

Applications of the extended uncertainty principle in AdS and dS spaces

B. Hamil¹, M. Merad^{2,a}, and T. Birkandan³

¹ Département de TC de SNV, Université Hassiba Benbouali, Chlef, Algeria

² Département des Sciences de la Matière, Faculté des Sciences Exactes, Université de Oum El Bouaghi, 04000 Oum El Bouaghi, Algeria

³ Department of Physics, Istanbul Technical University, 34469 Istanbul, Turkey

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Abstract. All commutation relations are modified in (anti)-de Sitter background and the Heisenberg uncertainty principle is changed to the so-called extended uncertainty principle (EUP). In this scenario, the commutators between position and momentum operators are functions of the position space variables, instead of a constant and the coordinate representation of the momentum operators for this model becomes coordinate dependent. In the AdS space, a lower bound on momentum uncertainty arises, which is not present in the dS space. In this paper, we present an exact solution of the D -dimensional free particle, the harmonic oscillator and pseudoharmonic oscillator in AdS and dS spaces. The eigenfunctions are determined for both cases and the energy eigenvalues are obtained.

1 Introduction

The idea of noncommutative spacetime is quite old. It has been incorporated in quantum fields by Snyder [1, 2] in order to regularize the divergences in quantum field theories. Snyder model admits a fundamental length scale and it is invariant under the Lorentz group action. However, this idea was largely forgotten due to the remarkable success of the renormalization theory in quantum electrodynamics. After the work of Kontsevich [3], Snyder model has attracted great attention. Modern candidates for a theory of quantum gravity and string theory suggests that the structure of spacetime may be noncommutative at scales close to the Planck length.

The Snyder model is based on the algebra generated by the positions X_μ , momenta P_μ and is given by

$$[X_\mu, P_\nu] = i\hbar(\eta_{\mu\nu} + \beta P_\mu P_\nu); \quad [P_\mu, P_\nu] = 0; \quad [X_\mu, X_\nu] = i\hbar\beta J_{\mu\nu}, \quad (1)$$

where $J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu$ are the generators of the Lorentz symmetry and β is a coupling constant which is generally assumed to be of the order of the Planck length. In Snyder model, the commutators between the operator of the positions X_μ and momenta P_μ are not constant but are functions of the phase space variables, and the Heisenberg uncertainty principle should be generalized by imposing the lower bound on position uncertainty. Some studies implies that the Snyder model can be interpreted as an example of doubly (deformed) special relativity (DSR) [4–6], namely, a theory where there exist two observer-independent scales, velocity which is identified by the speed of light, and length which is identified by the Planck length.

On the other hand, after the introduction of the cosmological constant Λ into Einstein's general theory of relativity to obtain static cosmological solutions, Dirac [7] suggested studying the equations of atomic physics in a spacetime with cosmological constant Λ which is called the de Sitter space. Therefore, in the presence of a cosmological constant, the ordinary Poincaré special relativity must be replaced by a de Sitter special relativity. It is interesting to note that the deformed de Sitter algebra has two invariant scales [8], the speed of light c and the radius $R = \sqrt{\pm 3/\Lambda}$, where the \pm sign is for AdS and dS, respectively. Recently, it has been shown by Guo *et al.* [9] that there is a correspondence between Snyder's model and the dS-invariant special relativity as well as the minimum uncertainty-like relation, which means that the physics close the Planck length ℓ_p and the radius $R = \sqrt{3/\Lambda}$ should be dual to

^a e-mail: meradm@gmail.com

each other. In addition, with a suitably chosen parametrization, Mignemi [10] showed that on the (A)dS background, the commutation relation should be modified by introducing corrections proportional to the cosmological constant. On (anti-) de Sitter background, the deformed commutation relations introduced by Mignemi is given by

$$[X_\mu, P_\nu] = i\hbar \left(\eta_{\mu\nu} + \frac{X_\mu X_\nu}{R^2} \right); \quad [X_\mu, X_\nu] = 0; \quad [P_\mu, P_\nu] = \frac{i\hbar}{R^2} L_{\mu\nu}, \quad (2)$$

where $\mu, \nu = 0, 1, 2, 3$, and $L_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu$. One of the fundamental consequences of the deformed commutation relations (2) is the modification of the properties of the quantum system under consideration, namely the wave functions and the eigenvalues. This is the main reason why the number of recent works on so-called quantum mechanics on AdS and dS spaces have undergone a significant growth [11–20].

The Schrödinger equation plays a central role in studying the energy and wave function of a non-relativistic physical system. In this Letter, we study the spatial part of the deformed Heisenberg algebra (2), by using a coordinate representation. In arbitrary dimensions (D), we solve the equations explicitly for the free-particle and harmonic-oscillator cases and obtain the quantum-mechanical eigenvalues and eigenfunctions in both the AdS and dS cases. Then, in the context of 3-dimensional non-relativistic deformed Heisenberg algebra, we study the Pseudoharmonic oscillator and obtain the results.

2 Quantum mechanics on (anti)-de Sitter spaces

2.1 AdS space

The formalism of quantum mechanics on AdS space has been extended to arbitrary dimensions (D) [15]. The modified Heisenberg algebra reads

$$[X_i, P_j] = i\hbar (\delta_{ij} + \alpha X_i X_j); \quad i, j = 1, 2, \dots, D, \quad (3)$$

where α is a small positive parameter. In the limit $\alpha \rightarrow 0$, we recover the canonical commutation relations of the standard quantum mechanics. As in the case of ordinary quantum mechanics, the commutation relation (3) leads to the following extended uncertainty principle (EUP):

$$(\Delta X_i) (\Delta P_i) \geq \frac{\hbar}{2} \left(1 + \alpha (\Delta X_i)^2 \right). \quad (4)$$

Solving this equation with respect to (ΔP_i) , we get

$$(\Delta P_i) \geq \frac{\hbar}{2} \left(\frac{1}{(\Delta X_i)} + \alpha (\Delta X_i) \right), \quad (5)$$

which leads to a minimum uncertainty in momentum (MUM)

$$(\Delta P_k)_{\min} = \frac{\hbar\sqrt{\alpha}}{2}, \quad \forall k. \quad (6)$$

In coordinate space, the simplest representation of the operators X_i and P_i is

$$X_i = x_i; \quad P_i = \frac{\hbar}{i} f(x) \frac{\partial}{\partial x_i} = (\delta_{ij} + \alpha x_i x_j) p_j, \quad (7)$$

where x_j and p_j satisfy the standard commutation relations of ordinary quantum mechanics. This representation yields the following commutation relation for the momentum operator:

$$[P_i, P_j] = i\hbar\alpha (x_i p_j - x_j p_i). \quad (8)$$

It can be easily shown that the new operators X_i and P_j are Hermitian with respect to the scalar product

$$\langle \phi | \psi \rangle = \int \frac{d^D \vec{r}}{(1 + \alpha r^2)^{\frac{1+D}{2}}} \phi^*(\vec{r}) \psi(\vec{r}); \quad \text{where } r = \sum_{i=1}^D x_i^2. \quad (9)$$

2.2 dS space

In dS space, the deformed Heisenberg algebra leading to the EUP in arbitrary dimensions (D) is defined by the following commutation relation:

$$[X_i, P_j] = i\hbar(\delta_{ij} - \alpha X_i X_j), \tag{10}$$

this commutation relation leads to the following EUP relations:

$$(\Delta X_i)(\Delta P_i) \geq \frac{\hbar}{2} \left(1 - \alpha (\Delta X_i)^2\right). \tag{11}$$

This relation does not give the minimal uncertainty in momentum. Solving this equation with respect to (ΔX_i) , we get

$$-\frac{(\Delta P_i)}{\alpha\hbar} - \frac{1}{\alpha} \sqrt{\alpha + \frac{(\Delta P_i)^2}{\hbar^2}} \leq (\Delta X_i) \leq -\frac{(\Delta P_i)}{\alpha\hbar} + \frac{1}{\alpha} \sqrt{\alpha + \frac{(\Delta P_i)^2}{\hbar^2}}, \tag{12}$$

in the limit $(\Delta P_i) \rightarrow 0$, (ΔX_i) are bounded as $-\frac{1}{\sqrt{\alpha}} \leq (\Delta X_i) \leq \frac{1}{\sqrt{\alpha}}$. A representation of X_i and P_i that satisfies (10), may be taken as

$$X_i = x_i; \quad P_i = \frac{\hbar}{i} (\delta_{ij} - \alpha x_i x_j) \frac{\partial}{\partial x_j}. \tag{13}$$

It is important to note that the scalar product, in this case, is not the usual one, but it is defined as

$$\langle \phi | \psi \rangle = \int \frac{d^D \vec{r}}{(1 - \alpha r^2)^{\frac{1+D}{2}}} \phi^*(\vec{r}) \psi(\vec{r}). \tag{14}$$

In the following sections, we will employ those relations to study the Schrödinger equation for a free particle, harmonic oscillator and pseudoharmonic-oscillator potential. The main goal of this study is to derive the expressions of the energy eigenvalues and the corresponding eigenfunctions in AdS and dS spaces.

3 Free particle

Let us first consider the Schrödinger equation for a free particle in a D -dimensional AdS space, with Hamiltonian

$$H = \frac{P^2}{2m}; \quad \text{where } P^2 = \sum_{j=1}^D P_j^2. \tag{15}$$

In D -dimensions, we have

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} - \frac{\hat{L}^2}{r^2}, \quad \text{and} \quad \sum_{j=1}^D x_j \frac{\partial}{\partial x_j} = r \frac{\partial}{\partial r}, \tag{16}$$

where \hat{L}^2 is a partial differential operator called the hyperangular momentum operator, and its definition is analogous to that of the 3-dimensional angular momentum. The operator \hat{L}^2 possesses the following spectrum:

$$\hat{L}^2 \mathcal{Y}_\ell^M(\Omega_D) = \ell(\ell + D - 2) \mathcal{Y}_\ell^M(\Omega_D), \tag{17}$$

where $\ell = 0, 1, 2, \dots$ and $\mathcal{Y}_\ell^M(\Omega_D)$ is the hyperspherical harmonics. For $D = 2$, we have $\mathcal{Y}_\ell^M(\Omega_2) = \frac{e^{iM\varphi}}{\sqrt{2\pi}}$, and for $D = 3$, we have $\mathcal{Y}_\ell^M(\Omega_3) = \mathcal{Y}_\ell^M(\theta, \varphi)$. Using eqs. (15), (16), (17), and using the separation of variables as $\psi = \mathcal{Y}_\ell^M(\Omega_D) R_\ell(r)$, the Schrödinger equation for a free particle in the D -dimensional AdS space reads

$$\left\{ \left[(1 + \alpha r^2) \frac{\partial}{\partial r} \right]^2 + \frac{D-1}{r} (1 + \alpha r^2) \frac{\partial}{\partial r} - \frac{\ell(\ell + D - 2)}{r^2} + \frac{2mE}{\hbar} \right\} R_\ell(r) = 0. \tag{18}$$

In order to find the explicit solution of eq. (18), it is more convenient to define the variable $\sqrt{\alpha}\rho = \tan^{-1} \sqrt{\alpha}r$, that yields

$$\left\{ \frac{\partial^2}{\partial \rho^2} + \frac{(D-1)\sqrt{\alpha}}{\tan \sqrt{\alpha}\rho} \frac{\partial}{\partial \rho} - \alpha \frac{\ell(\ell + D - 2)}{\tan^2(\sqrt{\alpha}\rho)} + \frac{2mE}{\hbar^2} \right\} R_\ell(\rho) = 0. \tag{19}$$

We can introduce an auxiliary function defined by

$$R_\ell(\rho) = \sin^\ell(\sqrt{\alpha}\rho) F_\ell(\rho). \quad (20)$$

Then eq. (19) for $F_\ell(\rho)$ becomes

$$\left\{ \frac{\partial^2}{\partial \rho^2} + \frac{\sqrt{\alpha}(2\ell + D - 1)}{\tan(\sqrt{\alpha}\rho)} \frac{\partial}{\partial \rho} - \alpha\ell + \frac{2mE}{\hbar^2} \right\} F_\ell(\rho) = 0. \quad (21)$$

Let us transform this equation to the form of hypergeometric equation. To do this, we introduce another transformation of the form $\chi = \frac{1 - \cos(\sqrt{\alpha}\rho)}{2}$ which casts eq. (21) into the form

$$\left\{ \chi(1 - \chi) \frac{\partial^2}{\partial \chi^2} + \left(\ell + \frac{D}{2} - (2\ell + D)\chi \right) \frac{\partial}{\partial \chi} - \ell + \frac{2mE}{\alpha\hbar^2} \right\} F_\ell(\chi) = 0. \quad (22)$$

This is a hypergeometric differential equation and it has two linearly independent solutions around $\chi = 0$, one regular and one singular, namely,

$$F_\ell(\chi) = N_1 F(A, B, C, \chi) + N_2 \chi^{1-C} F(A - C + 1, B - C + 1, 2 - C, \chi), \quad (23)$$

where the parameters A, B, C are given by

$$A = \frac{D-1}{2} + \ell + \sqrt{\left(\frac{D-1}{2}\right)^2 + \ell(\ell + D - 2) + \frac{2mE}{\alpha\hbar^2}}, \quad (24)$$

$$B = \frac{D-1}{2} + \ell - \sqrt{\left(\frac{D-1}{2}\right)^2 + \ell(\ell + D - 2) + \frac{2mE}{\alpha\hbar^2}}, \quad (25)$$

$$C = \frac{D}{2} + \ell. \quad (26)$$

The hypergeometric function $F(A, B, C, \chi)$ corresponds to a series solution of the hypergeometric equation, namely $F(A, B, C, \chi) = \sum_{n=0}^{\infty} a_n \chi^n$ with the recurrence relation,

$$a_{n+1} = \frac{(n+A)(n+B)}{(n+1)(n+C)} a_n, \quad (27)$$

where $a_0 = 1$ and the root of the indicial equation is set to zero [21]. When A or B is a negative integer, the numerator of (27) vanishes. Then the series,

$$F(A, B, C, \chi) = 1 + \frac{AB}{C} \chi + \frac{A(A+1)B(B+1)}{C(C+1)} \chi^2 + \dots, \quad (28)$$

is truncated to yield a polynomial of the degree n when $A = -n$ or $B = -n$. In both cases, we have

$$E_{n,\ell}^{AdS} = \frac{\alpha\hbar^2}{2m} (n^2 + n(2\ell + D - 1) + \ell), \quad (29)$$

and the final solution can be written as

$$\psi_{n,\ell}^{AdS} = \mathcal{N}_{n,\ell} \mathcal{Y}_\ell^M(\Omega_D) \sin^\ell(\sqrt{\alpha}\rho) F\left(D - 1 + 2\ell + n, -n, C, \frac{1 - \cos(\sqrt{\alpha}\rho)}{2}\right). \quad (30)$$

where $\mathcal{N}_{n,\ell}$ is the normalization constants. In dS space the energy is not quantized and the eigenfunctions can be written in terms of the hypergeometric function as

$$\psi_\ell^{dS} = \mathcal{C} \mathcal{Y}_\ell^M(\Omega_D) \sinh^\ell(\sqrt{\alpha}\eta) F\left(A', B', C, \frac{1 - \cosh(\sqrt{\alpha}\eta)}{2}\right). \quad (31)$$

with $\eta = \frac{\tanh^{-1}\sqrt{\alpha}r}{\sqrt{\alpha}}$ and,

$$A' = \frac{D-1}{2} + \ell + \sqrt{\left(\frac{D-1}{2}\right)^2 + \ell(\ell + D - 2) - \frac{2mE}{\alpha\hbar^2}}, \quad (32)$$

$$B' = \frac{D-1}{2} + \ell - \sqrt{\left(\frac{D-1}{2}\right)^2 + \ell(\ell + D - 2) - \frac{2mE}{\alpha\hbar^2}}. \quad (33)$$

4 Harmonic oscillator

The spectrum of the harmonic oscillator was obtained in [22] in 3 dimensions. The radial part of the Schrödinger equation for the harmonic oscillator in the D -dimensional AdS space can be written as

$$\left\{ \left[(1 + \alpha r^2) \frac{\partial}{\partial r} \right]^2 + \frac{D-1}{r} (1 + \alpha r^2) \frac{\partial}{\partial r} - \frac{\ell(\ell + D - 2)}{r^2} - \frac{m^2 \omega^2 r^2}{\hbar^2} + \frac{2mE}{\hbar^2} \right\} u_\ell(r) = 0. \tag{34}$$

The form of this equation is the same that holds for the harmonic oscillator in arbitrary dimensions with minimal length uncertainty relations [23], but with different coefficients, and it can be solved in the same way. We change the variable from r to ϱ , by setting $\varrho = \frac{\tan^{-1}(\sqrt{\alpha}r)}{\sqrt{\alpha}}$. Then eq. (34) can be written as

$$\left\{ \frac{\partial^2}{\partial \varrho^2} + \frac{(D-1)\sqrt{\alpha}}{\tan(\sqrt{\alpha}\varrho)} \frac{\partial}{\partial \varrho} - \alpha \frac{\ell(\ell + D - 2)}{\tan^2(\sqrt{\alpha}\varrho)} - \frac{m^2 \omega^2}{\alpha \hbar^2} \tan^2(\sqrt{\alpha}\varrho) + \frac{2mE}{\hbar^2} \right\} u_\ell(\varrho) = 0. \tag{35}$$

In order to solve the above equation, we perform a substitution

$$u_\ell(\varrho) = \sin^\ell(\sqrt{\alpha}\varrho) \cos^\kappa(\sqrt{\alpha}\varrho) \Phi_\ell(\varrho). \tag{36}$$

Using this function in eq. (35), we find the differential equation of the form

$$\left\{ \frac{\partial^2}{\partial \varrho^2} + \sqrt{\alpha} \left(\frac{(2\ell + D - 1)}{\tan(\sqrt{\alpha}\varrho)} - 2\kappa \tan(\sqrt{\alpha}\varrho) \right) \frac{\partial}{\partial \varrho} - (2\ell + D) \kappa \alpha + \alpha \left(\kappa(\kappa - 1) - \frac{m^2 \omega^2}{\alpha^2 \hbar^2} \right) \tan^2(\sqrt{\alpha}\varrho) - \alpha \ell + \frac{2mE}{\hbar^2} \right\} \Phi_\ell(\varrho) = 0. \tag{37}$$

Here, we choose κ to eliminate the term proportional to $\tan^2(\sqrt{\alpha}\varrho)$ by demanding

$$\kappa(\kappa - 1) - \frac{m^2 \omega^2}{\alpha^2 \hbar^2} = 0. \tag{38}$$

This equation leads to the following expression of κ :

$$\kappa_+ = \frac{1}{2} + \frac{m\omega}{\alpha\hbar} \sqrt{1 + \frac{\alpha^2 \hbar^2}{4m^2 \omega^2}}; \quad \kappa_- = \frac{1}{2} - \frac{m\omega}{\alpha\hbar} \sqrt{1 + \frac{\alpha^2 \hbar^2}{4m^2 \omega^2}}, \tag{39}$$

and the second solution κ_- leads to a non-physical wave function. Then, the equation for $\Phi_\ell(\varrho)$ simplifies to

$$\left\{ \frac{\partial^2}{\partial \varrho^2} + \sqrt{\alpha} \left(\frac{(2\ell + D - 1)}{\tan(\sqrt{\alpha}\varrho)} - 2\kappa \tan(\sqrt{\alpha}\varrho) \right) \frac{\partial}{\partial \varrho} - (2\ell + D) \kappa \alpha - \alpha \ell + \frac{2mE}{\hbar^2} \right\} \Phi_\ell(\varrho) = 0. \tag{40}$$

It is convenient to perform another change of variable $\xi = \sin^2(\sqrt{\alpha}\varrho)$, in terms of which eq. (37) can be written in the hypergeometric form

$$\left\{ \xi(1 - \xi) \frac{\partial^2}{\partial \xi^2} + \left[\left(\ell + \frac{D}{2} \right) - \left(\kappa + \ell + \frac{D}{2} + \frac{1}{2} \right) \xi \right] \frac{\partial}{\partial \xi} - \left(\frac{\ell}{2} + \frac{D}{4} \right) \kappa - \frac{\ell}{4} + \frac{mE}{2\alpha\hbar^2} \right\} \Phi_\ell(\xi) = 0. \tag{41}$$

The complete solution of eq. (41) around $\xi = 0$ can therefore be read as

$$\Phi_\ell(\xi) = N'_1 F(a; b; c; \xi) + N'_2 \xi^{1-c} F(a - c + 1; b - c + 1; 2 - c; \xi), \tag{42}$$

where the following parameters have been used:

$$a = \frac{\kappa}{2} + \frac{D-1}{4} + \frac{\ell}{2} + \sqrt{\frac{\kappa(\kappa - 1) + \ell(\ell + D - 2)}{4} + \frac{(D-1)^2}{16} + \frac{mE}{2\alpha\hbar^2}}, \tag{43}$$

$$b = \frac{\kappa}{2} + \frac{D-1}{4} + \frac{\ell}{2} - \sqrt{\frac{\kappa(\kappa - 1) + \ell(\ell + D - 2)}{4} + \frac{(D-1)^2}{16} + \frac{mE}{2\alpha\hbar^2}}, \tag{44}$$

$$c = \ell + \frac{D}{2}, \tag{45}$$

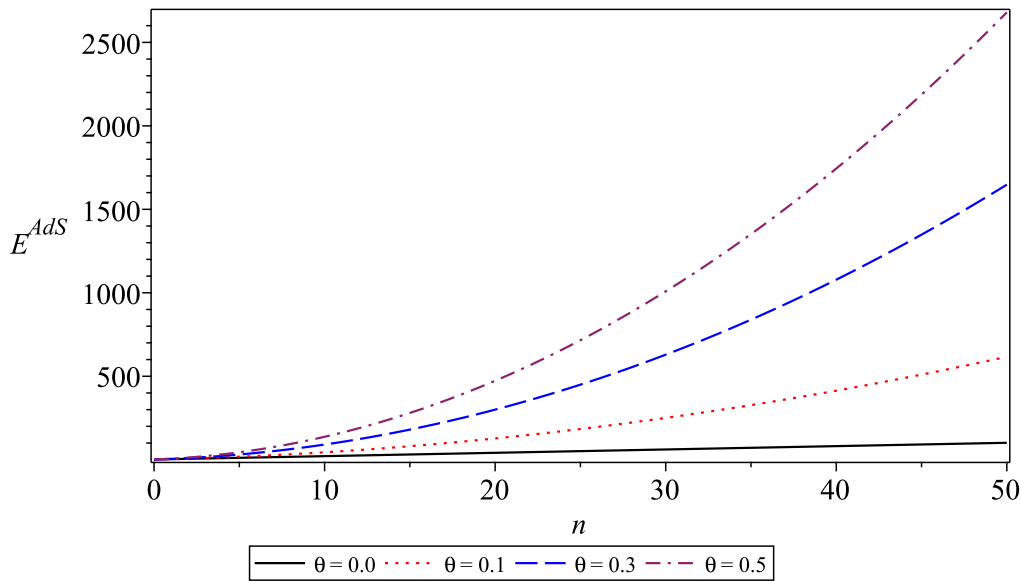


Fig. 1. Energy spectrum in AdS space as a function of n for different values of the parameter θ .

and N'_1, N'_2 are arbitrary coefficients. When a or b is a negative integer, the hypergeometric function $F(a; b; c, z)$ reduces to a polynomial of degree n ,

$$-n = \frac{\kappa}{2} + \frac{D-1}{4} + \frac{\ell}{2} - \sqrt{\frac{\kappa(\kappa-1) + \ell(\ell+D-2)}{4} + \frac{(D-1)^2}{16} + \frac{mE}{2\alpha\hbar^2}}. \tag{46}$$

Using the relationship between the Jacobi polynomial $P_n^{(\mu,\nu)}(x)$ and the hypergeometric function, namely,

$$P_n^{(\mu,\nu)}(x) = \frac{(-1)^n \Gamma(n+1+\nu)}{n!(1+\nu)} F\left(n+\mu+\nu+1; -n; 1+\nu, \frac{1+x}{2}\right), \tag{47}$$

we may further rewrite the wave functions for any n as

$$\psi_{n,\ell}(r, \Omega_D) = const. \times \mathcal{Y}_\ell^M(\Omega_D) r^\ell (1+\alpha r^2)^{-\frac{\ell+\kappa+}{2}} P_n^{(\ell+\frac{D-2}{2}, \frac{1}{2}+\kappa+)}\left(\frac{1-\alpha r^2}{1+\alpha r^2}\right). \tag{48}$$

The energy spectrum is then obtained as

$$E_{n,\ell}^{AdS} = \hbar\omega \left(2n + \ell + \frac{D}{2}\right) \sqrt{1 + \frac{\alpha^2 \hbar^2}{4m^2 \omega^2}} + \frac{2\alpha \hbar^2}{m} \left(n^2 + n \left(\ell + \frac{D}{2}\right)\right) + \frac{\alpha \hbar^2}{m} \left(\ell + \frac{D}{4}\right). \tag{49}$$

Notice that the spectrum energy depends on the square of the quantum number n , and the corrections to the spectrum are always positive. Moreover, the energy eigenvalues in AdS background are bigger than the energies in the ordinary quantum mechanics. The usual spectrum is recovered in the limit $\alpha = 0$. From the exact expression eq. (49), the explicit expressions of the energy eigenvalues of the 2D and 3D harmonic oscillators are found as

$$E_{n,\ell,D=2}^{AdS} = \hbar\omega (2n + \ell + 1) \sqrt{1 + \frac{\alpha^2 \hbar^2}{4m^2 \omega^2}} + \frac{2\alpha \hbar^2}{m} (n^2 + n(\ell + 1)) + \frac{\alpha \hbar^2}{m} \left(\ell + \frac{1}{4}\right), \tag{50}$$

$$E_{n,\ell,D=3}^{AdS} = \hbar\omega \left(2n + \ell + \frac{3}{2}\right) \sqrt{1 + \frac{\alpha^2 \hbar^2}{4m^2 \omega^2}} + \frac{2\alpha \hbar^2}{m} \left(n^2 + n \left(\ell + \frac{3}{2}\right)\right) + \frac{\alpha \hbar^2}{m} \left(\ell + \frac{3}{4}\right). \tag{51}$$

To show the effect of MUM on the energy eigenvalues, we calculate the energy eigenvalues $E^{AdS} \equiv \frac{E_{n,\ell,D=3}^{AdS}}{\hbar\omega}$ for different values of $\theta = \frac{\alpha \hbar}{m\omega}$ parameter. We choose $\theta = 0, 0.1, 0.3, 0.5$ and $\ell = 0$. The results are shown in fig. 1 and we note that the energy increases with quantum number n .

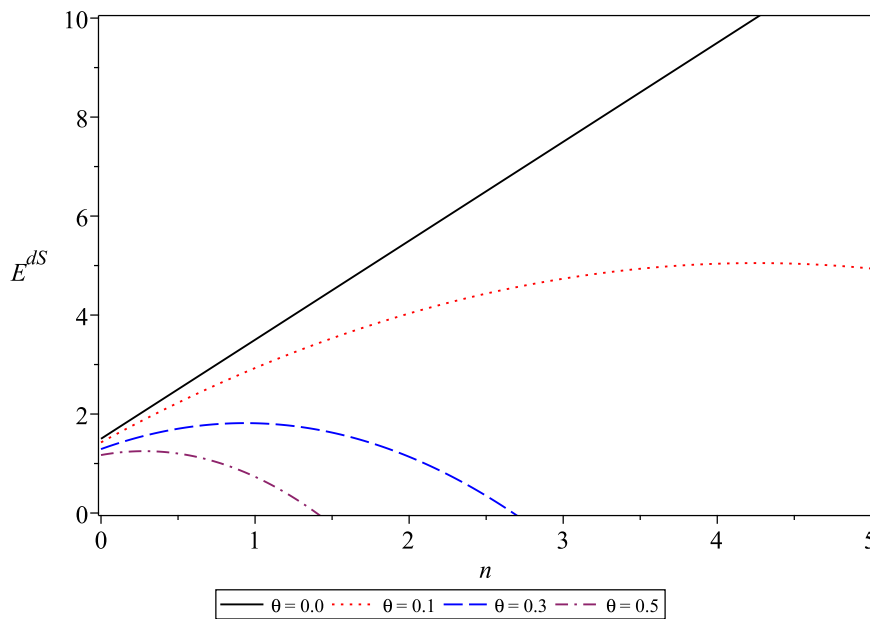


Fig. 2. Energy spectrum in dS space as a function of n for the different values of the parameter θ .

On the other hand, in dS space, the radial part of the Schrödinger equation for the harmonic oscillator is

$$\left\{ \left[(1 - \alpha r^2) \frac{\partial}{\partial r} \right]^2 + \frac{D - 1}{r} (1 - \alpha r^2) \frac{\partial}{\partial r} - \frac{\ell(\ell + D - 2)}{r^2} - \frac{m^2 \omega^2 r^2}{\hbar^2} + \frac{2mE}{\hbar^2} \right\} \Theta_\ell(r) = 0. \tag{52}$$

Performing a series of similar algebraic operations, the energy eigenvalues of the D -dimensional quantum harmonic oscillator in dS space are obtained as

$$E_{n,\ell}^{dS} = \hbar\omega \left(2n + \ell + \frac{D}{2} \right) \sqrt{1 + \frac{\alpha^2 \hbar^2}{4m^2 \omega^2}} - \frac{2\alpha \hbar^2}{m} \left(n^2 + n \left(\ell + \frac{D}{2} \right) \right) - \frac{\alpha \hbar^2}{m} \left(\ell + \frac{D}{4} \right). \tag{53}$$

It is remarkable that the above expression of energy spectrum in dS space yields smaller energy values than that of the ordinary quantum mechanics, and also the relation (53) would give negative eigenvalues when the quantum numbers n are large enough as the term $\frac{2\alpha \hbar^2 n^2}{m}$ decreases faster than the terms proportional to n . To preserve the bound $E \geq 0$, one must impose that $n \leq \frac{m\omega}{2\alpha \hbar} \sqrt{1 + \frac{\alpha^2 \hbar^2}{4m^2 \omega^2}} - \frac{2\alpha^2 \hbar^2}{m^2 \omega^2} \left(\ell + \frac{D}{4} \right) + \frac{\alpha^2 \hbar^2}{m^2 \omega^2} \left(\ell + \frac{D}{2} \right)^2 - \frac{1}{2} \left(\ell + \frac{D}{2} \right) + \frac{m\omega}{2\alpha \hbar} \sqrt{1 + \frac{\alpha^2 \hbar^2}{4m^2 \omega^2}}$. As a result of this behavior, only a finite number of energy levels are present. In fig. 2, we plot the energy eigenvalues $E^{dS} \equiv \frac{E_{n,\ell,D=3}}{\hbar\omega}$ as a function of the quantum number n for various values of θ , ($\theta = 0, 0.1, 0.3, 0.5, \ell = 0$). We note that the energy decreases with quantum number n .

5 Pseudoharmonic oscillator

The pseudoharmonic-oscillator potential (PHO) was initially studied by Gol'dman *et al.* [24]. This potential is used to describe the rotovibrational states of diatomic molecules and nuclear rotations and vibrations. Due to its importance in chemical physics and molecular physics, the Schrödinger equation for this potential has been widely studied by many authors [25–27]. The pseudoharmonic potential has the form

$$V(r) = D_e \left(\frac{r}{r_e} - \frac{r_e}{r} \right)^2, \tag{54}$$

where $D_e = \frac{1}{8K_e r_e}$ is the dissociation energy with the force constant K_e and r_e is the equilibrium internuclear distance of a given diatomic molecule. The PHO potential (54) contains both the harmonic and the inverse square interactions.

The Schrödinger equation for PHO potential in AdS space is given as

$$\left\{ \left[(1 + \alpha r^2) \frac{\partial}{\partial r} \right]^2 + \frac{2}{r} (1 + \alpha r^2) \frac{\partial}{\partial r} - \frac{\hat{L}^2 + \frac{2mD_e r_e^2}{\hbar^2}}{r^2} - \frac{2mD_e}{r_e^2 \hbar^2} r^2 + \frac{2m(E + 2D_e)}{\hbar^2} \right\} \Psi(r, \Omega) = 0. \quad (55)$$

Employing the transformation $\bar{\Psi}(r, \theta, \varphi) = \mathcal{Y}_\ell^M(\theta, \varphi) \Xi_\ell(r)$, the above equation reduces to the form

$$\left\{ \left[(1 + \alpha r^2) \frac{\partial}{\partial r} \right]^2 + \frac{2}{r} (1 + \alpha r^2) \frac{\partial}{\partial r} - \frac{\ell(\ell + 1) + \frac{2mD_e r_e^2}{\hbar^2}}{r^2} - \frac{2mD_e}{r_e^2 \hbar^2} r^2 + \frac{2m(E + 2D_e)}{\hbar^2} \right\} \Xi_\ell(r) = 0, \quad (56)$$

and the energy eigenvalues and the corresponding wave functions of the PHO potential can be determined immediately by transforming this equation (for $D = 3$) to a Schrödinger-like equation

$$\left\{ \left[(1 + \alpha r^2) \frac{\partial}{\partial r} \right]^2 + \frac{2}{r} (1 + \alpha r^2) \frac{\partial}{\partial r} - \frac{k(k + 1)}{r^2} - \frac{m^2 \omega^2 r^2}{\hbar^2} + \frac{2m\mathcal{E}}{\hbar^2} \right\} u_\ell(r) = 0, \quad (57)$$

with

$$\mathcal{E} = E + 2D_e; \quad \frac{2D_e}{r_e^2} = m\omega^2 \quad \text{and} \quad k(k + 1) = \ell(\ell + 1) + \frac{2mD_e r_e^2}{\hbar^2}, \quad (58)$$

$$k = -\frac{1}{2} + \sqrt{\left(\ell + \frac{1}{2}\right)^2 + \frac{2mD_e r_e^2}{\hbar^2}}. \quad (59)$$

The differential equation (57) corresponds to the Schrödinger equation of the harmonic-oscillator potential in AdS, with energy eigenvalues

$$\begin{aligned} E_{PHO}^{AdS} = & -2D_e + \frac{\hbar}{r_e} \sqrt{\frac{2D_e}{m}} \left(2n + 1 + \sqrt{\left(\ell + \frac{1}{2}\right)^2 + \frac{2mD_e r_e^2}{\hbar^2}} \right) \sqrt{1 + \frac{\alpha^2 r_e^2 \hbar^2}{8mD_e}} \\ & + \frac{2\alpha \hbar^2}{m} \left(n^2 + \left(n + \frac{1}{2}\right) \sqrt{\left(\ell + \frac{1}{2}\right)^2 + \frac{2mD_e r_e^2}{\hbar^2}} \right) + \frac{\alpha \hbar^2}{m} \left(2n + \frac{1}{4} \right). \end{aligned} \quad (60)$$

Due to the modification of the standard Heisenberg algebra, the above expression of energy spectrum depends on the deformation parameter α and its deviation grows quickly with the square of the quantum number n . Expanding the above expression for α , we obtain

$$E_{PHO}^{AdS} = E_{PHO} + \Delta E, \quad (61)$$

in the first order, where

$$E_{PHO} = -2D_e + \frac{\hbar}{r_e} \sqrt{\frac{2D_e}{m}} \left(2n + 1 + \sqrt{\left(\ell + \frac{1}{2}\right)^2 + \frac{2mD_e r_e^2}{\hbar^2}} \right), \quad (62)$$

are the unperturbed levels [26], and the correction term caused by MUM is obtained as

$$\Delta E = \frac{2\alpha \hbar^2}{m} \left(n^2 + \left(n + \frac{1}{2}\right) \sqrt{\left(\ell + \frac{1}{2}\right)^2 + \frac{2mD_e r_e^2}{\hbar^2}} \right) + \frac{\alpha \hbar^2}{m} \left(2n + \frac{1}{4} \right). \quad (63)$$

On other hand, according to the [28] the parameter $\vartheta^2 = \frac{2mD_e r_e^2}{\hbar^2}$ is very large for the majority of molecules. Expanding eq. (60) for powers of $\frac{1}{\vartheta}$, we get

$$\begin{aligned} E_{PHO}^{AdS} = & \left(n + \frac{1}{2} \right) \frac{4D_e}{\vartheta} + D_e \frac{\left(\ell + \frac{1}{2}\right)^2}{\vartheta^2} \\ & + \frac{\alpha \hbar^2}{m} \left[\vartheta(2n + 1) + \frac{1}{\vartheta} \left(n + \frac{1}{2} \right) \left(\ell + \frac{1}{2} \right)^2 + \left(2n^2 + 2n + \frac{1}{4} \right) \right] + \mathcal{O} \left(\frac{1}{\vartheta^3} \right). \end{aligned} \quad (64)$$

Table 1. Numerical data for diatomic molecules N₂ and CO.

Molecules	D_e (cm ⁻¹)	r_e (Å ⁰)	m (amu)
N ₂	96288.03528	1.0940	7.00335
CO	87471.42567	1.1282	6.860586

Table 2. Energy eigenvalues of the PHO potential for N₂, CO diatomic molecules for different values of n and ℓ in eV.

Molecules		N ₂		CO	
State n	ℓ	E_{PHO}	ΔE	E_{PHO}	ΔE
0	0	0.109182042289	$1.251257880144 \times 10^{-55}$	0.101955019262	$1.242649780446 \times 10^{-55}$
1	0	0.327421448290	$3.773766136708 \times 10^{-55}$	0.305745384042	$3.748357867489 \times 10^{-55}$
	1	0.327920156741	$3.773844455105 \times 10^{-55}$	0.306224073076	$3.748440047213 \times 10^{-55}$
2	0	0.545660854291	$6.319122960444 \times 10^{-55}$	0.509535748822	$6.277389984418 \times 10^{-55}$
	1	0.546159562742	$6.319253491105 \times 10^{-55}$	0.510014437856	$6.277526950626 \times 10^{-55}$
	2	0.547156948396	$6.319514544247 \times 10^{-55}$	0.510971784232	$6.277800873972 \times 10^{-55}$
3	0	0.763900260291	$8.887328351351 \times 10^{-55}$	0.713326113602	$8.829746131234 \times 10^{-55}$
	1	0.764398968743	$8.887511094277 \times 10^{-55}$	0.713804802636	$8.829937883924 \times 10^{-55}$
	2	0.765396354396	$8.887876568676 \times 10^{-55}$	0.714762149012	$8.830321376610 \times 10^{-55}$
	3	0.766892354765	$8.888424751653 \times 10^{-55}$	0.716198089357	$8.830896583904 \times 10^{-55}$
4	0	0.982139666292	$1.147838230943 \times 10^{-54}$	0.917116478382	$1.140542630794 \times 10^{-54}$
	1	0.982638374743	$1.147861726462 \times 10^{-54}$	0.917595167416	$1.140567284711 \times 10^{-54}$
	2	0.983635760397	$1.147908716028 \times 10^{-54}$	0.918552513792	$1.140616590913 \times 10^{-54}$
	3	0.985131760766	$1.147979196696 \times 10^{-54}$	0.919988454137	$1.140690546137 \times 10^{-54}$
	4	0.987126282143	$1.148073164052 \times 10^{-54}$	0.921902893416	$1.140789145487 \times 10^{-54}$

This result shows the influence of MUM on the energy levels of the PHO potential. The first term describes the harmonic vibration with frequency of $\omega = \frac{4D_e}{\hbar v}$, the second term describes the rotation with constant moment of inertia and the third term represents the corrections caused by the MUM.

In dS space, the energy of dissociation is

$$\begin{aligned}
 E_{PHO}^{dS} = & -2D_e + \frac{\hbar}{r_e} \sqrt{\frac{2D_e}{m}} \left(2n + 1 + \sqrt{\left(\ell + \frac{1}{2}\right)^2 + \frac{2mD_e r_e^2}{\hbar^2}} \right) \sqrt{1 + \frac{\alpha^2 r_e^2 \hbar^2}{8mD_e}} \\
 & - \frac{2\alpha \hbar^2}{m} \left(n^2 + \left(n + \frac{1}{2}\right) \sqrt{\left(\ell + \frac{1}{2}\right)^2 + \frac{2mD_e r_e^2}{\hbar^2}} \right) - \frac{\alpha \hbar^2}{m} \left(2n + \frac{1}{4} \right). \tag{65}
 \end{aligned}$$

The above relation shows that the energy levels of a diatomic molecule interacting with the PHO potential in dS space are smaller than the energies found in AdS space and the energy levels tend to negative values for large n .

Finally, we calculate the energy eigenvalues of the PHO potential for N₂ and CO diatomic molecules by means of eq. (60) with the potential parameter values given in table 1. The explicit values of the energy spectrum for different values of n , ℓ and $\alpha = 0.957 \times 10^{-34} \text{ m}^{-2}$ are shown in table 2. According to table 2, the effect of the deformation parameter on the energy spectrum is very small. The corrections are generally of the order 10^{-55} , thus it is unlikely that they are detectable experimentally.

6 Conclusion

We present a generalization of the exact solution of the D -dimensional Schrödinger equation for the free-particle and the harmonic-oscillator cases in the framework of AdS (anti-de Sitter) and dS (de Sitter) spaces. The D -dimensional wave functions are obtained analytically for both cases and the corresponding energy spectrum are studied.

For the free particle case, we obtained the Schrödinger equation in the form of a hypergeometric equation after a variable transformation. The energy eigenvalues have been found by applying the polynomial reduction condition of the hypergeometric function. In the case of the harmonic oscillator, we were able to write the wave functions in terms of Jacobi polynomials using a relation between the Jacobi polynomials and hypergeometric functions. We also presented graphically the behavior of the energy spectrum for different values of the deformation parameter in 3-dimensional AdS and dS spaces.

In the same context of this deformation, we studied the analytical solution of the pseudoharmonic potential in $D = 3$. The exact eigensolutions and the energy eigenvalues have been calculated for any angular momentum. Finally, in order to see the effect of the deformation on the physical systems and to compare them with the experimental results, a numerical table has been formed to present the binding energies of the Pseudoharmonic potential for N_2 and CO diatomic molecules with different values of n and ℓ . For these systems, we claimed that the energy corrections coming from the deformation parameter are unlikely to be detectable experimentally.

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