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A Hesitant Soft Fuzzy Rough Set and Its Applications

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ABSTRACT The difficulty of establishing a common membership degree is not because there is a margin of error or some possibility distribution values, but because there is a set of possible values. Based on hesitant fuzzy sets and soft sets, a hesitant soft fuzzy rough set model is proposed in this paper. Basic properties of hesitant soft fuzzy rough sets are investigated in detail. We obtain a decomposition theorem for a hesitant fuzzy binary relation, which states that every typical hesitant fuzzy binary relation on a set can be represented by a well-structured family of fuzzy binary relations on that set. Indeed, a hesitant fuzzy soft set can induce a hesitant fuzzy binary relation. Then we give the relationship between hesitant fuzzy rough sets and hesitant soft fuzzy rough sets. In addition, we prove a characterization theorem for the hesitant soft fuzzy rough set model, which shows that the lower and upper hesitant soft fuzzy rough approximations can be equivalently defined by using level sets of the hesitant fuzzy soft set. Finally, by analyzing the limitations and advantages in the existing literatures, we establish an approach to decision making problem based on the hesitant soft fuzzy rough set model proposed in this paper and give a practical example to illustrate the validity of the novel method.

INDEX TERMS Rough sets, hesitant fuzzy sets, soft sets, fuzzy binary relations, level sets.

I. INTRODUCTION

The contemporary concern about knowledge representation and information systems has put forward useful extensions of classical set theory such as fuzzy set theory and rough set theory. The concept of rough set was originally proposed by Pawlak [21] in 1982 as a formal tool for studying information systems characterized by insufficient and incomplete information. The starting point of this theory is an observation that objects having the same description are indiscernible with respect to available information. While the fuzzy set theory, introduced by Zadeh [36] in 1965, offers a wide variety of techniques for analyzing imprecise data. It soon evoked a natural question concerning possible connections between rough sets and fuzzy sets. It is generally accepted that these two theories are related, but distinct and complementary, to each other. Generally speaking, both theories address the problem of information granulation: the theory of fuzzy sets is centred upon fuzzy information granulation, whereas the rough set theory is focused on crisp information granulation.

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Pawlak's rough set can be described by a pair of crisp sets called the lower approximation and the upper approximation. The lower approximation is the greatest definable set contained in the given set of objects, while the upper approximation is the smallest definable set containing the given set. By using the concept of lower and upper approximations in rough set theory, knowledge hidden in information systems may be revealed and expressed in the form of decision rules. The rough set has been extended by miscellaneous ways including generalizing universes of discourse from one to two, objects from ordinary sets to fuzzy sets, relations from equivalence relations to other binary relations, and operators from conjuctions and disjuctions to fuzzy logical operators. In 1998, by applying a residual implication (for short, R-implication) to define the lower approximation operator, Morsi and Yakout [20] generalized the fuzzy rough sets, while the duality fails. In 2002, based on a border implication I (not necessarily a R-implication) and a triangular norm T, Radzikowska and Kerre [24] introduced (I, T)-fuzzy rough sets. For this model, the duality is partly holds. Later, Mi and Zhang [15] introduced (θ_T, σ_S)-generalized fuzzy rough sets, where θ_T is an residual implication based on a triangular norm T, and σ_S is the dual of θ's. Mi *et al.* [16] discussed (S, T)-generalized fuzzy rough sets, where T is a triangular norm and S is a triangular conorm. In 2009, Zhang *et al.* [41] proposed (I, T)-generalized interval-valued fuzzy rough sets. To modify the definition of the upper approximation operator such that the duality hold for a R-implication, in 2010, Yao *et al.* [35] defined a more general fuzzy rough sets so-called (I, J)-fuzzy rough sets, where I and J are two border implicators, which are real extensions of the fuzzy rough sets of Radzikowska and Kerre [24], and that of Mi and Zhang [15]. In 2013, Hu and Wong [11] investigated (S, T)-generalized interval-valued fuzzy rough sets and (θ_T, σ_S)-generalized interval-valued rough sets. In 2015, Gong and Zhang [9] defined the (I, J)-soft fuzzy rough set which generalized the (I, J)-fuzzy rough set of Yao et al. [35]. In 2018, based on two more extensive operations than left continuous triangular norms and right continuous triangular conorms, i.e., the binary operations $\odot$ and $\&$ on complete residuated and co-residuated lattice L, respectively, Qiao and Hu [23] proposed $(\odot, \&)$-fuzzy rough sets which generalized the (S, T)-generalized fuzzy rough sets [16] and (S, T)-generalized interval-valued fuzzy rough sets [11].

On the other hand, for the sake of attempting to better capture the possible subjectivity, uncertainty, imprecision of the evaluations, et cetera, Zadeh's introduction of fuzzy set (FS) was subsequently extended to different types for various applications. In recent years, Torra [28] introduced the concept of hesitant fuzzy set (HFS) as an extension of the FS in which the membership degree of a given element, called the hesitant fuzzy element (HFE), is defined as a set of possible values. This situation can be found in a group decision making problem. To clarify the necessity of introducing the HFS, consider a situation in which two decision makers discuss the membership degree of an element x to a set A, one wants to assign 0.3, but the other 0.5. Accordingly, the difficulty of establishing a common membership degree is not because there is a margin of error or some possibility distribution values, but because there is a set of possible values. In 2018, Alcantud and Torra proved decomposition theorems and extension principles for the hesitant fuzzy set [2]. In the HFS, the membership degree consists of several possible values reflecting the epistemic certainty but the epistemic uncertainty degree is ignored. Thus, Zhu *et al.* [44] proposed an extension of the HFS–dual hesitant fuzzy set (DHFS), where both the membership and non-membership degrees contain a set of possible values. Furthermore, all the fuzzy set, the IVFS and the IFS can be treated as the particular cases of the DHFS. The DHFS, by comparison, is able to reflect the gradual epistemic uncertainty to ill-known objects more granularly. Since the hesitant fuzzy set and its extensions indeed describe the thoughts of experts better because of a better tolerance, they have been widely applied in practical decision making processes such as the interval-valued intuitionistic hesitant fuzzy set [42], the generalized hesitant fuzzy set (GHFS) [22], the hesitant interval-valued fuzzy set (HIVFS) [31], the probabilistic hesitant fuzzy set (PHFS) [43] and the probabilistic dual hesitant fuzzy set (PDHFS) [10].

While probability theory, fuzzy set theory, rough set theory, and other mathematical tools are well known and often useful approaches to describing uncertainty, each of these theories has its inherent difficulties as pointed out in [17], [18]. The reason for these difficulties is the inadequacy of the parametrization tool of the theory. In 1999, Molodtsov [17] introduced the concept of soft set, which can be seen as a new mathematical tool for dealing with uncertainties. This so-called soft set theory is free from the difficulties affecting existing methods. In 2015, Tripathy and Arun [29] introduced the notion of characteristic function of a soft set and rectified basic operations on soft sets. In 2018, Molodtsov [19] introduced the concept of equivalence of soft sets and discussed the correct operations and correct relationships for soft sets on the basis of equivalence. However, in the practical model, the parameters in the soft set are vague words or sentences involve vague words. Considering this point, Maji *et al.* [12] introduced the notion of fuzzy soft set by combining the fuzzy set and the soft set. Roy and Maji [25] presented a fuzzy soft set theoretic approach towards a decision making problem. Yang *et al.* [33] introduced the concept of interval-valued fuzzy soft set and a decision making problem is analyzed by the interval-valued fuzzy soft set. In 2011, Gong *et al.* [8] introduced the interval-valued intuitionistic fuzzy soft set and described its application to multi-parameter group decision-making problems. In 2014, Wang *et al.* [30] introduced the hesitant fuzzy soft set (HFSS) by combining the notion of hesitancy with Molodtsov's soft set. As a further generalization, the interval-valued hesitant fuzzy soft set, the weighted interval-valued hesitant fuzzy soft set and their applications in decision making problem were presented by Zhang *et al.* [37].

Recently, a framework to combine fuzzy sets, rough sets and soft sets all together was provided by Feng *et al.* [5], [6], which gives rise to several interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. This study is a research hotspot, and it presents a potentially interesting research direction. In [6], Feng *et al.* proved that Pawlak's rough set model can be viewed as a special instance of the soft rough set. Subsequently, Meng *et al.* [14] further discussed the relationship between the soft rough set and the soft rough fuzzy set and introduced the soft fuzzy rough set. Sun and Ma [27] gave a new approach to decision making problems, they defined a new fuzzy soft set named as pseudo fuzzy soft set by exchanging the role of the universe and the parameter set, and a soft fuzzy rough set based on the pseudo fuzzy soft set. Although fuzzy rough set theory or soft fuzzy rough set theory can handle some decision making problems and quantify the ideas of decision makers by using a crisp number, one of the main features of decision-making activities should be described in hesitancy situations. When facing the problem, the decision-makers can not offer a comprehensive, accurate and flexible solution by using the fuzzy rough set or soft fuzzy rough set. But if

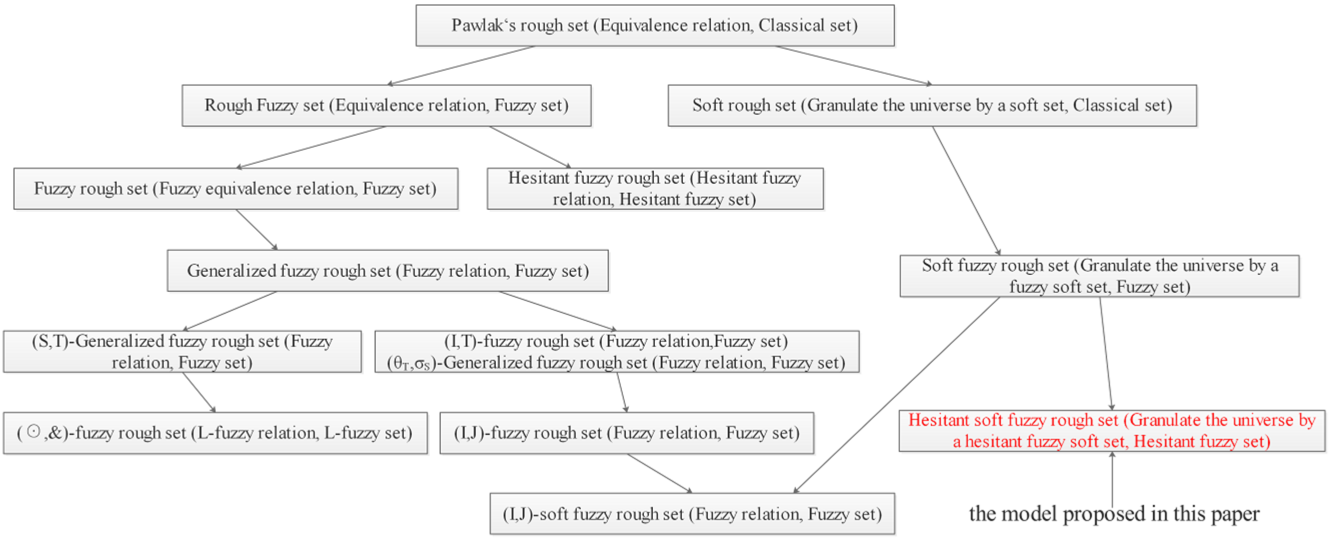

FIGURE 1. The extension process of rough sets. The first item in the bracket is relation and the second represents object.

the basic features of decision-making activities are described by several numbers within [0,1], we can avoid such a situation. So it is very natural for us to extend concepts from fuzzy rough set theory or soft fuzzy rough set theory to their generalizations in hesitant fuzzy set theory. In 2014, the fusion of hesitant fuzzy set and rough set–hesitant fuzzy rough set, is firstly explored by Yang *et al.* [34], and both constructive and axiomatic approaches are considered. And the interval-valued hesitant fuzzy rough set was further investigated by Zhang *et al.* [38]. In 2017, Zhang *et al.* [39] introduced the hesitant fuzzy rough set over two universes and its application in decision making. In 2018, the hesitant fuzzy compatible rough set over two different universes and its application in hesitant fuzzy soft set based decision making were investigated by Zhang and He [40]. In this paper, we mainly are devoted to establishing a new hybrid model called a hesitant soft fuzzy rough set which is extended from soft fuzzy rough set theory to hesitant fuzzy set theory. The Fig. 1 intuitively shows that the new model discussed in this paper is the extension of the already existing models from the mathematical point of view. In addition, we investigate a practical application of hesitant soft fuzzy rough sets in decision making.

The paper is organized as follows. Section 2 presents some basic results of rough sets, soft sets and hesitant fuzzy sets. In Section 3, we introduce a hesitant soft fuzzy rough set model and investigated basic properties. Then we obtain a decomposition theorem for a hesitant fuzzy binary relation and give the relationship between hesitant fuzzy rough sets and hesitant soft fuzzy rough sets. In addition, we prove a characterization theorem for the hesitant soft fuzzy rough set model. In Section 4, a decision making approach to a hesitant fuzzy soft set based on the hesitant soft fuzzy rough set model is established and an example is given to illustrate the efficiency of the approach. Section 5 concludes this paper.

II. PRELIMINARIES AND NOTATIONS

In this section, we recall some basic notions and previous results that will be used in the later parts of this paper.

A. FUZZY LOGICAL CONNECTIVES

A triangular norm (t-norm for short) is a binary operator $T : [0, 1]^2 \to [0, 1]$ such that for all $x, y, z \in [0, 1]$, the conditions hold (T1) $y \leqslant z$ implies $T(x, y) \leqslant T(x, z)$ (monotonicity), (T2) $T(x, y) = T(y, x)$(commutativity), (T3) $T(T(x, y), z) = T(x, T(y, z))$ (associativity), (T4) $T(x, 1) = 1$ (boundary condition). A t-norm is said to be continuous (left-continuous) if it is a continuous (left-continuous) function. The three important continuous t-norms are: the minimum $T_M(x, y) = x \wedge y$, the algebraic product $T_P(x, y) = xy$ and the Łukasiewicz t-norm $T_L(x, y) = 0 \vee (x + y - 1)$,

A triangular conorm (t-conorm or s-norm, for short) is a binary operator $S : [0, 1]^2 \to [0, 1]$ such that for all $x, y, z \in [0, 1]$, the conditions (T1), (T2), (T3) and (S4) hold, where (S4) $S(x, 0) = x$ (boundary condition). The well-known continuous t-conorms are: the maximum $S_M(x, y) = x \vee y$, the probabilistic sum $S_P(x, y) = x + y - xy$ and the bounded sum $S_L(x, y) = (x + y) \wedge 1$.

Definition 1: A negation $N : [0, 1] \to [0, 1]$ is a decreasing function such that $N(1) = 0$ and $N(0) = 1$. A negation N is called involutive (weakly involutive) if $N(N(X)) = x$ ($N(N(X)) \geqslant x$) for all $x \in [0, 1]$. The standard negation $N(x) = 1 - x$ is involutive.

Note that every involutive negation is continuous. For any continuous negation N we have $N(\vee x_i) = \wedge N(x_i)$ and $N(\wedge x_i) = \vee N(x_i)$.

Definition 2: A function $I : [0, 1]^2 \to [0, 1]$ is said to be an implication if it satisfies the conditions $I(0, 0) = I(0, 1) = I(1, 1) = 1$ and $I(1, 0) = 0$. We call an implication I a border implication if for all $x \in [0, 1]$, $I(1, x) = x$.

An implication I is said to be left monotonic (right monotonic, respectively) if it is decreasing in its first component (increasing in its second component). An implication I is said to be hybrid monotonic if it is both left monotonic and right monotonic.

The most important classes of implications are R-, S- and QL-implications. Let T be a left-continuous t-norm, S be a t-conorm and N a negation, then an implication I is called

(i) a R-implication (residual implication) based on T if for all $(x, y) \in [0, 1]^2$, we have $I(x, y) = \bigvee\{c \in [0, 1] : T(x, c) \leqslant y\}$.

(ii) a S-implication based on S and N if for all $(x, y) \in [0, 1]^2$, we have $I(x, y) = S(N(x), y)$.

(iii) a QL-implication (quantum logic implication) based on T, S and N if for all $(x, y) \in [0, 1]^2$, we have $I(x, y) = S(N(x), T(x, y))$.

One can easily check the fact that every R-, S- and QL-implication is a border implication.

Proposition 1 (See [24]): Every R-implication and every S-implication is hybrid monotonic and every QL-implication is right monotonic.

Given a negation N and a border implication I, define an N-dual operation of I, $\theta_{I,N} : [0, 1]^2 \to [0, 1]$ satisfies $\theta_{I,N} = N(I(N(x), N(y)))$. Then we have $\theta_{I,N}(1, 0) = \theta_{I,N}(1, 1) = \theta_{I,N}(0, 0) = 0$ and $\theta_{I,N}(0, 1) = 1$. Moreover, if N is an involution, then $\theta_{I,N}(0, x) = N(I(1, N(x))) = N(N(x)) = x$.

For example, we way S is the N-dual of T, if for all $x, y \in [0, 1]$, $S(x, y) = N(T(N(x), N(y)))$. Furthermore, if N is the standard negation, i.e., $S(x, y) = 1 - T(1 - x, 1 - y)$, then we say S is the dual of T.

B. SOFT FUZZY ROUGH SETS

Let U be a nonempty set called the universe of discourse, $\mathcal{P}(U)$ be the set of all subsets of U and $\mathcal{F}(U)$ be the set of all fuzzy sets in the universe U. R denotes a relation on U. When R is an equivalence relation on U, the pair $P = (U, R)$ is called a Pawlak approximation space. R will generate a partition $U/R = \{[x]_R : x \in U\}$ on U, where $[x]_R = \{y \in U : (x, y) \in R\}$ is the equivalence class with respect to R containing x. These equivalence classes are referred to as R-elementary sets which are the basic building blocks (concepts) of our knowledge about reality. The main question addressed by rough sets is how to represent subset X of U by means of elements of the quotient set U/R. For an approximation space $P = (U, R)$, for each $X \subseteq U$, the lower rough approximation operator $\underline{R}$ and the upper rough approximation operator $\overline{R}$ of X with respect to P are defined as [21]

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}, \tag{1}$$
$$\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}. \tag{2}$$

A subset $X \subseteq U$ is said to be definable in a given approximation space P if $\underline{R}(X) = \overline{R}(X)$; otherwise, X is called a rough set. Note that sometimes the pair $(\underline{R}X, \overline{R}X)$ is also referred to as the rough set of X with respect to P.

Dubois and Prade [4] introduced the lower and upper approximations of fuzzy sets in a Pawlak approximation space, and obtained a new notion called rough fuzzy sets.

Definition 3 (See [4]): Let $P = (U, R)$ be a Pawlak approximation space and $\mu \in \mathcal{F}(U)$. The lower and upper rough approximations of μ in P are denoted by $\underline{R}(\mu)$ and $\overline{R}(\mu)$, respectively, which are fuzzy subsets in U defined by

$$\underline{R}(\mu)(x) = \wedge\{\mu(y) : y \in [x]_R\}, \tag{3}$$
$$\overline{R}(\mu)(x) = \vee\{\mu(y) : y \in [x]_R\}, \tag{4}$$

for all $x \in U$. The operators $\underline{R}$ and $\overline{R}$ are called the lower and upper rough approximation operators on fuzzy sets, respectively. μ is called definable in P if $\underline{R}(\mu) = \overline{R}(\mu)$; otherwise μ is called a rough fuzzy set.

Clearly, rough fuzzy sets are natural extensions of rough sets.

The equivalence classes are the building blocks for the construction of the lower and upper approximations. By replacing the equivalence relation by an arbitrary relation, different kinds of generalizations of Pawlak rough set model were obtained. Dubois and Prade [4] were among the first who investigated the problem of a fuzzification of a rough set, the concept of fuzzy rough set were proposed by replacing crisp binary relations with fuzzy relations in the universe.

Given a nonempty universe U, when R is a fuzzy binary relation on U, i.e., $R \in \mathcal{F}(U \times U)$, the pair $PF = (U, R)$ is called a fuzzy approximation space. For all $x, y, z \in U$, if R satisfies reflexivity ($R(x, x) = 1$) and symmetry ($R(x, y) = R(y, x)$), we say R is a fuzzy similarity relation. If R satisfies reflexivity, symmetry and transitivity ($R(x, y) \geqslant \min\{R(x, z), R(z, y)\}$), we say R is a fuzzy equivalence relation.

Definition 4 (See [4]): Let $PF = (U, R)$ be a fuzzy approximation space and R a fuzzy equivalence relation. Given a fuzzy set $\mu \in \mathcal{F}(U)$, the lower and upper rough approximations of $x \in U$ with respect to μ are defined as

$$\underline{R}(\mu)(x) = \inf_{y \in U} \max\{1 - R(x, y), \mu(y)\}, \tag{5}$$
$$\overline{R}(\mu)(x) = \sup_{y \in U} \min\{R(x, y), \mu(y)\}. \tag{6}$$

The operators $\underline{R}$ and $\overline{R}$ are called the lower and upper fuzzy rough approximation operators, respectively. The pair $(\underline{R}(\mu), \overline{R}(\mu))$ is called a fuzzy rough set.

Dubois and Prade also pointed out that the rough fuzzy set is a special case of the fuzzy rough set in the universe. In 1998, Morsi and Yakout [20] generalized the operator from min to a t-norm, introduced T-equivalence relation, i.e., R satisfies reflexivity, symmetry and T-transitivity ($R(x, y) \geqslant T(R(x, z), R(z, y))$, where T is a t-norm), and built the generalized fuzzy rough set model. In 2002, Radzikowska and Kerre [24] introduced (I, T)-fuzzy rough sets, where T is a triangular norm. In 2010, Yao *et al.* [35] defined a more general fuzzy rough sets so-called (I, J)-fuzzy rough sets, where I and J are two border implicators, as follows:

Definition 5 (See [35]): Let $PF = (U, R)$ be a fuzzy approximation space, R a fuzzy equivalence relation, and I, J be border implications. Then the lower fuzzy rough approximation operator $\underline{R}_I$ and the upper fuzzy rough approximation operator $\overline{R}^J$ are given by

$$\underline{R}_I(\mu)(x) = \bigwedge_{y\in U} I(R(x,y), \mu(y)), \tag{7}$$

$$\overline{R}^J(\mu)(x) = \bigvee_{y\in U} \theta_{J,N}(N(R(x,y)), \mu(y)) = \bigvee_{y\in U} N(J(R(x,y), N(\mu(y)))), \tag{8}$$

where $\mu \in \mathcal{F}(U)$ and N is a negation. The pair $(\underline{R}_I(\mu)(x), \overline{R}^J(\mu)(x))$ is called a (I, J)-fuzzy rough set.

Remark 1: Let T be a t-norm, and S be the N-dual s-norm of T. If N is an involution, and J is a S-implication, i.e., $J(x, y) = S(N(x), y)$, then $\theta_{J,N}(N(R(x, y)), \mu(y)) = T(R(x, y), \mu(y))$, thus, the (I, J)-fuzzy rough set reduces to a (I, T)-fuzzy rough set.

Let U be a nonempty set of the universe and E be the set of all possible parameters under consideration with respect to U. Usually, parameters are attributes, characteristics, or properties of objects in U. A pair $S = (F, A)$ is called a soft set [17] over U, where $A \subseteq E$ and F is a mapping given by $F : A \to P(U)$. In other words, the soft set is not a type of set, but a parameterized family of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A).

Note that a soft set $S = (F, A)$ over U is called a full soft set (see [5], [6]) if $\bigcup_{a\in A} F(a) = U$.

Example 1: Suppose that there are four cars in the universe U given by $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $A = \{a_1, a_2, a_3, a_4, a_5\}$ is the set of parameters, which stand for being beautiful, being cheap, being safe, being comfortable and being in strong power, respectively. In this case, to define a soft set means to point out beautiful cars, cheap cars and so on. The soft set (F, E) may describe the 'attractiveness of the cars' which Mr. X is going to buy.

Suppose that $F(a_1) = \{x_2, x_4\}$, $F(a_2) = \{x_1, x_3\}$, $F(a_3) = \{x_3, x_4, x_5\}$, $F(a_4) = \{x_1, x_3, x_5\}$, $F(a_5) = \{x_2\}$. Then the soft set (F, A) is a parameterized family $\{F(a_i) : 1 \leqslant i \leqslant 5\}$ of subsets of U and gives us a collection of approximate descriptions of an object, i.e.,

$$(F, A) = \{(a_1, \{x_2, x_4\}), (a_2, \{x_1, x_3\}), (a_3, \{x_3, x_4, x_5\}), (a_4, \{x_1, x_3, x_5\}), (a_5, \{x_2\})\}.$$

Remark 2 (See [17]): Zadeh's fuzzy set may be considered as a special case of the soft set.

Let μ be a fuzzy set and $F(\alpha) = \{x \in U : \mu(x) \geq \alpha\}$ $(\forall \alpha \in [0, 1])$ be the α-level set of μ. If we know the family $\{F(\alpha) : \alpha \in [0, 1]\}$, we can calculate $\mu(x)$ by means of the formula $\mu(x) = \sup_{x\in F(\alpha)} \alpha$, that is, $\mu = \bigcup_{\alpha\in[0,1]} \alpha F(\alpha)$. This observation is usually summarized by a decomposition theorem in fuzzy set theory, which establishes a one-to-one correspondence between a fuzzy and a family of crisp sets satisfying certain conditions. Thus, fuzzy set μ may be considered as the soft set $(F, [0, 1])$.

Remark 3 (See [1]): Pawlak's rough set model may be considered as a special case of the soft set.

Suppose that $P = (U, R)$ is a Pawlak approximation space and $X \subseteq U$. Let $R(X) = (\underline{R}X, \overline{R}X)$ be the rough set of X with respect to R. Consider two predicates $p_1(x), p_2(x)$, which mean "$[x]_R \subseteq X$" and "$[x]_R \cap X \neq \emptyset$", respectively. The predicates $p_1(x), p_2(x)$ may be treated as elements of a parameter set, that is, $E = \{p_1(x), p_2(x)\}$. Then we can define a set-valued mapping

$$F : \quad E \to \mathcal{P}(U),$$
$$p_i(x) \mapsto F(p_i(x)) = \{x \in U : p_i(x) \textit{ is true}\},$$

where $i = 1, 2$. It follows that the rough set $R(X)$ may be considered a soft set (F, E) with the following representation $(F, E) = \{(p_1(x), \underline{R}X), (p_2(x), \overline{R}X)\}$.

Definition 6 (See [25]): A pair $S = (f, A)$ is called a fuzzy soft set over U, where $A \subseteq E$ and f is a mapping given by $f : A \to \mathcal{F}(U)$.

In the definition of a fuzzy soft set, fuzzy sets in the universe U are used as substitutes for the crisp subsets of U. Hence, every soft set may be considered as a fuzzy soft set.

Example 2: Mr. X thinks x_1 is a little expensive and this fuzzy information cannot be expressed only by two crisp numbers, that is, 0 and 1, a membership degree can be used instead, which is associated with each element and represented by a real number in the interval [0, 1]. The fuzzy soft set (f, A) can describe the 'attractiveness of the cars' which Mr. X is going to buy under the fuzzy information.

$$f(a_1) = \{\frac{x_1}{0.2}, \frac{x_2}{0.5}, \frac{x_3}{0.3}, \frac{x_4}{0.3}, \frac{x_5}{0.4}, \frac{x_6}{0.6}\},$$
$$f(a_2) = \{\frac{x_1}{0.6}, \frac{x_2}{0.5}, \frac{x_3}{0.6}, \frac{x_4}{0.7}, \frac{x_5}{0.4}, \frac{x_6}{0.3}\},$$
$$f(a_3) = \{\frac{x_1}{0.4}, \frac{x_2}{0.6}, \frac{x_3}{0.8}, \frac{x_4}{0.3}, \frac{x_5}{0.4}, \frac{x_6}{0.7}\},$$
$$f(a_4) = \{\frac{x_1}{0.3}, \frac{x_2}{0.2}, \frac{x_3}{0.5}, \frac{x_4}{0.7}, \frac{x_5}{0.5}, \frac{x_6}{0.8}\},$$
$$f(a_5) = \{\frac{x_1}{0.6}, \frac{x_2}{0.3}, \frac{x_3}{0.5}, \frac{x_4}{0.4}, \frac{x_5}{0.7}, \frac{x_6}{0.3}\}.$$

Motivated by Dubois and Prade's original idea about the rough fuzzy set, Feng *et al.* [5], [6] introduced the lower and upper approximations of soft sets in a Pawlak approximation space, and the concept of soft rough set and soft rough fuzzy set were proposed. Furthermore, Feng *et al.* [6] proved that Pawlak's rough set model can be viewed as a special instance of the soft rough set. Meng *et al.* [14] introduced the lower and upper soft fuzzy rough approximations of fuzzy sets and pointed out that soft fuzzy rough sets are extensions of soft rough fuzzy sets. In 2015, Gong and Zhang [9] defined the (I, J)-soft fuzzy rough set which generalized the (I, J)-fuzzy rough set of Yao *et al.* [35].

Definition 7 (See [5], [6]): Let $S = (F, A)$ be a soft set over U. Then the pair $SP = (U, S)$ is called a soft approximation space. Based on the soft approximation space P, we define the following two operations

$$\underline{apr}_{SP}(X) = \{x \in U : \exists\, a \in A(x \in F(a) \subseteq X)\}, \tag{9}$$

$$\overline{apr}_{SP}(X) = \{x \in U : \exists\, a \in A(x \in F(a), F(a) \cap X \neq \emptyset)\}, \tag{10}$$

assigning to every subset $X \subseteq U$. $\underline{apr}_{SP}(X)$ and $\overline{apr}_{SP}(X)$ are called the lower and upper soft rough approximations of X in S, respectively. If $\underline{apr}_{SP}(X) = \overline{apr}_{SP}(X)$, X is said to be soft definable; otherwise X is called a soft rough set.

Obviously, $\underline{apr}_{SP}(X)$ and $\overline{apr}_{SP}(X)$ can be expressed equivalently as

$$\underline{apr}_{SP}(X) = \bigcup_{a\in A}\{F(a) : F(a) \subseteq X\},$$

$$\overline{apr}_{SP}(X) = \bigcup_{a\in A}\{F(a) : F(a) \cap X \neq \emptyset\}.$$

Definition 8 ([14]): Let $S = (f, A)$ be a fuzzy soft set over U. Then the pair $SF = (U, S)$ is called a soft fuzzy approximation space. For a fuzzy set $\mu \in \mathcal{F}(U)$, the lower and upper soft fuzzy rough approximations of μ with respect to SF are denoted by $\underline{Apr}_{SF}(\mu)$ and $\overline{Apr}_{SF}(\mu)$, respectively, which are fuzzy sets in U given by

$$\underline{Apr}_{SF}(\mu)(x) = \bigwedge_{a\in A}\left((1 - f(a)(x)) \vee \left(\bigwedge_{y\in U}((1 - f(a)(y)) \vee \mu(y))\right)\right), \tag{11}$$

$$\overline{Apr}_{SF}(\mu)(x) = \bigvee_{a\in A}\left(f(a)(x) \wedge \left(\bigvee_{y\in U}(f(a)(y) \wedge \mu(y))\right)\right), \tag{12}$$

for all $x \in U$. The operators $\underline{Apr}_{SF}(\mu)$ and $\overline{Apr}_{SF}(\mu)$ are called the lower and upper soft fuzzy rough approximations on fuzzy sets, respectively. If $\underline{Apr}_{SF}(\mu) = \overline{Apr}_{SF}(\mu)$, μ is said to be soft fuzzy definable; otherwise μ is called a soft fuzzy rough set.

C. HESITANT FUZZY SOFT SETS

When giving the membership degree of an element, the difficulty of establishing the membership degree is not because we have a margin of error, or some possibility distribution on the possibility values, but because we have several possible values. For such cases, Torra [28] proposed the hesitant fuzzy set (HFS) as a generalization form of the fuzzy set (FS).

Let U be a fixed set, a hesitant fuzzy set (HFS) on U is in terms of a function h_E that when applied to U returns a subset of [0, 1]. To be easily understood, we express the HFS by a mathematical symbol

$$E = \{\langle x, h_E(x)\rangle | x \in U\},$$

where $h_E(x)$ is a set of some values in [0, 1], representing the possible membership degrees of the element $x \in U$ to the set E. For convenience, we call $h = h_E(x)$ a hesitant fuzzy element (HFE) and $H(U)$ the set of HFSs on U. In particular, if $h_E(x)$ is a non-empty and finite subset of [0, 1], HFS is called a typical hesitant fuzzy set (THFS) (see [3]).

For each typical hesitant fuzzy set h_E on U, let

$$h_E(x) = \{h_E^1(x), \cdots, h_E^{l_M(x)}(x)\},$$

where $h_M^1(x) < \cdots < h_E^{l_E(x)}(x)$ and $l_E(x) = |h_E(x)|$ is the cardinality of the HFE $h_E(x)$.

Definition 9 (See [28]): Let U be the universe of discourse, $\forall F, G \in H(U)$, then

(i) the complement of F is denoted by F^c such that $\forall x \in U$,

$$h_{F^c}(x) = \sim h_F(x) = \{1 - h : \forall h \in h_F(x)\};$$

(ii) the intersection of F and G is denoted by $F \cap G$ such that $\forall x \in U$,

$$h_{F\cap G}(x) = h_F(x) \barwedge h_G(x) = \{h \in h_F(x) \cup h_G(x) : h \leq \min\{h_F^+(x), h_G^+(x)\}\};$$

(iii) the intersection of F and G is denoted by $F \cap G$ such that $\forall x \in U$,

$$h_{F\cup G}(x) = h_F(x) \veebar h_G(x) = \{h \in h_F(x) \cup h_G(x) : h \geq \max\{h_F^-(x), h_G^-(x)\}\};$$

where $h_F^+(x)$ is the upper bound of F, i.e.,

$$h_F^+(x) = \max\{h : h \in h_F(x)\},$$

and $h_F^-(x)$ is the lower bound of F, i.e.,

$$h_F^-(x) = \min\{h : h \in h_F(x)\}.$$

Proposition 2 (See [34]): Let F, G and H be HFSs on U, then for any $x, y, z \in U$, the following properties hold:

(1) Idempotent:

$$h_F(x) \barwedge h_F(x) = h_F(x), \quad h_F(x) \veebar h_F(x) = h_F(x).$$

(2) Commutativity:

$$h_F(x) \barwedge h_G(y) = h_G(y) \barwedge h_F(x), \quad h_F(x) \veebar h_G(y) = h_G(y) \veebar h_F(x).$$

(3) Associativity:

$$h_F(x) \barwedge (h_G(y) \barwedge h_H(z)) = (h_F(x) \barwedge h_G(y)) \barwedge h_H(z),$$
$$h_F(x) \veebar (h_G(y) \veebar h_H(z)) = (h_F(x) \veebar h_G(y)) \veebar h_H(z).$$

(4) Distributivity:

$$h_F(x) \barwedge (h_G(y) \veebar h_H(z)) = (h_F(x) \barwedge h_G(y)) \veebar (h_F(x) \barwedge h_H(z)),$$
$$h_F(x) \veebar (h_G(y) \barwedge h_H(z)) = (h_F(x) \veebar h_G(y)) \barwedge (h_F(x) \veebar h_H(z)).$$

(5) De Morgan's laws:

$$\sim (h_F(x) \barwedge h_G(y)) = (\sim h_F(x)) \veebar (\sim h_G(y)),$$
$$\sim (h_F(x) \veebar h_G(y)) = (\sim h_F(x)) \barwedge (\sim h_G(y)).$$

(6) Double negation law:

$$\sim (\sim h_F(x)) = h_F(x).$$

For a HFE h, $s(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma$ is called the score function of h, where $\#h$ is the number of the values in h. For two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Let F and G be HFSs on U. We say $F \subseteq G$ if and only if $h_F(x) \preceq h_G(x)$ for any $x \in U$, i.e., $h_F^-(x) \leq h_F^-(x)$ and $h_F^+(x) \leq h_F^+(x)$. The (α, k)-level set and strong (α, k)-level set associated with E are defined, respectively, as

$$_{\alpha,k}h_E = \{x \in U : |\{h \in h_E(x) : h \geq \alpha\}| \geq k\}$$

and

$$_{\alpha^+,k}h_E = \{x \in U : |\{h \in h_E(x) : h > \alpha\}| \geq k\}$$

for all $\alpha \in [0, 1]$ and for all $k \in \{1, 2, \cdots\}$.

Given a universe U, a hesitant fuzzy relation on U is a hesitant fuzzy set such that $R \in H(U \times U)$, i.e., $R = \{\langle (x, y), h_R(x, y) \rangle : (x, y) \in U \times U\}$, where $h_R(x, y)$ is a set of the values in $[0, 1]$, which is used to denote the possible membership degrees of the relationships between x and y. R is referred to as serial if and only if $\forall x \in U$, there is a $y \in U$ such that $h_R(x, y) = 1$; R is referred to as reflexive if and only if $h_R(x, x) = 1$ holds for each $x \in U$; R is referred to as symmetric if and only if $h_R(x, y) = h_R(y, x)$ $(\forall x, y \in U)$; R is referred to as transitive if and only if $h_R(x, y) \barwedge h_R(y, z) \preceq h_R(x, z)$ $(\forall x, y, z \in U)$ (see [34]). If a hesitant fuzzy relation R on U is reflexive, symmetric and transitive, we say R is a hesitant fuzzy equivalent relation on U.

Definition 10 (See [34]): Let R be a hesitant fuzzy relation on U. The pair $PH = (U, R)$ is called a hesitant fuzzy approximation space. Given a hesitant fuzzy set $E \in H(U)$, the lower approximation and upper approximations of E are denoted by $\underline{R}(h_E)$ and $\overline{R}(h_E)$, respectively, which are hesitant fuzzy sets in U defined by

$$\underline{R}(h_E)(x) = \overline{\bigwedge}_{y \in U} \{h_{R^c}(x, y) \veebar h_E(y)\}, \tag{13}$$
$$\overline{R}(h_E)(x) = \underline{\bigvee}_{y \in U} \{h_R(x, y) \barwedge h_E(y)\}. \tag{14}$$

for all $x \in U$. The operators $\underline{R}$ and $\overline{R}$ are called the lower and upper hesitant fuzzy rough approximation operators, respectively. The pair $(\underline{R}(h_E), \overline{R}(h_E))$ is called a hesitant fuzzy rough set.

Definition 11 (See [2]): Let $\mathbb{E} = \{h_{E(i)}\}_{i \in J}$ be a family of hesitant fuzzy sets on U, indexed by the set of indices J. Then the HFS associated with $\mathbb{E}$, denoted by either $h_{\mathbb{E}}$ or $\bigcup_{i \in J} h_{E(i)}$, is defined as

$$h_{\mathbb{E}} : \quad U \to \mathcal{P}([0, 1]),$$
$$x \mapsto \bigcup_{i \in J} h_{E(i)}(x).$$

Theorem 1 (See [2]): Let h_E be a typical hesitant fuzzy set on U. Then h_E is the HFS associated with the family of fuzzy sets $f = \{_kH\}_{k \in \mathbb{N}^+}$, i.e.,

$$h_E = \bigcup_{k=1,2,\cdots} {_kH},$$

where $_1H(x) = \max\{\alpha \in [0, 1] : x \in_{\alpha,1} h_E\} = h_E^{l_E(x)}(x)$ for each $x \in U$; if $_1H, \cdots, _kH$ are known, then $_{k+1}H(x) = \max\{\alpha \in [0, 1] : x \in_{\alpha,k+1} h_E\}$, if $x \in_{\alpha,k+1} h_E$ some $\alpha \in [0, 1]$, and $_{k+1}H(x) =_k H(x)$ otherwise.

Theorem 1 produces a decomposition of any THFS in terms of the simplest THFSs, which are the fuzzy sets.

Example 3: Let $U = \{x_1, x_2\}$, $E = \{\langle x_1, \{0.3, 0.6, 0.7\}\rangle, \langle x_2, \{0.4, 0.5\}\rangle\}$. Then

$$_{\alpha,1}h_E = \begin{cases} \{x_1, x_2\}, & \alpha \leq 0.5, \\ \{x_1\}, & 0.5 < \alpha \leq 0.7, \\ \emptyset, & otherwise, \end{cases}$$

$$_{\alpha,2}h_E = \begin{cases} \{x_1, x_2\}, & \alpha \leq 0.4, \\ \{x_1\}, & 0.4 < \alpha \leq 0.6, \\ \emptyset, & otherwise, \end{cases}$$

$$_{\alpha,3}h_E = \begin{cases} \{x_1\}, & \alpha \leq 0.3, \\ \emptyset, & otherwise, \end{cases}$$

$_{\alpha,4}h_E = \emptyset$ for each $\alpha \in [0, 1]$. Thus, we have

$$\begin{aligned} {_1H} : \quad & U \to [0, 1] \\ & x_1 \mapsto 0.7, \\ & x_2 \mapsto 0.5, \\ {_2H} : \quad & U \to [0, 1] \\ & x_1 \mapsto 0.6, \\ & x_2 \mapsto 0.4, \\ {_3H} : \quad & U \to [0, 1] \\ & x_1 \mapsto 0.3, \\ & x_2 \mapsto 0.4, \end{aligned}$$

and $_3H =_4 H =_5 H = \cdots$. Therefore,

$$h_E = \bigcup_{k=1,2,\cdots} {_kH} = {_1H} \cup {_2H} \cup {_3H}.$$

By combining the notion of hesitancy with Molodtsov's soft set, Wang et al. [30] introduced the hesitant fuzzy soft set.

Definition 12: A pair (f, A) is called a hesitant fuzzy soft set (HFSS) over U, where $A \subseteq E$ and f is a mapping given by $f : A \to H(U)$.

TABLE 1. Tabular representation of a hesitant fuzzy soft set.

A/U	x_1	x_2	$\cdots$	x_n
a_1	$\{h_{11}\}$	$\{h_{12}\}$	$\cdots$	$\{h_{1n}\}$
a_2	$\{h_{21}\}$	$\{h_{22}\}$	$\cdots$	$\{h_{2n}\}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
a_m	$\{h_{m1}\}$	$\{h_{m2}\}$	$\cdots$	$\{h_{mn}\}$

The hesitant fuzzy soft set is a parameterized family of hesitant fuzzy subsets of U. For convenience, for $x \in U$, we call $f(a)(x)$ a hesitant fuzzy soft element (HFSE).

Example 4: Consider Example 1. Mr. X found it was hard to give a single value to express his opinion about the cars with respect to different criteria. For example, Mr. X thinks that the degree of car x_1 satisfies that criterion a_1 'beautiful' is 0.3 or 0.2. Then the HFSS (f, A) can describe the 'attractiveness of the cars' which Mr. X is going to buy under the hesitant fuzzy information.

$$f(a_1)=\{\frac{x_1}{0.2, 0.3}, \frac{x_2}{0.5, 0.6}, \frac{x_3}{0.3}, \frac{x_4}{0.3, 0.5}, \frac{x_5}{0.4, 0.5}, \frac{x_6}{0.6, 0.7}\},$$
$$f(a_2) = \{\frac{x_1}{0.4, 0.6, 0.7}, \frac{x_2}{0.5, 0.7, 0.8}, \frac{x_3}{0.6, 0.8}, \frac{x_4}{0.7, 0.9}, \frac{x_5}{0.3, 0.4, 0.5}, \frac{x_6}{0.3}\},$$
$$f(a_3)=\{\frac{x_1}{0.2, 0.4}, \frac{x_2}{0.6, 0.7}, \frac{x_3}{0.8, 0.9}, \frac{x_4}{0.3, 0.5}, \frac{x_5}{0.4, 0.6}, \frac{x_6}{0.7}\},$$
$$f(a_4) = \{\frac{x_1}{0.3, 0.5, 0.6}, \frac{x_2}{0.2}, \frac{x_3}{0.5}, \frac{x_4}{0.6, 0.7}, \frac{x_5}{0.5, 0.6}, \frac{x_6}{0.8}\},$$
$$f(a_5) = \{\frac{x_1}{0.6}, \frac{x_2}{0.2, 0.3, 0.5}, \frac{x_3}{0.5, 0.7}, \frac{x_4}{0.2, 0.4}, \frac{x_5}{0.5, 0.7}, \frac{x_6}{0.3, 0.5}\}.$$

To stored a hesitant fuzzy soft set in the computer, a hesitant fuzzy soft set can be represented by Table 1, where $\{h_{ij}\} \subseteq [0, 1], i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$.

III. HESITANT SOFT FUZZY ROUGH SETS

In this section, we define a hesitant soft fuzzy rough set model and investigate basic properties of this model in detail. Then we obtain a decomposition theorem for a hesitant fuzzy binary relation and give the relationship between hesitant fuzzy rough sets and hesitant soft fuzzy rough sets. Furthermore, we prove a characterization theorem for the hesitant soft fuzzy rough set model.

Let $S = (f, A)$ be a hesitant fuzzy soft set over U. We call $S = (f, A)$ a full hesitant fuzzy soft set if $\bigvee_{a\in A} h_{f(a)}(x) = 1$ for all $x \in U$, and then $HSF = (U, S)$ is said to be a full hesitant soft fuzzy approximation space. If for all $a \in A, x \in U$, $f(a, x)$ is a typical hesitant fuzzy set, then we call $S = (f, A)$ a typical hesitant fuzzy soft set (THFSS).

We define the (strong) (α, k)-level set of the hesitant fuzzy soft set $S = (f, A)$ with respect to attribute a as follows:

$$S_{\alpha,k(a)} = \{x \in U : |\{h \in h_{f(a)}(x) : h \geq \alpha\}| \geq k\},$$
$$S_{\alpha^+,k(a)} = \{x \in U : |\{h \in h_{f(a)}(x) : h > \alpha\}| \geq k\},$$

where $\alpha \in [0, 1]$, $a \in A$. Obviously, if $k = 1$,

$$S_{\alpha,1(a)} = \{x \in U : h^-_{f(a)}(x) \geq \alpha\},$$
$$S_{\alpha^+,1(a)} = \{x \in U : h^-_{f(a)}(x) > \alpha\}.$$

By applying (strong) (α, k)-level set of fuzzy soft set $S = (f, A)$, a set of objects with respect to an attribute can be found. From Table 2, if we want to obtain a set of objects of which all the attribute values with respect a_i are not less than 0.5, then we will find these objects as $\{x \in U : h^-_{f(a_i)}(x) \geq 0.5\}$. Furthermore, if we get a set of objects of which all the attribute values with respect a_i are not less than 0.5 and a_2 are more than 0.6, then the selected objects as $\{x \in U : h^-_{f(a_i)}(x) \geq 0.5\} \cap \{x \in U : h^-_{f(a_2)}(x) > 0.6\}$.

Clearly, from the definition of hesitant fuzzy soft set we known that the mapping $f : A \to H(U)$ is a binary hesitant fuzzy relation defined between the universe U with with the parameter set A. That is, for any $a_i \in A$ and $x_j \in U$, $f(a_i)(x_j) \in H(A \times U)$. In general, $f(a_i)(x_j)$ does not satisfy reflexive, symmetric and transitive. Therefore, $f(a_i)(x_j)$ is an arbitrary hesitant fuzzy relation.

Theorem 2: Let $S = (f, A)$ be a hesitant fuzzy soft set over U. Then S determines a hesitant fuzzy binary relation $R_S \in H(A \times U)$, which is defined by

$$h_{R_S}(a, y) = h_{f(a)}(y),$$

where $a \in A$, $y \in U$.
Conversely, assume that R is a binary relation from A to U. If we define a set-valued mapping $h_{f_R} : A \to H(U)$ by

$$h_{f_R}(a)(y) = h_R(a, y),$$

where $a \in A$, then $S_R = (h_{f_R}, A)$ is a hesitant fuzzy soft set over U, and $S_{R_S} = S$, $R_{S_R} = R$.

Proof: It is obtained directly by the definition of hesitant fuzzy soft set.

Based on the concept of hesitant fuzzy soft set, we define the hesitant soft fuzzy rough set as follows:

Definition 13: Let $S = (f, A)$ be a hesitant fuzzy soft set over U. Then the pair $HSF = (U, S)$ is called a hesitant soft fuzzy approximation space. For a hesitant fuzzy set $E \in H(U)$, the lower and upper hesitant soft fuzzy rough approximations of E with respect to HSF are denoted by $\underline{Apr}_{HSF}$ and $\overline{Apr}_{HSF}$, respectively, which are hesitant fuzzy sets in U given by

$$\underline{Apr}_{HSF}(h_E)(x) = \overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \veebar \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \veebar h_E(y))\right)\right), \tag{15}$$

$$\overline{Apr}_{HSF}(h_E)(x) = \underline{\bigvee}_{a\in A}\left(h_{f(a)}(x) \barwedge \left(\underline{\bigvee}_{y\in U}(h_{f(a)}(y) \barwedge h_E(y))\right)\right), \tag{16}$$

for all $x \in U$. The operators $\underline{Apr}_{HSF}$ and $\overline{Apr}_{HSF}$ are called the lower and upper hesitant soft fuzzy rough

TABLE 2. Tabular representation of a hesitant fuzzy soft set.

A/U	x_1	x_2	x_3	x_4	x_5	x_6
a_1	0.2,0.3	0.5,0.6	0.3	0.3,0.5	0.4,0.5	0.6,0.7
a_2	0.4,0.6,0.7	0.5,0.7,0.8	0.6,0.8	0.7,0.9	0.3,0.4,0.5	0.3
a_3	0.2,0.4	0.6,0.7	0.8,0.9	0.3,0.5	0.4,0.6	0.7
a_4	0.3,0.5,0.6	0.2	0.5	0.6,0.7	0.5,0.6	0.8
a_5	0.6	0.2,0.3,0.5	0.5,0.7	0.2,0.4	0.5,0.7	0.3,0.5

approximation operators on hesitant fuzzy sets, respectively. If $\underline{Apr}_{HSF}(h_E) = \overline{Apr}_{HSF}(h_E)$, E is said to be hesitant soft fuzzy definable; otherwise E is called a hesitant soft fuzzy rough set. The pair $(\underline{Apr}_{HSF}(h_E), \overline{Apr}_{HSF}(h_E))$ is also called a hesitant soft fuzzy rough set of E with respect to HSF.

Remark 5: If $S = (f, A)$ is a fuzzy soft set and E is a fuzzy set, then $\barwedge$ and $\veebar$ reduce to the minimal and maximal operations, and $(\underline{Apr}_{HSF}(h_E), \overline{Apr}_{HSF}(h_E))$ is a soft fuzzy rough set.

Example 5: Consider Example 4. Let $S = (f, A)$ be a hesitant fuzzy soft set on U (Table 2).
Give a hesitant fuzzy subset $E \in H(U)$ as follows:

$$E = \{\langle x_1, \{0.2, 0.3\}\rangle, \langle x_2, \{0.5\}\rangle, \langle x_3, \{0.4, 0.6\}\rangle, \langle x_4, \{0.7, 0.8, 0.9\}\rangle, \langle x_5, \{0.1\}\rangle, \langle x_6, \{0.9\}\rangle\}.$$

By Definition 13, the lower and upper hesitant soft fuzzy rough approximations of E, respectively, as follows:

$$\underline{Apr}_{HSF}(h_E)(x) = \overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \veebar \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \veebar h_E(y))\right)\right),$$

$$\overline{Apr}_{HSF}(h_E)(x) = \underline{\bigvee}_{a\in A}\left(h_{f(a)}(x) \barwedge \left(\underline{\bigvee}_{y\in U}(h_{f(a)}(y) \barwedge h_E(y))\right)\right).$$

for all $x \in U$.
Since

$$\overline{\bigwedge}_{x_i\in U}(h_{f^c(a_1)}(x_i) \veebar h_E(x_i)) = (\{0.7, 0.8\} \veebar \{0.2, 0.3\}) \barwedge (\{0.4, 0.5\} \veebar \{0.5\}) \barwedge (\{0.7\} \veebar \{0.4, 0.6\}) \barwedge (\{0.5, 0.7\} \veebar \{0.7, 0.8, 0.9\}) \barwedge (\{0.5, 0.6\} \veebar \{0.1\}) \barwedge (\{0.3, 0.4\} \veebar \{0.9\}) = \{0.7, 0.8\} \barwedge \{0.5\} \barwedge \{0.7\} \barwedge \{0.7, 0.8, 0.9\} \barwedge \{0.5, 0.6\} \barwedge \{0.9\} = \{0.5\},$$

$$\overline{\bigwedge}_{x_i\in U}(h_{f^c(a_2)}(x_i) \veebar h_E(x_i)) = \{0.3, 0.4, 0.6\} \barwedge \{0.5\} \barwedge \{0.4, 0.6\} \barwedge \{0.7, 0.8, 0.9\} \barwedge \{0.5, 0.6, 0.7\} \barwedge \{0.9\} = \{0.3, 0.4, 0.5\},$$

$$\overline{\bigwedge}_{x_i\in U}(h_{f^c(a_3)}(x_i) \veebar h_E(x_i)) = \{0.6, 0.8\} \barwedge \{0.5\} \barwedge \{0.4, 0.6\} \barwedge \{0.7, 0.8, 0.9\} \barwedge \{0.4, 0.6\} \barwedge \{0.9\} = \{0.4, 0.5\},$$

$$\overline{\bigwedge}_{x_i\in U}(h_{f^c(a_4)}(x_i) \veebar h_E(x_i)) = \{0.4, 0.5, 0.7\} \barwedge \{0.8\} \barwedge \{0.5, 0.6\} \barwedge \{0.7, 0.8, 0.9\} \barwedge \{0.4, 0.5\} \barwedge \{0.9\} = \{0.4, 0.5\},$$

$$\overline{\bigwedge}_{x_i\in U}(h_{f^c(a_5)}(x_i) \veebar h_E(x_i)) = \{0.4\} \barwedge \{0.5, 0.7, 0.8\} \barwedge \{0.4, 0.5, 0.6\} \barwedge \{0.7, 0.8, 0.9\} \barwedge \{0.3, 0.5\} \barwedge \{0.9\} = \{0.3, 0.4\},$$

then we obtain the lower hesitant soft fuzzy rough approximations of E as follows:

$$\underline{Apr}_{HSF}(h_E)(x_1) = (\{0.7, 0.8\} \veebar \{0.5\}) \barwedge (\{0.3, 0.4, 0.6\} \veebar \{0.3, 0.4, 0.5\}) \barwedge (\{0.6, 0.8\} \veebar \{0.4, 0.5\}) \barwedge (\{0.4, 0.5, 0.7\} \veebar \{0.4, 0.5\}) \barwedge (\{0.4\} \veebar \{0.3, 0.4\}) = \{0.7, 0.8\} \barwedge \{0.3, 0.4, 0.5, 0.6\} \barwedge \{0.6, 0.8\} \barwedge \{0.4, 0.5, 0.7\} \barwedge \{0.4\} = \{0.3, 0.4\},$$

$$\underline{Apr}_{HSF}(h_E)(x_2) = \{0.5\} \barwedge \{0.3, 0.4, 0.5\} \barwedge \{0.4, 0.5\} \barwedge \{0.8\} \barwedge \{0.5, 0.7, 0.8\} = \{0.3, 0.4, 0.5\},$$

$$\underline{Apr}_{HSF}(h_E)(x_3) = \{0.7\} \barwedge \{0.3, 0.4, 0.5\} \barwedge \{0.4, 0.5\} \barwedge \{0.5\} \barwedge \{0.3, 0.4, 0.5\} = \{0.3, 0.4, 0.5\},$$

$$\underline{Apr}_{HSF}(h_E)(x_4) = \{0.5, 0.7\} \barwedge \{0.3, 0.4, 0.5\} \barwedge \{0.5, 0.7\} \barwedge \{0.4, 0.5\} \barwedge \{0.6, 0.8\} = \{0.3, 0.4, 0.5\},$$

$$\underline{Apr}_{HSF}(h_E)(x_5) = \{0.5, 0.6\} \barwedge \{0.5, 0.6, 0.7\} \barwedge \{0.4, 0.5, 0.6\} \barwedge \{0.4, 0.5\} \barwedge \{0.3, 0.4, 0.5\} = \{0.3, 0.4, 0.5\},$$

$$\underline{Apr}_{HSF}(h_E)(x_6) = \{0.5\} \barwedge \{0.7\} \barwedge \{0.4, 0.5\} \barwedge \{0.4, 0.5\} \barwedge \{0.5, 0.7\} = \{0.4, 0.5\}.$$

Similarly,

$$\underline{\bigvee}_{x_i\in U}(h_{f(a_1)}(x_i) \barwedge h_E(x_i)) = (\{0.2, 0.3\} \barwedge \{0.2, 0.3\}) \veebar (\{0.5, 0.6\} \barwedge \{0.5\}) \veebar (\{0.3\} \barwedge \{0.4, 0.6\}) \veebar (\{0.3, 0.5\} \barwedge \{0.7, 0.8, 0.9\}) \veebar (\{0.4, 0.5\} \barwedge \{0.1\}) \veebar (\{0.6, 0.7\} \barwedge \{0.9\}) = \{0.2, 0.3\} \veebar \{0.5\} \veebar \{0.3\} \veebar \{0.3, 0.5\} \veebar \{0.1\}$$

$$
\begin{aligned}
&underline\vee \{0.6, 0.7\}\\
&= \{0.6, 0.7\},\\
&\underline{\bigvee}_{x_i\in U} (h_{f(a_2)}(x_i) \barwedge h_E(x_i))\\
&= \{0.2, 0.3\} \veebar \{0.5\} \veebar \{0.4, 0.6\} \veebar \{0.7, 0.8, 0.9\} \veebar \{0.1\}\\
&\veebar \{0.3\}\\
&= \{0.7, 0.8, 0.9\},\\
&\underline{\bigvee}_{x_i\in U} (h_{f(a_3)}(x_i) \barwedge h_E(x_i))\\
&= \{0.2, 0.3\} \veebar \{0.5\} \veebar \{0.4, 0.6\} \veebar \{0.3, 0.5\} \veebar \{0.1\}\\
&\veebar \{0.7\}\\
&= \{0.7\},\\
&\underline{\bigvee}_{x_i\in U} (h_{f(a_4)}(x_i) \barwedge h_E(x_i))\\
&= \{0.2, 0.3\} \veebar \{0.2\} \veebar \{0.4, 0.5\} \veebar \{0.6, 0.7\} \veebar \{0.1\}\\
&\veebar \{0.8\}\\
&= \{0.8\},\\
&\underline{\bigvee}_{x_i\in U} (h_{f(a_5)}(x_i) \barwedge h_E(x_i))\\
&= \{0.2, 0.3\} \veebar \{0.2, 0.3, 0.5\} \veebar \{0.4, 0.5, 0.6\} \veebar \{0.2, 0.4\}\\
&\veebar \{0.1\} \veebar \{0.3, 0.5\}\\
&= \{0.4, 0.5, 0.6\},
\end{aligned}
$$

then we obtain the upper hesitant soft fuzzy rough approximations of E as follows:

$$
\begin{aligned}
&\overline{Apr}_{HSF}(h_E)(x_1)\\
&= (\{0.2, 0.3\} \barwedge \{0.6, 0.7\}) \veebar (\{0.4, 0.6, 0.7\}\\
&\barwedge \{0.7, 0.8, 0.9\})\\
&\veebar (\{0.2, 0.4\} \barwedge \{0.7\}) \veebar (\{0.3, 0.5, 0.6\} \barwedge \{0.8\})\\
&\veebar (\{0.6\} \barwedge \{0.4, 0.5, 0.6\})\\
&= \{0.2, 0.3\} \veebar \{0.4, 0.6, 0.7\} \veebar \{0.2, 0.4\}\\
&\veebar \{0.3, 0.5, 0.6\} \veebar \{0.4, 0.5, 0.6\}\\
&= \{0.4, 0.5, 0.6, 0.7\},\\
&\overline{Apr}_{HSF}(h_E)(x_2)\\
&= \{0.5, 0.6\} \veebar \{0.5, 0.7, 0.8\} \veebar \{0.6, 0.7\} \veebar \{0.2\}\\
&\veebar \{0.2, 0.3, 0.4, 0.5\}\\
&= \{0.6, 0.7, 0.8\},\\
&\overline{Apr}_{HSF}(h_E)(x_3)\\
&= \{0.3\} \veebar \{0.6, 0.7, 0.8\} \veebar \{0.7\} \veebar \{0.5\} \veebar \{0.4, 0.5, 0.6\}\\
&= \{0.7, 0.8\},\\
&\overline{Apr}_{HSF}(h_E)(x_4)\\
&= \{0.3, 0.5\} \veebar \{0.7, 0.8, 0.9\} \veebar \{0.3, 0.5\} \veebar \{0.6, 0.7\}\\
&\veebar \{0.2, 0.4\}\\
&= \{0.7, 0.8, 0.9\},\\
&\overline{Apr}_{HSF}(h_E)(x_5)\\
&= \{0.4, 0.5\} \veebar \{0.3, 0.4, 0.5\} \veebar \{0.4, 0.6\} \veebar \{0.5, 0.6\}\\
&\veebar \{0.4, 0.5, 0.6\}\\
&= \{0.5, 0.6\},\\
&\overline{Apr}_{HSF}(h_E)(x_6)\\
&= \{0.6, 0.7\} \veebar \{0.3\} \veebar \{0.7\} \veebar \{0.8\} \veebar \{0.3, 0.4, 0.5\}\\
&= \{0.8\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\underline{Apr}_{HSF}(h_E)\\
&=\{\langle x_1, \{0.3, 0.4\}\rangle, \langle x_2, \{0.3, 0.4, 0.5\}\rangle, \langle x_3, \{0.3, 0.4, 0.5\}\rangle,\\
&\langle x_4, \{0.3, 0.4, 0.5\}\rangle, \langle x_5, \{0.3, 0.4, 0.5\}\rangle, \langle x_6, \{0.4, 0.5\}\rangle\},\\
&\overline{Apr}_{HSF}(h_E)\\
&= \{\langle x_1, \{0.4, 0.5, 0.6, 0.7\}\rangle, \langle x_2, \{0.6, 0.7, 0.8\}\rangle,\\
&\langle x_3, \{0.7, 0.8\}\rangle,\\
&\langle x_4, \{0.7, 0.8, 0.9\}\rangle, \langle x_5, \{0.5, 0.6\}\rangle, \langle x_6, \{0.8\}\rangle\}.
\end{aligned}
$$

Theorem 3: Let $S = (f, A)$ be a hesitant fuzzy soft set over U, $HSF = (U, S)$ be a hesitant soft fuzzy approximation space. Then for any $F, G \in H(U)$, we have

(1) $\underline{Apr}_{HSF}(h_U) = \overline{Apr}_{HSF}(h_U) = U$,
$\underline{Apr}_{HSF}(\emptyset) = \overline{Apr}_{HSF}(h_\emptyset) = \emptyset$.

(2) $\underline{Apr}_{HSF}(h_{F^c}) = h_{(\overline{Apr}_{HSF}(h_F))^c}$,
$\overline{Apr}_{HSF}(h_{F^c}) = h_{(\underline{Apr}_{HSF}(h_F))^c}$.

(3) $F \subseteq G \Rightarrow \underline{Apr}_{HSF}(h_F) \subseteq \underline{Apr}_{HSF}(h_G)$,
$\overline{Apr}_{HSF}(h_F) \subseteq \overline{Apr}_{HSF}(h_G)$.

(4) $\underline{Apr}_{HSF}(h_{F\cap G}) = \underline{Apr}_{HSF}(h_F) \cap \underline{Apr}_{HSF}(h_G)$,
$\overline{Apr}_{HSF}(h_{F\cup G}) = \overline{Apr}_{HSF}(h_F) \cup \overline{Apr}_{HSF}(h_G)$.

(5) $\underline{Apr}_{HSF}(h_{F\cup G}) \subseteq \underline{Apr}_{HSF}(h_F) \cup \underline{Apr}_{HSF}(h_G)$,
$\overline{Apr}_{HSF}(h_{F\cap G}) \supseteq \overline{Apr}_{HSF}(h_F) \cap \overline{Apr}_{HSF}(h_G)$.

Proof: (1) For any $x \in U$, $h_U(x) = 1$, $h_\emptyset(x) = 0$. Then

$$
\begin{aligned}
&\underline{Apr}_{HSF}(h_U)(x)\\
&\quad= \overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \veebar \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \veebar h_U(y))\right)\right)\\
&\quad= \overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \veebar \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \veebar \{1\})\right)\right)\\
&\quad= \overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \veebar h_{f^c(a)}(y)\right) \veebar \{1\}\right)\\
&\quad= \overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}(h_{f^c(a)}(x) \veebar h_{f^c(a)}(y))\right) \veebar \{1\}\\
&\quad= \{1\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\overline{Apr}_{HSF}(h_\emptyset)(x)\\
&\quad= \underline{\bigvee}_{a\in A}\left(h_{f(a)}(x) \barwedge \left(\underline{\bigvee}_{y\in U}(h_{f(a)}(y) \barwedge h_\emptyset(y))\right)\right)\\
&\quad= \underline{\bigvee}_{a\in A}\left(h_{f(a)}(x) \barwedge \left(\underline{\bigvee}_{y\in U}(h_{f(a)}(y) \barwedge \{0\})\right)\right)\\
&\quad= \underline{\bigvee}_{y\in U}\left(\underline{\bigvee}_{a\in A}\left(h_{f(a)}(x) \barwedge h_{f(a)}(y)\right) \barwedge \{0\}\right)\\
&\quad= \underline{\bigvee}_{y\in U}\left(\underline{\bigvee}_{a\in A}(h_{f(a)}(x) \barwedge h_{f(a)}(y))\right) \barwedge \{0\}\\
&\quad= \{0\}.
\end{aligned}
$$

Therefore, $\underline{Apr}_{HSF}(h_U) = U$, $\overline{Apr}_{HSF}(h_{\emptyset}) = \emptyset$. Similarly, we can obtain other equations.

(2) For any $x \in U$,

$$
\begin{aligned}
&\underline{Apr}_{HSF}(h_{F^c})(x)\\
&= \overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \,\underline{\vee}\, h_{F^c}(y))\right)\right)\\
&= \overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \,\underline{\vee}\, h_{F^c}(y)\right)\\
&= \overline{\bigwedge}_{y\in U}\left(\sim \left(\left(\underline{\bigvee}_{a\in A}(h_{f(a)}(x) \,\overline{\wedge}\, h_{f(a)}(y))\right) \overline{\wedge}\, h_F(y)\right)\right)\\
&=\sim \left(\underline{\bigvee}_{y\in U}\left(\underline{\bigvee}_{a\in A}(h_{f(a)}(x) \,\overline{\wedge}\, h_{f(a)}(y))\right) \overline{\wedge}\, h_F(y)\right)\\
&=\sim \left(\underline{\bigvee}_{a\in A}\left(h_{f(a)}(x) \,\overline{\wedge} \left(\underline{\bigvee}_{y\in U}(h_{f(a)}(y) \,\overline{\wedge}\, h_F(y))\right)\right)\right)\\
&= h_{(\overline{Apr}_{HSF}(h_F))^c}(x).
\end{aligned}
$$

Therefore, $\underline{Apr}_{HSF}(h_{F^c}) = h_{(\overline{Apr}_{HSF}(h_F))^c}$. The other equation can be proved in the similar way.

(3) Since $F \subseteq G$, i.e., $h_F(x) \preceq h_G(x)$ for all $x \in U$, then

$$
\begin{aligned}
&\underline{Apr}_{HSF}(h_F)(x)\\
&= \overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee} \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \,\underline{\vee}\, h_F(y))\right)\right)\\
&= \overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_F(y)\right)\\
&\preceq \overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_G(y)\right)\\
&= \overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee} \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \,\underline{\vee}\, h_G(y))\right)\right)\\
&= \underline{Apr}_{HSF}(h_G)(x).
\end{aligned}
$$

Therefore, $\underline{Apr}_{HSF}(h_F) \subseteq \underline{Apr}_{HSF}(h_G)$. Similarly, we can obtain $\overline{Apr}_{HSF}(h_F) \subseteq \overline{Apr}_{HSF}(h_G)$.

(4) For any $x \in U$,

$$
\begin{aligned}
&\underline{Apr}_{HSF}(h_{F\cap G})(x)\\
&= \overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee} \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \,\underline{\vee}\, h_{F\cap G}(y))\right)\right)\\
&= \overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_{F\cap G}(y)\right)\\
&= \overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, (h_F(y) \,\overline{\wedge}\, h_G(y))\right)\\
&= \overline{\bigwedge}_{y\in U}\left(\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_F(y)\right)\right.\\
&\qquad \left.\overline{\wedge} \left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_G(y)\right)\right)\\
&= \left(\overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_F(y)\right)\right)\\
&\overline{\wedge} \left(\overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_G(y)\right)\right)\\
&= \left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee} \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \,\underline{\vee}\, h_F(y))\right)\right)\right)\\
&\overline{\wedge} \left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee} \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \,\underline{\vee}\, h_G(y))\right)\right)\right)\\
&= \underline{Apr}_{HSF}(h_F)(x) \cap \underline{Apr}_{HSF}(h_G)(x).
\end{aligned}
$$

Therefore, $\underline{Apr}_{HSF}(h_{F\cap G}) = \underline{Apr}_{HSF}(h_F) \cap \underline{Apr}_{HSF}(h_G)$. Similarly, $\overline{Apr}_{HSF}(h_{F\cup G}) = \overline{Apr}_{HSF}(h_F) \cup \overline{Apr}_{HSF}(h_G)$.

(5) For any $x \in U$, since $h_F(x) \preceq h_F(x) \,\underline{\vee}\, h_G(x)$, then

$$
\begin{aligned}
&\underline{Apr}_{HSF}(h_{F\cup G})(x)\\
&= \overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee} \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \,\underline{\vee}\, h_{F\cup G}(y))\right)\right)\\
&= \overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_{F\cup G}(y)\right)\\
&= \overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, (h_F(y) \,\underline{\vee}\, h_G(y))\right)\\
&= \overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_F(y) \,\underline{\vee}\, h_G(y)\right)\\
&\succeq \overline{\bigwedge}_{y\in U}\left(\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_F(y)\right)\right.\\
&\qquad \left.\underline{\vee} \left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_G(y)\right)\right)\\
&= \left(\overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_F(y)\right)\right)\\
&\underline{\vee} \left(\overline{\bigwedge}_{y\in U}\left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\, h_{f^c(a)}(y)\right) \underline{\vee}\, h_G(y)\right)\right)\\
&= \left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee} \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \,\underline{\vee}\, h_F(y))\right)\right)\right)\\
&\underline{\vee} \left(\overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee} \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \,\underline{\vee}\, h_G(y))\right)\right)\right)\\
&= \underline{Apr}_{HSF}(h_F)(x) \cup \underline{Apr}_{HSF}(h_G)(x).
\end{aligned}
$$

Therefore, $\underline{Apr}_{HSF}(h_{F\cup G}) \subseteq \underline{Apr}_{HSF}(h_F) \cup \underline{Apr}_{HSF}(h_G)$. Since $h_F(x) \succeq h_F(x) \,\overline{\wedge}\, h_G(x)$ for any $x \in U$, then we can prove $\overline{Apr}_{HSF}(h_{F\cap G}) \supseteq \overline{Apr}_{HSF}(h_F) \cap \overline{Apr}_{HSF}(h_G)$ in the similar way.

Now we wonder, $\forall a \in A$, what is the relationship between x and y if we know $f(a,x)$ and $f(a,y)$. To that purpose, we define a hesitant fuzzy binary relation on U induced by a hesitant fuzzy soft set as follows:

Definition 14: Let $S = (f, A)$ be a hesitant fuzzy soft set over U. Then S induces a hesitant fuzzy binary relation $R^S \in H(U \times U)$, which is defined by

$$R^S = \{\langle (x,y), h_{R^S}(x,y)\rangle \mid (x,y) \in U \times U\},$$

where $h_{R^S}(x,y) = \underline{\bigvee}_{a\in A}(h_{f(a)}(x) \,\overline{\wedge}\, h_{f(a)}(y))$.

Theorem 4: Let $S = (f, A)$ be a hesitant fuzzy soft set over U and R be the hesitant fuzzy binary relation induced by S. Then for any $E \in H(U)$,

$$
\begin{aligned}
\underline{R}(h_E) &= \underline{Apr}_{HSF}(h_E),\\
\overline{R}(h_E) &= \overline{Apr}_{HSF}(h_E).
\end{aligned}
$$

Proof: For any $x \in U$,

$$
\begin{aligned}
&\underline{R}(h_E)(x)\\
&= \overline{\bigwedge}_{y\in U}\{h_{R^c}(x,y) \underline{\vee} h_E(y)\}\\
&= \overline{\bigwedge}_{y\in U}\left\{\sim\left(\underline{\bigvee}_{a\in A}(h_{f(a)}(x)\,\overline{\wedge}\,h_{f(a)}(y))\right) \underline{\vee}\, h_E(y)\right\}\\
&= \overline{\bigwedge}_{y\in U}\left\{\overline{\bigwedge}_{a\in A}(h_{f^c(a)}(x) \underline{\vee}\, h_{f^c(a)}(y))\right) \underline{\vee}\, h_E(y)\right\}\\
&= \overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \underline{\vee} \left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \underline{\vee}\, h_E(y))\right)\right)\\
&= \underline{Apr}_{HSF}(h_E)(x),\\
&\overline{R}(h_E)(x)\\
&= \underline{\bigvee}_{y\in U}\{h_R(x,y)\,\overline{\wedge}\,h_E(y)\}\\
&= \underline{\bigvee}_{y\in U}\left\{\underline{\bigvee}_{a\in A}(h_{f(a)}(x)\,\overline{\wedge}\,h_{f(a)}(y))\,\overline{\wedge}\,h_E(y)\right\}\\
&= \underline{\bigvee}_{a\in A}\left(h_{f(a)}(x)\,\overline{\wedge}\left(\underline{\bigvee}_{y\in U}(h_{f(a)}(y)\,\overline{\wedge}\,h_E(y))\right)\right)\\
&= \overline{Apr}_{HSF}(h_E)(x).
\end{aligned}
$$

Example 6: Consider Example 5. According to Definition 14, S induces a hesitant fuzzy binary relation R^S, which is defined by

$$R^S = \{\langle (x,y), h_{R^S}(x,y)\rangle | (x,y) \in U \times U\},$$

where $h_{R^S}(x,y) = \underline{\bigvee}_{a\in A}(h_{f(a)}(x)\,\overline{\wedge}\,h_{f(a)}(y))$. Thus, by calculating $R^S(x_i,x_j)$ for all pairs of the universe U, we obtain the hesitant fuzzy relation matrix $R^S(x_i,x_j)_{6\times 6}$ as follows:

$$
R^S = \begin{pmatrix}
0.6,0.7 & 0.4,0.5,0.6,0.7 & 0.5,0.6,0.7 & 0.4,0.5,0.6,0.7 & 0.5,0.6 & 0.3,0.4,0.5,0.6\\
0.4,0.5,0.6,0.7 & 0.6,0.7,0.8 & 0.6,0.7,0.8 & 0.5,0.7,0.8 & 0.4,0.5,0.6 & 0.6,0.7\\
0.5,0.6,0.7 & 0.6,0.7,0.8 & 0.8,0.9 & 0.6,0.7,0.8 & 0.5,0.6,0.7 & 0.7\\
0.4,0.5,0.6,0.7 & 0.5,0.7,0.8 & 0.6,0.7,0.8 & 0.7,0.9 & 0.5,0.6 & 0.6,0.7\\
0.5,0.6 & 0.4,0.5,0.6 & 0.5,0.6,0.7 & 0.5,0.6 & 0.5,0.6,0.7 & 0.5,0.6\\
0.3,0.4,0.5,0.6 & 0.6,0.7 & 0.7 & 0.6,0.7 & 0.5,0.6 & 0.8
\end{pmatrix} \quad (17)
$$

By Definition 10, the lower and upper hesitant fuzzy rough approximations of E in terms of the hesitant fuzzy relation R^S, respectively, as follows:

$$\underline{R^S}(h_E)(x) = \overline{\bigwedge}_{x_i\in U}\{h_{(R^S)^c}(x,x_i) \underline{\vee}\, h_E(x_i)\},$$

$$\overline{R^S}(h_E)(x) = \underline{\bigvee}_{x_i\in U}\{h_R(x,x_i)\,\overline{\wedge}\,h_E(x_i)\},$$

for all $x \in U$. Thus, we have

$$
\begin{aligned}
&\underline{R^S}(h_E)(x_1)\\
&= \overline{\bigwedge}_{x_i\in U}\{h_{(R^S)^c}(x_1,x_i) \underline{\vee}\, h_E(x_i)\},\\
&= (\{0.3,0.4\} \underline{\vee} \{0.2,0.3\})\,\overline{\wedge}\,(\{0.3,0.4,0.5,0.6\} \underline{\vee} \{0.5\})\\
&\overline{\wedge}\,(\{0.3,0.4,0.5\} \underline{\vee} \{0.4,0.6\})\,\overline{\wedge}\,(\{0.3,0.4,0.5,0.6\}\\
&\underline{\vee} \{0.7,0.8,0.9\})\,\overline{\wedge}\,(\{0.4,0.5\} \underline{\vee} \{0.1\})\\
&\overline{\wedge}\,(\{0.4,0.5,0.6,0.7\} \underline{\vee} \{0.9\})\\
&= \{0.3,0.4\}\,\overline{\wedge}\,\{0.5,0.6\}\,\overline{\wedge}\,\{0.4,0.5,0.6\}\,\overline{\wedge}\,\{0.7,0.8,0.9\}\\
&\overline{\wedge}\,\{0.4,0.5\}\,\overline{\wedge}\,\{0.9\}\\
&= \{0.3,0.4\},\\
&\underline{R^S}(h_E)(x_2)\\
&= \overline{\bigwedge}_{x_i\in U}\{h_{(R^S)^c}(x_2,x_i) \underline{\vee}\, h_E(x_i)\},\\
&= \{0.3,0.4,0.5,0.6\}\,\overline{\wedge}\,\{0.5\}\,\overline{\wedge}\,\{0.4,0.6\}\,\overline{\wedge}\,\{0.7,0.8,0.9\}\\
&\overline{\wedge}\,\{0.4,0.5,0.6\}\,\overline{\wedge}\,\{0.9\}\\
&= \{0.3,0.4,0.5\},\\
&\underline{R^S}(h_E)(x_3)\\
&= \overline{\bigwedge}_{x_i\in U}\{h_{(R^S)^c}(x_3,x_i) \underline{\vee}\, h_E(x_i)\},\\
&= \{0.3,0.4,0.5\}\,\overline{\wedge}\,\{0.5\}\,\overline{\wedge}\,\{0.4,0.6\}\,\overline{\wedge}\,\{0.7,0.8,0.9\}\\
&\overline{\wedge}\,\{0.3,0.4,0.5\}\,\overline{\wedge}\,\{0.9\}\\
&= \{0.3,0.4,0.5\},\\
&\underline{R^S}(h_E)(x_4)\\
&= \overline{\bigwedge}_{x_i\in U}\{h_{(R^S)^c}(x_4,x_i) \underline{\vee}\, h_E(x_i)\},\\
&= \{0.3,0.4,0.5,0.6\}\,\overline{\wedge}\,\{0.5\}\,\overline{\wedge}\,\{0.4,0.6\}\,\overline{\wedge}\,\{0.7,0.8,0.9\}\\
&\overline{\wedge}\,\{0.4,0.5\}\,\overline{\wedge}\,\{0.9\}\\
&= \{0.3,0.4,0.5\},\\
&\underline{R^S}(h_E)(x_5)\\
&= \overline{\bigwedge}_{x_i\in U}\{h_{(R^S)^c}(x_5,x_i) \underline{\vee}\, h_E(x_i)\},\\
&= \{0.4,0.5\}\,\overline{\wedge}\,\{0.5,0.6\}\,\overline{\wedge}\,\{0.4,0.5,0.6\}\,\overline{\wedge}\,\{0.7,0.8,0.9\}\\
&\overline{\wedge}\,\{0.3,0.4,0.5\}\,\overline{\wedge}\,\{0.9\}\\
&= \{0.3,0.4,0.5\},\\
&\underline{R^S}(h_E)(x_6)\\
&= \overline{\bigwedge}_{x_i\in U}\{h_{(R^S)^c}(x_6,x_i) \underline{\vee}\, h_E(x_i)\},\\
&= \{0.4,0.5,0.6,0.7\}\,\overline{\wedge}\,\{0.5\}\,\overline{\wedge}\,\{0.4,0.6\}\,\overline{\wedge}\,\{0.7,0.8,0.9\}\\
&\overline{\wedge}\,\{0.4,0.5\}\,\overline{\wedge}\,\{0.9\}\\
&= \{0.4,0.5\},
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\overline{R^S}(h_E)(x_1)\\
&= \underline{\bigvee}_{x_i\in U}\{h_{R^S}(x_1,x_i)\,\overline{\wedge}\,h_E(x_i)\}\\
&= (\{0.6,0.7\}\,\overline{\wedge}\,\{0.2,0.3\}) \underline{\vee} (\{0.4,0.5,0.6,0.7\}\,\overline{\wedge}\,\{0.5\})\\
&\underline{\vee} (\{0.5,0.6,0.7\}\,\overline{\wedge}\,\{0.4,0.6\}) \underline{\vee} (\{0.4,0.5,0.6,0.7\}\\
&\overline{\wedge}\,\{0.7,0.8,0.9\}) \underline{\vee} (\{0.5,0.6\}\,\overline{\wedge}\,\{0.1\})
\end{aligned}
$$

$$\begin{aligned}
&\veebar (\{0.3, 0.4, 0.5, 0.6\} \barwedge \{0.9\}) \\
&= \{0.2, 0.3\} \veebar \{0.4, 0.5\} \veebar \{0.4, 0.5, 0.6\} \\
&\veebar \{0.4, 0.5, 0.6, 0.7\} \veebar \{0.1\} \\
&= \{0.4, 0.5, 0.6, 0.7\},
\end{aligned}$$

$$\begin{aligned}
&\overline{R^S}(h_E)(x_2) \\
&= \bigvee_{x_i \in U} \{h_{R^S}(x_2, x_i) \barwedge h_E(x_i)\} \\
&= \{0.2, 0.3\} \veebar \{0.5\} \veebar \{0.4, 0.6\} \veebar \{0.5, 0.7, 0.8\} \veebar \{0.1\} \\
&\veebar \{0.6, 0.7\} \\
&= \{0.6, 0.7, 0.8\},
\end{aligned}$$

$$\begin{aligned}
&\overline{R^S}(h_E)(x_3) \\
&= \bigvee_{x_i \in U} \{h_{R^S}(x_3, x_i) \barwedge h_E(x_i)\} \\
&= \{0.2, 0.3\} \veebar \{0.5\} \veebar \{0.4, 0.6\} \veebar \{0.6, 0.7, 0.8\} \veebar \{0.1\} \\
&\veebar \{0.7\} \\
&= \{0.7, 0.8\},
\end{aligned}$$

$$\begin{aligned}
&\overline{R^S}(h_E)(x_4) \\
&= \bigvee_{x_i \in U} \{h_{R^S}(x_4, x_i) \barwedge h_E(x_i)\} \\
&= \{0.2, 0.3\} \veebar \{0.5\} \veebar \{0.4, 0.6\} \veebar \{0.7, 0.8, 0.9\} \veebar \{0.1\} \\
&\veebar \{0.6, 0.7\} \\
&= \{0.7, 0.8, 0.9\},
\end{aligned}$$

$$\begin{aligned}
&\overline{R^S}(h_E)(x_5) \\
&= \bigvee_{x_i \in U} \{h_{R^S}(x_5, x_i) \barwedge h_E(x_i)\} \\
&= \{0.2, 0.3\} \veebar \{0.4, 0.5\} \veebar \{0.4, 0.5, 0.6\} \veebar \{0.5, 0.6\} \\
&\veebar \{0.1\} \veebar \{0.5, 0.6\} \\
&= \{0.5, 0.6\},
\end{aligned}$$

$$\begin{aligned}
&\overline{R^S}(h_E)(x_6) \\
&= \bigvee_{x_i \in U} \{h_{R^S}(x_6, x_i) \barwedge h_E(x_i)\} \\
&= \{0.2, 0.3\} \veebar \{0.5\} \veebar \{0.4, 0.6\} \veebar \{0.6, 0.7\} \veebar \{0.1\} \veebar \{0.8\} \\
&= \{0.8\}.
\end{aligned}$$

Therefore,

$$\underline{R^S}(h_E) = \underline{Apr}_{HSF}(h_E), \quad \overline{R^S}(h_E) = \overline{Apr}_{HSF}(h_E).$$

Proposition 3: Let $S = (f, A)$ be a hesitant fuzzy soft set over U and R be the hesitant fuzzy binary relation induced by S. If R is reflexive, then for any $E \in H(U)$, we have $\underline{Apr}_{HSF}(h_E) \subseteq h_E \subseteq \overline{Apr}_{HSF}(h_E)$.

Proof: For any $x \in U$, since R is reflexive, then $h_{R^c}(x, x) = \{0\}$. According to Theorem 4, we have

$$\begin{aligned}
&\underline{Apr}_{HSF}(h_E)(x) \\
&= \bigwedge_{y \in U} \{h_{R^c}(x, y) \veebar h_E(y)\} \\
&= \left(h_{R^c}(x, x) \veebar h_F(x)\right) \overline{\bigwedge} \left(\overline{\bigwedge}_{y \neq x} \{h_{R^c}(x, y) \veebar h_E(y)\}\right) \\
&= h_F(x) \overline{\bigwedge} \left(\overline{\bigwedge}_{y \neq x} \{h_{R^c}(x, y) \veebar h_E(y)\}\right) \\
&\preceq h_E(x).
\end{aligned}$$

Therefore, $\underline{Apr}_{HSF}(h_E) \subseteq h_E$. Similarly, we can prove $h_E \subseteq \overline{Apr}_{HSF}(h_E)$.

Proposition 4: Let $S = (f, A)$ be a hesitant fuzzy soft set over U and R be the hesitant fuzzy binary relation induced by S. Given $E \in H(U)$, then for all $x \in U$, we have

$$\begin{aligned}
h^-_{\underline{Apr}_{HSF}(h_E)}(x) &= \min\{\max\{h^-_{R^c}(x, y), h^-_E(y)\} : y \in U\}, \\
h^+_{\underline{Apr}_{HSF}(h_E)}(x) &= \min\{\max\{h^+_{R^c}(x, y), h^+_E(y)\} : y \in U\}. \\
h^-_{\overline{Apr}_{HSF}(h_E)}(x) &= \max\{\min\{h^-_R(x, y), h^-_E(y)\} : y \in U\}, \\
h^+_{\overline{Apr}_{HSF}(h_E)}(x) &= \max\{\min\{h^+_R(x, y), h^+_E(y)\} : y \in U\}.
\end{aligned}$$

Proof: For any $x \in U$, according to Theorem 6, we have $\underline{Apr}_{HSF}(h_E)(x) = \bigwedge_{y \in U} \{h_{R^c}(x, y) \veebar h_E(y)\}$, therefore, by the definition of $\barwedge$ and $\veebar$, we obtain $h^-_{\underline{Apr}_{HSF}(h_E)}(x) = \min\{\max\{h^-_{R^c}(x, y), h^-_E(y)\} : y \in U\}$, $h^+_{\underline{Apr}_{HSF}(h_E)}(x) = \min\{\max\{h^+_{R^c}(x, y), h^+_E(y)\} : y \in U\}$. The other equations can be proved in the similar way.

We denote the (α, t)-level set and strong (α, t)-level set associated with R^S, respectively, as

$$\begin{aligned}
{}_{\alpha,t}h_{R^S} &= \{(x, y) \in U \times U : |\{h \in h_R(x, y) : h \geq \alpha\}| \geq t\}, \\
{}_{\alpha^+,t}h_{R^S} &= \{(x, y) \in U \times U : |\{h \in h_R(x, y) : h > \alpha\}| \geq t\},
\end{aligned}$$

for all $\alpha \in [0, 1]$ and for all $t \in \mathbb{N}^+$.

Theorem 5: Let $S = (f, A)$ be a hesitant fuzzy soft set over U and R^S be the hesitant fuzzy binary relation induced by S. If R^S is typical, then R^S is the hesitant fuzzy binary relation on U associated with the family of fuzzy binary relations $R = \{{}_tR\}_{t \in \mathbb{N}^+}$ on U, i.e.,

$$h_{R^S} = \bigcup_{t=1,2,\cdots} {}_tR,$$

where ${}_1R(x, y) = \max\{\alpha \in [0, 1] : (x, y) \in_{\alpha,1} h_{R^S}\} = h^{l_{R^S}(x,y)}_{R^S}(x, y)$ for each $(x, y) \in U \times U$; if ${}_1R, \cdots, {}_tR$ are known, then ${}_{t+1}R(x, y) = \max\{\alpha \in [0, 1] : (x, y) \in_{\alpha,t+1} h_{R^S}\}$, if $(x, y) \in_{\alpha,t+1} h_R$ some $\alpha \in [0, 1]$, and ${}_{t+1}R(x, y) =_t R(x, y)$ otherwise.

Proof: Since R^S is a typical hesitant fuzzy set, the conclusion can be easily obtained by Theorem 1.

Theorem 5 produces a decomposition of any hesitant fuzzy binary relation in terms of the fuzzy binary relations.

Example 7: Consider Example 6. For the hesitant fuzzy soft set $S = (f, A)$, and the induced hesitant fuzzy binary relation R^S (see Equation 17). According to Theorem 5,

$$h_{R^S} = \bigcup_{t=1}^{4} {}_tR^S,$$

where

$${}_1R^S = \begin{pmatrix} 0.7 & 0.7 & 0.7 & 0.7 & 0.6 & 0.6 \\ 0.7 & 0.8 & 0.8 & 0.8 & 0.6 & 0.7 \\ 0.7 & 0.8 & 0.9 & 0.8 & 0.7 & 0.7 \\ 0.7 & 0.8 & 0.8 & 0.9 & 0.6 & 0.7 \\ 0.6 & 0.6 & 0.7 & 0.6 & 0.7 & 0.6 \\ 0.6 & 0.7 & 0.7 & 0.7 & 0.6 & 0.8 \end{pmatrix},$$

$$
{}_2R^S = \begin{pmatrix} 0.6 & 0.6 & 0.6 & 0.6 & 0.5 & 0.5 \\ 0.6 & 0.7 & 0.7 & 0.7 & 0.5 & 0.6 \\ 0.6 & 0.7 & 0.8 & 0.7 & 0.6 & 0.7 \\ 0.6 & 0.7 & 0.7 & 0.7 & 0.5 & 0.6 \\ 0.5 & 0.5 & 0.6 & 0.5 & 0.6 & 0.5 \\ 0.5 & 0.6 & 0.7 & 0.6 & 0.5 & 0.8 \end{pmatrix},
$$

$$
{}_3R^S = \begin{pmatrix} 0.6 & 0.5 & 0.5 & 0.5 & 0.5 & 0.4 \\ 0.5 & 0.6 & 0.6 & 0.5 & 0.4 & 0.6 \\ 0.5 & 0.6 & 0.8 & 0.6 & 0.5 & 0.7 \\ 0.5 & 0.5 & 0.6 & 0.7 & 0.5 & 0.6 \\ 0.5 & 0.4 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.4 & 0.6 & 0.7 & 0.6 & 0.5 & 0.8 \end{pmatrix},
$$

$$
{}_4R^S = \begin{pmatrix} 0.6 & 0.4 & 0.5 & 0.4 & 0.5 & 0.3 \\ 0.4 & 0.5 & 0.6 & 0.5 & 0.4 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.6 & 0.5 & 0.7 \\ 0.4 & 0.5 & 0.5 & 0.7 & 0.5 & 0.6 \\ 0.5 & 0.4 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.3 & 0.6 & 0.7 & 0.6 & 0.5 & 0.8 \end{pmatrix}.
$$

Remark 4: If $S = (f, A)$ is a full fuzzy soft set, then $\overline{\wedge}$ and $\underline{\vee}$ reduce to $\wedge$ and $\vee$, respectively, and R^S is a fuzzy equivalence relation.

Proof: For any $x \in U$, the reflexivity and symmetry of R^S can be easily obtained.

Since $S = (f, A)$ is a full fuzzy soft set, then $\bigvee_{a\in A} h_{f(a)}(x) = 1$ for all $x \in U$, thus, for every $x \in U$, there exists $a_0 \in A$ such that $h_{f(a_0)}(x) = 1$. Therefore, $\forall x, y, z \in U$,

$$
\begin{aligned}
&\bigwedge(h_{R^S}(x, z), h_{R^S}(z, y)) \\
&= \bigwedge\left(\bigvee_{a\in A}(h_{f(a)}(x) \wedge h_{f(a)}(z)), \bigvee_{a\in A}(h_{f(a)}(z) \wedge h_{f(a)}(y))\right) \\
&= \bigwedge\left(h_{f(a_0)}(x) \wedge h_{f(a_0)}(z), h_{f(a_0)}(z) \wedge h_{f(a_0)}(y)\right) \\
&= h_{f(a_0)}(x) \wedge h_{f(a_0)}(y)) \\
&\leq \bigvee_{a\in A}(h_{f(a)}(x) \wedge h_{f(a)}(y))) \\
&= h_{R^S}(x, y).
\end{aligned}
$$

It follows that R^S is transitive. Therefore, R^S is a fuzzy equivalence relation.

Remark 5: If $S = (f, A)$ is a full fuzzy soft set, then $t = 1$, and for any $\alpha \in [0, 1]$, ${}_tR_\alpha = R_\alpha$ and ${}_tR_{\alpha^+} = R_{\alpha^+}$ construct a partition of the universe U, respectively, i.e.,

$$
U/R_\alpha = \{[x]_{R_\alpha} : x \in U\}, \; U/R_{\alpha^+} = \{[x]_{R_{\alpha^+}} : x \in U\},
$$

where

$$
\begin{aligned}
[x]_{R_\alpha} &= \{y \in U : h_R(x, y) \geq \alpha\} \\
&= \{y \in U : \bigvee_{a\in A}(h_{f(a)}(x) \wedge h_{f(a)}(y)) \geq \alpha\}, \\
&= \{y \in U : \bigvee_{a\in A}(f(a)(x) \wedge f(a)(y)) \geq \alpha\},
\end{aligned}
$$

$$
\begin{aligned}
[x]_{R_{\alpha^+}} &= \{y \in U : h_R(x, y) > \alpha\} \\
&= \{y \in U : \bigvee_{a\in A}(h_{f(a)}(x) \wedge h_{f(a)}(y)) > \alpha\}, \\
&= \{y \in U : \bigvee_{a\in A}(f(a)(x) \wedge f(a)(y)) > \alpha\}.
\end{aligned}
$$

Theorem 6: Let $S = (f, A)$ be a hesitant fuzzy soft set, R^S be the hesitant fuzzy binary relation induced by S. If R^S is a typical hesitant fuzzy equivalence relation on U, then ${}_tR$ is the fuzzy equivalence relation on U for all $t \in \mathbb{N}^+$, and ${}_tR_\beta$, ${}_tR_{\beta^+}$ ($t \in \mathbb{N}^+$) are the crisp equivalence relations on U for all $\beta \in [0, 1]$.

Proof: Since R^S is a typical hesitant fuzzy binary relation on U, then according to Theorem 5,

$$
h_{R^S} = \bigcup_{t=1,2,\cdots} {}_tR,
$$

where ${}_tR$ ($\forall t \in \mathbb{N}^+$) is a fuzzy binary relation. Since R^S is a hesitant fuzzy equivalence relation, then $\forall x, y, z \in U$, we have (i) R^S is reflexive, i.e., $h_{R^S}(x, x) = 1$, it implies that ${}_tR(x, x) = 1$, $\forall t \in \mathbb{N}^+$; (ii) R^S is symmetric, i.e., $h_{R^S}(x, y) = h_{R^S}(y, x)$, thus, ${}_tR(x, y) =_t R(y, x), \forall t \in \mathbb{N}^+$; (iii) R^S is transitive, i.e., $h_{R^S}(x, y) \,\overline{\wedge}\, h_{R^S}(y, z) \preceq h_{R^S}(x, z)$, which implies that $\min\{h^+_{R^S}(x, y), h^+_{R^S}(y, z)\} \leq h^+_{R^S}(x, z)$ and $\min\{h^-_{R^S}(x, y), h^-_{R^S}(y, z)\} \leq h^-_{R^S}(x, z)$, thus, we obtain ${}_tR(x, y) \geqslant \min\{{}_tR(x, z),{}_t R(z, y)\}$, $\forall t \in \mathbb{N}^+$. Therefore, ${}_tR$ is the fuzzy equivalence relation on U for all $t \in \mathbb{N}^+$.

On the other hand, $\forall \beta \in [0, 1]$, for the β-level set and the strong β-level set of fuzzy binary relation ${}_tR$ ($t \in \mathbb{N}^+$), i.e.,

$$
\begin{aligned}
{}_tR_\beta &= \{(x, y) : {}_tR(x, y) \geq \beta\}, \\
{}_tR_{\beta^+} &= \{(x, y) : {}_tR(x, y) > \beta\},
\end{aligned}
$$

since ${}_tR$ is the fuzzy equivalence relation, ${}_tR_\beta$, ${}_tR_{\beta^+}$ are reflexive and symmetric are easily obtained. Since ${}_tR$ is transitive, i.e., ${}_tR(x, y) \geqslant \min\{{}_tR(x, z), {}_tR(z, y)\}$ for any $x, y, z \in U$, then for all $\beta \in [0, 1]$, if $(x, y) \in {}_tR_\beta$ and $(y, z) \in {}_tR_\beta$, i.e., ${}_tR_\beta(x, y) \geq \beta$ and ${}_tR_\beta(y, z) \geq \beta$, we have ${}_tR_\beta(x, z) \geq \beta$, i.e., $(x, z) \in {}_tR_\beta$, it implies that ${}_tR_\beta$ is transitive. Therefore, ${}_tR_\beta$ ($t \in \mathbb{N}^+$) is the crisp equivalence relation on U for all $\beta \in [0, 1]$.

Similarly, ${}_tR_{\beta^+}$ ($t \in \mathbb{N}^+$) is the crisp equivalence relation on U for all $\beta \in [0, 1]$.

Remark 6: $\forall t \in \mathbb{N}^+$, ${}_tR_\beta$ and ${}_tR_{\beta^+}$ construct a partition of the universe U for all $\beta \in [0, 1]$, respectively, i.e.,

$$
U/{}_tR_\beta = \{[x]_{{}_tR_\beta} : x \in U\}, \; U/{}_tR_{\beta^+} = \{[x]_{{}_tR_{\beta^+}} : x \in U\},
$$

where

$$
\begin{aligned}
[x]_{{}_tR_\beta} &= \{y \in U : {}_tR(x, y) \geq \beta\}, \\
[x]_{{}_tR_{\beta^+}} &= \{y \in U : {}_tR(x, y) > \beta\}.
\end{aligned}
$$

Theorem 7: Let $S = (f, A)$ be a full fuzzy soft set, R^S be the typical hesitant fuzzy binary relation induced by S.

Then for $E \in TH(U)$, we have

$$\begin{aligned}
\underline{Apr}_{HSF}(h_E) &= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \underline{{}_tR_{1-\beta}}({}_kH_\beta))\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \underline{{}_tR_{(1-\beta)^+}}({}_kH_\beta))\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \underline{{}_tR_{1-\beta}}({}_kH_{\beta^+}))\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \underline{{}_tR_{(1-\beta)^+}}({}_kH_{\beta^+})),
\end{aligned}$$

and

$$\begin{aligned}
\overline{Apr}_{HSF}(h_E) &= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \overline{{}_tR_{\beta}}({}_kH_\beta))\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \overline{{}_tR_{\beta^+}}({}_kH_\beta))\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \overline{{}_tR_{\beta}}({}_kH_{\beta^+}))\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \overline{{}_tR_{\beta^+}}({}_kH_{\beta^+})),
\end{aligned}$$

where $h_{R^S} = \bigcup_{t\in\mathbb{N}^+} {}_tR$, $h_E = \bigcup_{k\in\mathbb{N}^+} {}_kH$.

Proof: For any $x \in U$,

$$\begin{aligned}
&\bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \underline{{}_tR_{1-\beta}}({}_kH_\beta)(x))\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\sup\{\beta\in[0,1] : x \in \underline{{}_tR_{1-\beta}}({}_kH_\beta)\}\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\sup\{\beta\in[0,1] : [x]_{{}_tR_{1-\beta}} \subseteq {}_kH_\beta\}\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\sup\{\beta\in[0,1] : \textit{for all } y,\\
&\qquad y \in [x]_{{}_tR_{1-\beta}} \Longrightarrow y \in {}_kH_\beta\}\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\sup\{\beta\in[0,1] : \textit{for all } y,\\
&\qquad {}_tR(x,y) \geq 1-\beta \Longrightarrow y \in {}_kH_\beta\}\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\sup\{\beta\in[0,1] : \textit{for all } y,\\
&\qquad y \notin {}_kH_\beta \Longrightarrow {}_tR(x,y) < 1-\beta\}\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigwedge\{1 - {}_tR(x,y) : y \notin {}_kH_\beta\}\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigwedge\{{}_kH(y) \vee (1 - {}_tR(x,y)) : y \in U\}\\
&= \bigwedge\left\{\left(\bigcup_{k\in\mathbb{N}^+} {}_kH(y)\right) \vee \left(1-\left(\bigcup_{t\in\mathbb{N}^+} {}_tR(x,y)\right)\right) : y \in U\right\}\\
&= \overline{\bigwedge}_{y\in U}\left\{h_E(y) \,\underline{\vee}\, (1 - h_{R^S}(x,y))\right\}\\
&= \overline{\bigwedge}_{y\in U}\left\{h_E(y) \,\underline{\vee}\left(1 - \underline{\bigvee}_{a\in A}(h_{f(a)}(x) \,\overline{\wedge}\, h_{f(a)}(y))\right)\right\}\\
&= \overline{\bigwedge}_{a\in A}\left(h_{f^c(a)}(x) \,\underline{\vee}\left(\overline{\bigwedge}_{y\in U}(h_{f^c(a)}(y) \,\underline{\vee}\, h_E(y))\right)\right)\\
&= \underline{Apr}_{HSF}(h_E)(x).
\end{aligned}$$

Therefore, we have

$$\underline{Apr}_{HSF}(h_E) = \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \underline{{}_tR_{1-\beta}}({}_kH_\beta)).$$

For any $x \in U$,

$$\begin{aligned}
&\bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \overline{{}_tR_{\beta}}({}_kH_\beta)(x))\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\sup\{\beta\in[0,1] : x \in \overline{{}_tR_{\beta}}({}_kH_\beta)\}\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\sup\{\beta\in[0,1] : [x]_{{}_tR_\beta} \cap {}_kH_\beta \neq \emptyset\}\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\sup\{\beta\in[0,1] : \textit{there exists a } y \textit{ such that}\\
&\qquad y \in [x]_{{}_tR_\beta} \textit{ and } y \in {}_kH_\beta\}\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\sup\{\beta\in[0,1] : \textit{there exists a } y \textit{ such that}\\
&\qquad {}_tR(x,y) \geq \beta \textit{ and } {}_kH(y) \geq \beta\}\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\sup\{{}_tR(x,y) : y \in {}_kH_\beta\}\\
&= \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee\{{}_kH(y) \wedge {}_tR(x,y) : y \in U\}\\
&= \bigvee\left\{\left(\bigcup_{k\in\mathbb{N}^+} {}_kH(y)\right) \wedge \left(\bigcup_{t\in\mathbb{N}^+} {}_tR(x,y)\right) : y \in U\right\}\\
&= \underline{\bigvee}_{y\in U}\left\{h_E(y) \,\overline{\wedge}\, h_{R^S}(x,y)\right\}\\
&= \underline{\bigvee}_{y\in U}\left\{h_E(y) \,\overline{\wedge}\left(\underline{\bigvee}_{a\in A}(h_{f(a)}(x) \,\overline{\wedge}\, h_{f(a)}(y))\right)\right\}\\
&= \underline{\bigvee}_{a\in A}\left(h_{f(a)}(x) \,\overline{\wedge}\left(\underline{\bigvee}_{y\in U}(h_{f(a)}(y) \,\overline{\wedge}\, h_E(y))\right)\right)\\
&= \overline{Apr}_{HSF}(h_E)(x).
\end{aligned}$$

Therefore, we have

$$\overline{Apr}_{HSF}(h_E) = \bigcup_{k\in\mathbb{N}^+}\bigcup_{t\in\mathbb{N}^+}\bigvee_{\beta\in[0,1]}(\beta \wedge \overline{{}_tR_{\beta}}({}_kH_\beta)).$$

Similarly, the other equations can be proved.

Theorem 7 indicates that the lower approximation $\underline{Apr}_{HSF}(h_E)$ and upper approximation $\overline{Apr}_{HSF}(h_E)$ can be equivalently defined by using the (strong) level sets of a hesitant fuzzy soft set.

IV. APPLICATION OF HESITANT SOFT FUZZY ROUGH SETS BASED DECISION MAKING

In this section, we establish an approach to decision making problem based on the hesitant soft fuzzy rough set model proposed in this paper.

For decision making in an imprecise environment, some of these problems are essentially humanistic and thus subjective in nature (e.g. human understanding and vision systems);

there actually does not exist a unique or uniform criterion for evaluating the alternatives. Therefore, every existing decision approach could inevitably have their limitations and advantages more or less. In fact, all the existing approaches to decision making based on soft set and its extensions theory have solved kinds of decision problem effectively. In 2007, Roy and Maji [25] first give the decision method based on fuzzy soft set theory. In 2010, Feng *et al.* [7] analyzed the limitations of the decision method proposed by Roy and Maji in detail and established a novel approach by using the level soft sets to solve the fuzzy soft set based decision making problems. In 2014, Wang *et al.* [30] apply this level soft sets method to hesitant fuzzy soft set based decision making problems. In 2018, Zhang and He [40] investigated the hesitant fuzzy compatible rough set over two different universes and its application in hesitant fuzzy soft set based decision making, where for a given threshold value, a hesitant fuzzy soft set could be converted into a hesitant fuzzy compatible relation.

Though the Feng's et al. methods have overcome the limitations exiting in the Roy and Maji's method, but there need to chose the thresholds in advance by decision makers. By considering different types of thresholds, they can derive different level soft sets from the original fuzzy soft set. In general, the final optimal decisions based on different level soft sets could be different, that is, the results will be dependent to the threshold values to some extent. In the following, we establish a new approach to decision making based on hesitant soft fuzzy rough set theory. This approach will using the data information provided by the decision making problem only and does not need any additional available information provided by decision makers or other ways. Therefore, the final optimal decision could be more objective and also avoid the paradox results for the same decision problem induced by the effect of the subjective factors influenced by different experts.

Suppose that the universe $U = \{x_1, x_2, \cdots, x_m\}$ is an initial universe of objects, and $A = \{a_1, a_2, \cdots, a_n\}$ is a set of parameters. Let (f, A) be a hesitant fuzzy soft set over U. For a ceratin decision evaluation problem, one want to find out the decision alternative in universe with the evaluation value as larger as possible on every evaluate index. Thus, we first constructive an optimistic optimum normal decision object E on the evaluation universe U as follows:

$$E = \{\langle x, \{\max h_{f(a_1)(x)}, \max h_{f(a_2)(x)}, \cdots, \max h_{f(a_{|A|})(x)}\}\rangle | x \in U\},$$

where $|A|$ denotes the cardinality of the parameter set A. Similarly, pessimistic optimum normal decision object E can be constructed.

Secondly, calculate the hesitant soft fuzzy rough lower approximation $\underline{Apr}_{HSF}(h_E)$ and hesitant soft fuzzy rough upper approximation $\overline{Apr}_{HSF}(h_E)$ of the optimum normal decision object E by Definition 13. Since the rough lower approximation and upper approximation are two most close to the approximated set of the universe, we obtain two most close values $\underline{Apr}(h_E)(x_i)$ and $\overline{Apr}_{HSF}(h_E)(x_i)$ to the decision alternative $x_i \in U$. So we redefine the choice value c_i, which used by the existing decision making based fuzzy soft set, for the decision alternative x_i on the universe U as follows

$$c_i = s(\underline{Apr}(h_E)(x_i)) + s(\overline{Apr}_{HSF}(h_E)(x_i)), \ x_i \in U,$$

where $s(\cdot)$ denotes the score function of a hesitant fuzzy element. Finally, take the object $x_i \in U$ in universe U with the maximum choice value c_i as the optimum decision for the given decision making problem. If there exists two or more object $x_i \in U$ with the same maximum choice value c_i, then one of them can be chosen randomly as the optimum decision for the given decision making problem.

We present the decision algorithm as follows:

Step 1. Input the hesitant fuzzy soft set $S = (F, A)$.
Step 2. Compute the optimistic optimum normal decision object E.
Step 3. Compute the hesitant soft fuzzy rough lower approximation $\underline{Apr}_{HSF}h_E$ and hesitant soft fuzzy rough upper approximation $\overline{Apr}h_E$.
Step 4. Compute the choice value $c_i = s(\underline{Apr}(h_E)(x_i)) + s(\overline{Apr}_{HSF}(h_E)(x_i)), \ x_i \in U$.
Step 5. The decision is $x_k \in U$ if $c_k = \max c_i, \ i = 1, 2, \cdots, |U|$.
Step 6. If k has more than one value, then any one of x_k may be chosen.

To illustrate our method, let us consider the following example.

Example 8: Assume that a company wants to fill a position. Let $U = \{x_1, x_2, \cdots, x_5\}$ be a set of five candidates who apply for the position. Suppose that the set of candidates U can be characterized by a set of parameters $A = \{a_1, a_2, a_3, a_4\}$, where $a_j (j = 1, 2, 3, 4)$ stands for "computer knowledge", "higher education", "skilled foreign", "languages" and "experience", respectively. Now suppose that the company organizes two experts to evaluate five candidates under four parameters. In that case, the characteristics of five candidates under four parameters are represented by a hesitant fuzzy soft set. Its tabular representation is shown in Table 3 which is cited from [39].

TABLE 3. Tabular representation of a hesitant fuzzy soft set.

A/U	x_1	x_2	x_3	x_4	x_5
a_1	0.3,0.6	0.4,0.5	0.2,0.4	0.8,0.9	0.1,0.2
a_2	0.7,0.8	0.1,0.2	0.1,0.3	0.7,0.8	0.6,0.9
a_3	0.2,0.3	0.1,0.5	0.5,0.8	0.4,0.5	0.1,0.4
a_4	0.8,0.9	0.4,0.5	0.4,0.4	0.1,0.3	0.3,0.7

Then we can obtain the optimistic optimum normal decision object E as follows:

$$\begin{aligned} E = \{&\langle x_1, \{0.3, 0.6, 0.8, 0.9\}\rangle, \langle x_2, \{0.2, 0.5\}\rangle, \\ &\langle x_3, \{0.3, 0.4, 0.8\}\rangle, \langle x_4, \{0.3, 0.5, 0.8, 0.9\}\rangle, \\ &\langle x_5, \{0.2, 0.4, 0.7, 0.9\}\rangle\}. \end{aligned}$$

According to Definition 13, we have

$$\begin{aligned}
&\underline{Apr}_{HSF}(h_E)\\
&\quad = \{\langle x_1, \{0.2, 0.3, 0.4, 0.5, 0.6\}\rangle, \langle x_2, \{0.5, 0.6\}\rangle,\\
&\quad \langle x_3, \{0.3, 0.4, 0.5, 0.6\}\rangle, \langle x_4, \{0.2, 0.3, 0.4, 0.5, 0.6\}\rangle,\\
&\quad \langle x_5, \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7\}\rangle\},\\
&\overline{Apr}_{HSF}(h_E)\\
&\quad = \{\langle x_1, \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}\rangle,\\
&\quad \langle x_2, \{0.3, 0.4, 0.5\}\rangle,\\
&\quad \langle x_3, \{0.3, 0.4, 0.5, 0.8\}\rangle,\\
&\quad \langle x_4, \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}\rangle,\\
&\quad \langle x_5, \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}\rangle\}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
c_1 &= s(\underline{Apr}_{HSF}(h_E)(x_1)) + s(\overline{Apr}_{HSF}(h_E)(x_1))\\
&= 0.4 + 0.6 = 1.0,\\
c_2 &= s(\underline{Apr}_{HSF}(h_E)(x_2)) + s(\overline{Apr}_{HSF}(h_E)(x_2))\\
&= 0.55 + 0.4 = 0.95,\\
c_3 &= s(\underline{Apr}_{HSF}(h_E)(x_3)) + s(\overline{Apr}_{HSF}(h_E)(x_3))\\
&= 0.45 + 0.5 = 0.95,\\
c_4 &= s(\underline{Apr}_{HSF}(h_E)(x_4)) + s(\overline{Apr}_{HSF}(h_E)(x_4))\\
&= 0.4 + 0.6 = 1.0,\\
c_5 &= s(\underline{Apr}_{HSF}(h_E)(x_5)) + s(\overline{Apr}_{HSF}(h_E)(x_5))\\
&= 0.45 + 0.6 = 1.05.
\end{aligned}$$

From the above results, it is easy to see that the maximum choice value is $c_5 = 1.05$, scored by x_5, and then x_5 is the most suitable candidate for the position.

V. CONCLUSION

In this paper, we construct a hesitant soft fuzzy rough set model and investigate basic properties in detail. Based on the decomposition theorem for a hesitant fuzzy binary relation, which states that every typical hesitant fuzzy binary relation on a set can be represented by a well-structured family of fuzzy binary relations on that set, we give the relationship between hesitant fuzzy rough sets and hesitant soft fuzzy rough sets. The characterization theorem for the hesitant soft fuzzy rough set model is also given. Finally, we develop a decision making approach to a hesitant fuzzy soft set by using the new model and use a numerical example to illustrate the validity.

We believe that the new model will extend the application scope of rough set theory and help us to gain new insights into the mathematical structures of fuzzy sets, soft sets, rough sets and hesitant fuzzy sets. Our further research is to extend this model to multi-granulation and present an axiomatic characterization.

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