect of changing α on the behaviour of solution trajectories is displayed in Fig. 2 at fixed values of $x(0) = 0.5$ and $\rho = 1/2$. We consider three values for α: 0.5, 0.75 and 1.0. It is clearly seen that the required time for the solution trajectories to reach the equilibrium point $x_2 = 1$ decreases as α increases. In other words, the rate of convergence to the steady state solution is proportional with α for the considered choice of ρ. This is obvious because of the fact that the Mittag-Leffler function is completely monotonic.

Fig. 3 shows the solution trajectories at $\alpha = 1/2$ and $x(0) = 4$, while ρ is changing from 1/2 to 3/2. It is obvious that the rate of convergence to the steady state solution, $x_2 = 1$, is inversely proportional with ρ. In addition, the value $x'(0)$ is increasing as ρ decreases.

Example 2. Consider the following modified cubic logistic model

$$({}_{t_0}D^{\alpha,\rho}x)(t) = rx(t)\left(1 - \frac{(x(t))}{k}\right)(x(t) - m), \quad t > 0, \ x(0) = x_0, \tag{34}$$

where $r = 1/2, \ m = 1$ and $k = 10$. It can be easily verified that this equation has three equilibria given by $x_1 = 0, \ x_2 = 1$ and $x_3 = m$.

Fig. 4 shows the solution trajectories as the initial point, at $t = 0$, changes in the set $\{0.5, 1.2, 4, 8, 12\}$ when $\alpha = 1/2$ and $\rho = 1/2$. One can clearly guess that the solution trajectories converges to

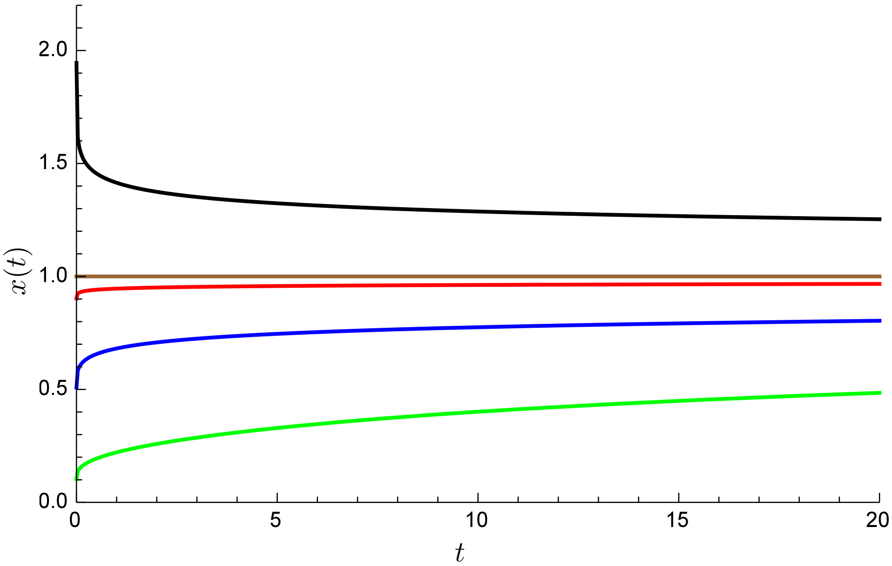

Fig. 1. Graphs of the solution trajectories for Example 1 at $\alpha = 1/2$ and $\rho = 1/2$ for different values of initial condition: green, $x(0) = 0.1$; blue, $x(0) = 0.5$; red, $x(0) = 0.9$; black, $x(0) = 2.0$.

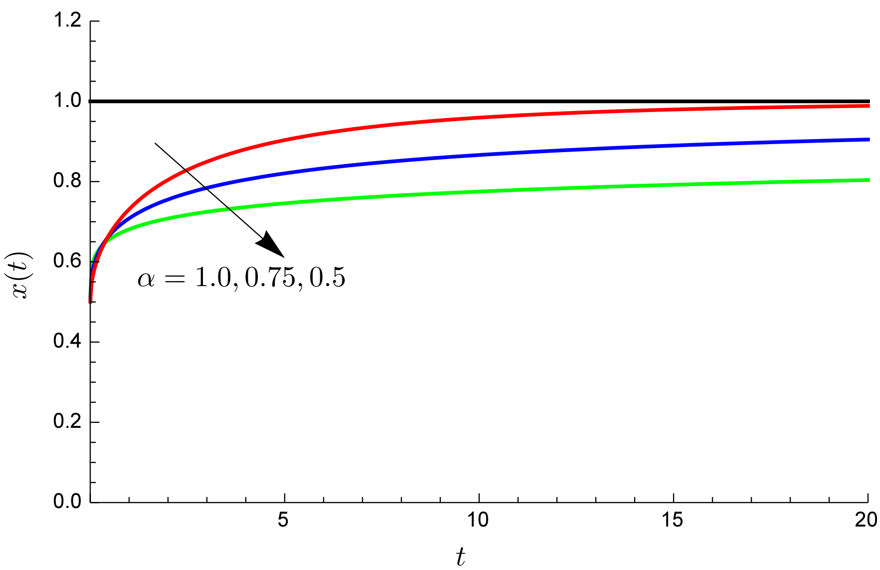

Fig. 2. Graphs of the solution trajectories for Example 1 at $x(0) = 4$ and $\rho = 1/2$ for different values of α: green, $\alpha = 0.5$; blue, $\alpha = 0.75$; red, $\alpha = 1.0$.

$x_1 = 0$ asymptotically for $0 < x_0 < 1$, while they converge to $x_3 = 10$ asymptotically for $x_0 > 1$. Thus, we conclude that $x_1 = 0$ and $x_3 = 10$ are asymptotically stable equilibrium solutions whereas $x_2 = 1$ is unstable equilibrium solution. Similar to the findings of the previous example, the rate of convergence of solution trajectories to the equilibrium solutions depends on the initial points. In other words, the rate of convergence to a steady state is higher as the initial point closer to the value of the steady state.

Table 2 displays the error bounds, $E(t)$, for different initial points of the solution trajectories which clearly indicates the accuracy of the present results. Notice that $E(t)$ in this example is defined by

$$E(t) = \max_{t\in I} |Res(t)|,$$

where

$$Res(t) = ({}_{t_0}D^{\alpha,\rho}x)(t) - rx(t)\left(1 - \frac{x(t)}{k}\right)(x(t) - m).$$

Table 2
Absolute errors of solution trajectories for Example 2 at $\alpha = 1/2$ and $\rho = 1/2$ for different values of initial condition, $x(0)$.

$x(0)$	$E(t)$
0.5	1.82939×10^{-11}
1.2	1.98952×10^{-12}
4.0	1.23030×10^{-9}
8.0	8.01894×10^{-10}
12.0	7.08892×10^{-8}

Fig. 5 displays the effect of changing the value of α on the behaviour of solution trajectories at $x(0) = 4$ and $\rho = 1/2$. We consider three values for α: 0.5, 0.75 and 1.0. Obviously, the required time for the solution trajectories to reach the equilibrium point $x_3 = 10$ decreases as α increases. In other words, the rate of

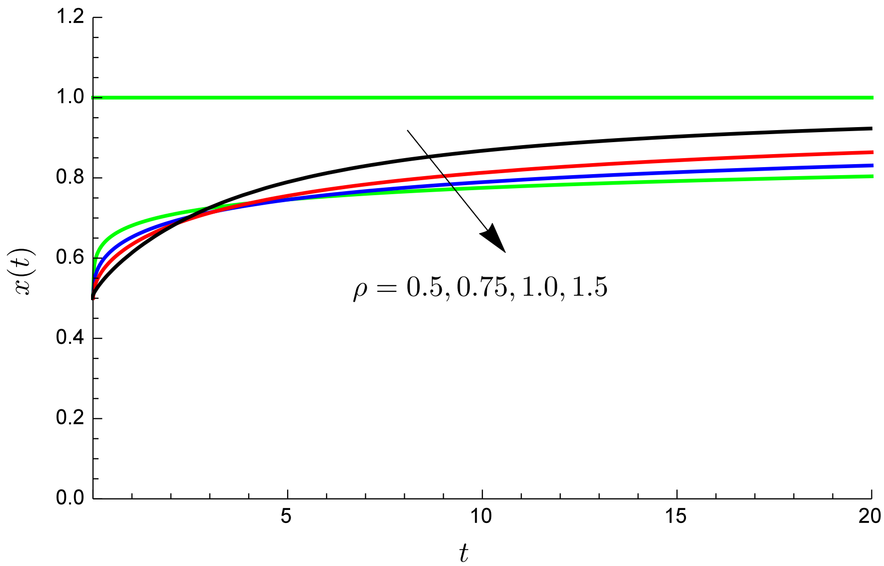

Fig. 3. Graphs of the solution trajectories for Example 1 at $\alpha = 1/2$ and $x(0) = 4$ for different values of ρ: black line, $\rho = 0.5$; red line, $\rho = 0.75$; blue line, $\rho = 1.0$; green line, $\rho = 1.5$.

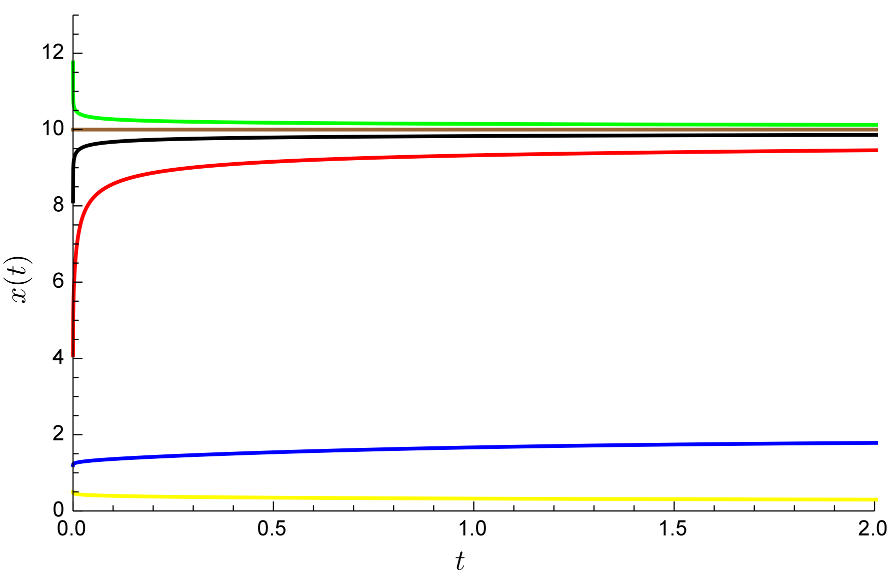

Fig. 4. Graphs of the solution trajectories for Example 2 at $\alpha = 1/2$ and $\rho = 1/2$ for different values of initial condition, $x(0)$: yellow line, $x(0) = 0.5$; blue line, $x(0) = 1.2$; red line, $x(0) = 4.0$; black line, $x(0) = 8.0$; green line, $x(0) = 12.0$.

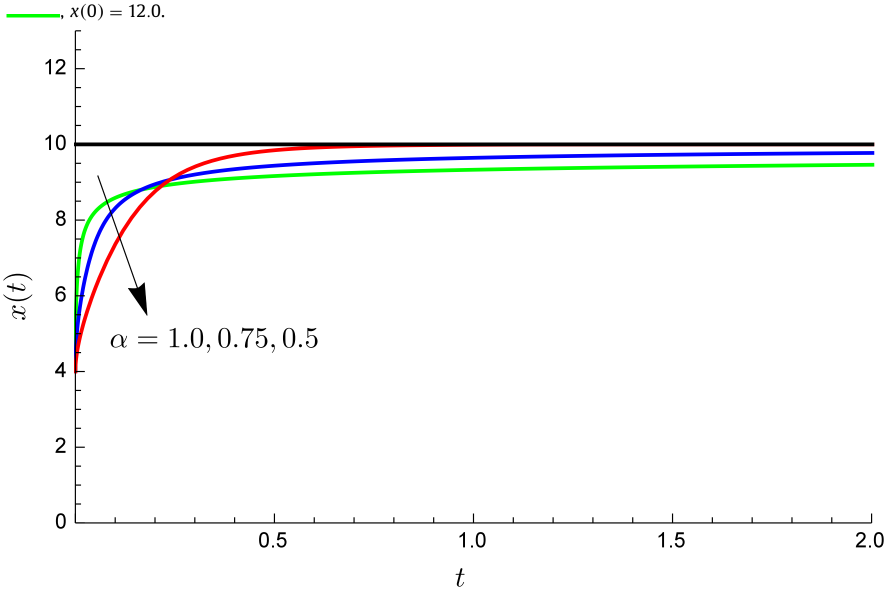

Fig. 5. Graphs of the solution trajectories for Example 2 at $x(0) = 4$ and $\rho = 1/2$ for different values of α: red line, $\alpha = 0.5$; blue line, $\alpha = 0.75$; green line, $\alpha = 1.0$.

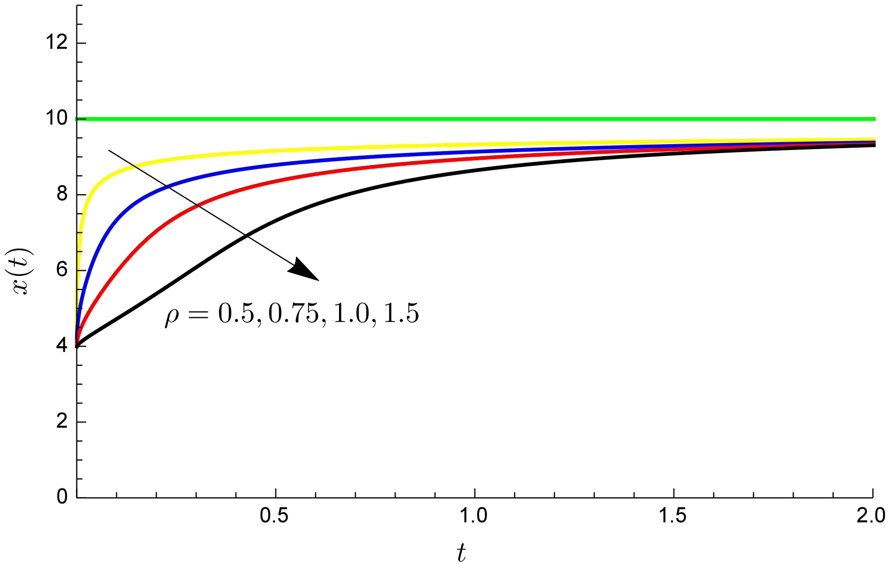

Fig. 6. Graphs of the solution trajectories for Example 2 at $\alpha = 1/2$ and $x(0) = 4$ for different values of ρ: yellow, $\rho = 0.5$; blue, $\rho = 0.75$; red, $\rho = 1.0$; black, $\rho = 1.5$.

convergence to the steady state solution is proportional with α. In addition, the slope of the tangent line at $t = 0$, $x'(0)$, is dramatically increasing with the decreasing of α.

The solution trajectories at $\alpha = 1/2$ and $x(0) = 4$, while ρ is changing from 1/2 to 3/2 are displayed in Fig. 6. It is obvious that the rate of convergence to the steady state solution, $x_3 = 10$, is inversely proportional with ρ. In addition, the value $x'(0)$ is increasing as ρ decreases.

6. Conclusion

In this article, we analysed the logistic equation that contains Caputo fractional operators generated by the conformable derivative. This fractional derivative involves two parameters: α, the order of the derivative and $\rho > 0$ that emerges from the conformable derivative. Existence and uniqueness results and stability were discussed. In addition, numerical examples were considered to demonstrate these results. It was seen that even the fact that ρ appears in the solution of the perturbed system, being positive, it does not affect the stability of the models. To assert this numerically, we gave examples when $\rho < 1$ and $\rho > 1$. When $\rho = 1$, we reobtain the results in [23,24]. It is worth mentioning that for the sake of comparison we took the values of α considered in [23,24].

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