Chapter 8

Applications to Nash Equilibrium

Contents

Look up at the stars and not down at your feet.

Stephen Hawking (1942–2018)

Chapter points

- The result of this approach is to produce a rigorous mathematical analysis for models at the interplay between Nash equilibria and mathematical physics.
- The arguments combine refined analytic tools, variational analysis, fixed point theory, and iterative methods.
- Applications include periodic problems with variational or nonvariational structure as well as problems driven by singular operators.

8.1 INTRODUCTION

Many problems describing models in the real world can be reduced to fixed point problems of the type

$$u = N(u),$$

where N is a nonlinear operator.

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In many cases, the problem has a variational structure, namely it is equivalent to finding the critical points of the associated “energy” functional E, that is $E'(u) = 0$. Thus, the fixed points of the operator N appear as critical points of the functional E. The critical points could be minima, maxima or saddle points, conferring to the fixed points a variational property. Thus, it makes sense to ask whether a fixed point of N is a minimum, or a maximum or a saddle point of E. Problems of this type become more interesting in the case of a system

$$\begin{cases} u = N_1(u, v) \\ v = N_2(u, v), \end{cases} \tag{8.1}$$

which does not have a variational form, but each of its component equations has a variational structure. More precisely, there exist functionals E_1 and E_2 such that the system (8.1) is equivalent to

$$\begin{cases} E_{11}(u, v) = 0 \\ E_{22}(u, v) = 0, \end{cases}$$

where $E_{11}(u, v)$ is the partial derivative of E_1 with respect to u, and $E_{22}(u, v)$ is the partial derivative of E_2 with respect to v.

A nontrivial problem is to see how the fixed points (u, v) of the operator $N = (N_1, N_2)$ are connected to the variational properties of the two functionals. One possible situation, which fits to physical principles, is that a fixed point (u^*, v^*) of N is a Nash-type equilibrium of the functionals E_1, E_2, that is,

$$E_1\left(u^*, v^*\right) = \min_u E_1\left(u, v^*\right)$$
$$E_2\left(u^*, v^*\right) = \min_v E_2\left(u^*, v\right).$$

(Note that the relations above correspond to a symmetric form of (2.11) in Chapter 2, for the particular case of $n = 2$, in the sense that “min” is taken instead of “max”.)

In the next section, we will focus on this problem in relationship with the Nash equilibrium for Perov contractions. An iterative scheme for finding a Nash-type equilibrium is introduced and its convergence is studied. The result is illustrated with an application to periodic solutions for a second-order differential system, which describes the oscillations of two pendulums.

We first establish a minimum property for classical contractions in the abstract setting of Hilbert space. We start with the case of contractions on the whole space.

Theorem 8.1. *Let X be a Hilbert space and $N : X \to X$ be a contraction with the unique fixed point u^*. Assume that there exists a C^1-functional E bounded from below such that*

$$E'(u) = u - N(u) \quad \text{for all } u \in X. \tag{8.2}$$

Then u^* minimizes the functional E, that is,

$$E\left(u^*\right) = \inf_X E.$$

Proof. By the Bishop-Phelps theorem (Theorem A.4 in Appendix A), there is a sequence (u_n) with

$$E(u_n) \to \inf_X E \quad \text{and} \quad E'(u_n) \to 0. \tag{8.3}$$

Let $v_n := E'(u_n) = u_n - N(u_n)$. We have $v_n \to 0$ and

$$\begin{aligned} \left|u_{n+p} - u_n\right| &\le \left|N\left(u_{n+p}\right) - N(u_n)\right| + \left|v_{n+p} - v_n\right| \\ &\le a\left|u_{n+p} - u_n\right| + \left|v_{n+p} - v_n\right|. \end{aligned}$$

Here, $a \in [0, 1)$ is the contraction constant of N. Hence

$$\left|u_{n+p} - u_n\right| \le \frac{1}{1-a}\left|v_{n+p} - v_n\right|.$$

Since (v_n) is a convergent sequence, this implies that (u_n) is Cauchy, too. It follows that $u_n \to \overline{u}$ for some $\overline{u}$. Now relation (8.3) yields

$$E(\overline{u}) = \inf_X E \quad \text{and} \quad E'(\overline{u}) = 0.$$

Relation $E'(\overline{u}) = 0$ shows that $\overline{u}$ is a fixed point of N, and since N has a unique fixed point, $\overline{u} = u^*$.

An analogue result holds for contractions on a ball $\overline{B}_R = \{u \in X : |u| \le R\}$ of the Hilbert space X. ☐

Theorem 8.2. *Let X be a Hilbert space and $N : \overline{B}_R \to X$ be a contraction satisfying the Leray-Schauder condition*

$$u \ne \lambda N(u) \quad \text{for } |u| = R \text{ and } \lambda \in (0, 1). \tag{8.4}$$

Let u^* denote the unique fixed point of N (guaranteed by the nonlinear alternative). Assume that there exists a C^1-functional E bounded from below on $\overline{B}_R$ such that

$$E'(u) = u - N(u) \quad \text{for all } u \in \overline{B}_R.$$

Then u^* minimizes the functional E on $\overline{B}_R$, that is,

$$E\left(u^*\right) = \inf_{\overline{B}_R} E.$$

Proof. As a consequence of Schechter's critical point theorem in a ball (Theorem A.5 in Appendix A), there is a sequence (u_n) of elements from $\overline{B}_R$, with

$$E(u_n) \to \inf_{\overline{B}_R} E \quad \text{and}$$

$$\text{either} \quad E'(u_n) \to 0, \quad \text{or}$$

$$E'(u_n) - \frac{\left(E'(u_n), u_n\right)}{R^2} u_n \to 0, \quad |u_n| = R, \quad \left(E'(u_n), u_n\right) \le 0.$$

In the first case, when $E'(u_n) \to 0$, we repeat the argument developed in the proof of Theorem 8.1.

In the second case, since $\left(E'(u_n), u_n\right) = R^2 - (N(u_n), u_n)$ and N is bounded as a contraction, we may pass to a subsequence in order to assume the convergence

$$\mu_n := -\frac{\left(E'(u_n), u_n\right)}{R^2} \to \mu \ge 0.$$

Furthermore, if $v_n := E'(u_n) + \mu_n u_n$, then

$$(1+\mu) u_n = v_n + N(u_n) + z_n$$

with $z_n = (\mu - \mu_n) u_n$. Therefore $z_n \to 0$. Next, we observe that

$$(1+\mu) \left|u_{n+p} - u_n\right| \le \left|v_{n+p} - v_n\right| + a \left|u_{n+p} - u_n\right| + \left|z_{n+p} - z_n\right|$$

and so

$$\left|u_{n+p} - u_n\right| \le \frac{1}{1+\mu-a} \left(\left|v_{n+p} - v_n\right| + \left|z_{n+p} - z_n\right|\right).$$

This implies that (u_n) is Cauchy. Let $\overline{u}$ be its limit. Then

$$E(\overline{u}) = \inf_{\overline{B}_R} E \quad \text{and} \quad E'(\overline{u}) + \mu \overline{u} = 0,$$

where $|\overline{u}| = R$ and $\mu \ge 0$. We claim that the case $\mu > 0$ is not possible. Indeed, if we assume that $\mu > 0$, then from $\overline{u} - N(\overline{u}) + \mu \overline{u} = 0$ we would have $\overline{u} = \frac{1}{1+\mu} N(\overline{u})$ which has been excluded by the Leray-Schauder condition (8.4). Hence $\mu = 0$, $E'(\overline{u}) = 0$, that is, $\overline{u} = N(\overline{u})$. Again the uniqueness of the fixed point guarantees $\overline{u} = u^*$. □

We recall that a sequence (u_n) with $(E(u_n))$ converging and $E'(u_n) \to 0$ is called a *Palais-Smale sequence*, while the property of a functional of existing a convergent subsequence for each Palais-Smale sequence, is named the *Palais-Smale condition*. Thus, Theorem 8.1 asserts that if E' is represented by (8.2), then E satisfies even more than the Palais-Smale condition, in the sense that the Palais-Smale sequences are entirely (not only some of their subsequences) convergent.

We point out that if in Theorem 8.2, the operator N is assumed to be more general condensing, then the minimizing sequence (u_n) has a subsequence converging to the absolute minimum of E on $\overline{B}_R$. Indeed, if (u_n) satisfies $E'(u_n) \to 0$, then using a measure α of noncompactness with respect to whom N is condensing, we find

$$\begin{aligned} \alpha(\{u_n\}) &= \alpha\left(\left\{E'(u_n) + N(u_n)\right\}\right) \\ &\leq \alpha\left(\left\{E'(u_n)\right\}\right) + \alpha(\{N(u_n)\}) \\ &= \alpha(\{N(u_n)\}). \end{aligned} \tag{8.5}$$

If $\{u_n\}$ is not relatively compact, that is, $\alpha(\{u_n\}) > 0$, then by the condensing property, $\alpha(\{N(u_n)\}) < \alpha(\{u_n\})$, which in view of (8.5) yields the contradiction $\alpha(\{u_n\}) < \alpha(\{u_n\})$. Hence $\{u_n\}$ is relatively compact, as desired.

If we focus on critical points of a functional E, and not on the fixed points of an operator N, then we can state the following more general result.

Theorem 8.3. *Let X be a Banach space with norm $|.|$ and E be a C^1-functional bounded from below with E' strongly monotone, that is*

$$\left(E'(u) - E'(v), u - v\right) \geq a|u - v|^2 \quad \textit{for all } u, v \in X,$$

and some $a > 0$. Then there exists $u^* \in X$ with

$$E\left(u^*\right) = \inf_X E \quad \textit{and} \quad E'\left(u^*\right) = 0.$$

Proof. As in the proof of Theorem 8.1, let (u_n) be such that

$$E(u_n) \to \inf_X E \quad \text{and} \quad E'(u_n) \to 0.$$

Denote $v_n := E'(u_n)$. We have $v_n \to 0$ in X', and

$$\begin{aligned} a\left|u_{n+p} - u_n\right|^2 &\leq \left(E'\left(u_{n+p}\right) - E'(u_n), u_{n+p} - u_n\right) \\ &= \left(v_{n+p} - v_n, u_{n+p} - u_n\right) \leq \left|v_{n+p} - v_n\right| \left|u_{n+p} - u_n\right|. \end{aligned}$$

It follows that

$$\left|u_{n+p} - u_n\right| \leq \frac{1}{a}\left|v_{n+p} - v_n\right|$$

and the assertion follows as above. □

Similarly, we have the following generalization of Theorem 8.2.

Theorem 8.4. *Let X be a Hilbert space and E be a C^1 functional such that E' is strongly monotone on $\overline{B}_R$ and*

$$E'(u) + \mu u \neq 0 \quad \textit{for } |u| = R \textit{ and } \mu > 0.$$

Then there exists u^* with

$$E\left(u^*\right)=\inf_{\overline{B}_R} E \quad \textit{and} \quad E'\left(u^*\right)=0.$$

Proof. With the notations from the proof of Theorem 8.2, we have

$$\begin{aligned}
&a\left|u_{n+p}-u_n\right|^2 \\
&\leq \left(E'\left(u_{n+p}\right)-E'\left(u_n\right), u_{n+p}-u_n\right) \\
&=\left(v_{n+p}-v_n, u_{n+p}-u_n\right)-\mu\left(u_{n+p}-u_n, u_{n+p}-u_n\right) \\
&\quad+\left(\mu-\mu_{n+p}\right)\left(u_{n+p}, u_{n+p}-u_n\right)-\left(\mu-\mu_n\right)\left(u_n, u_{n+p}-u_n\right).
\end{aligned}$$

Hence

$$(a+\mu)\left|u_{n+p}-u_n\right| \leq \left|v_{n+p}-v_n\right|+R\left(\left|\mu-\mu_{n+p}\right|+\left|\mu-\mu_n\right|\right),$$

which implies that (u_n) is a Cauchy sequence. □

8.2 NASH EQUILIBRIUM FOR PEROV CONTRACTIONS

Let $(X_i, |\,.\,|_i)$, $i=1,2$, be Hilbert spaces identified to their dual spaces and let $X=X_1\times X_2$. Consider the system

$$\begin{cases} u=N_1(u,v) \\ v=N_2(u,v) \end{cases}$$

where $(u,v)\in X$. Assume that each equation of the system has a variational form, that is, there exist continuous functionals $E_1, E_2 : X \to \mathbb{R}$ such that $E_1(.,v)$ is Fréchet differentiable for every $v\in X_2$, $E_2(u,.)$ is Fréchet differentiable for every $u\in X_1$, and

$$\begin{aligned}
E_{11}(u,v)&=u-N_1(u,v) \\
E_{22}(u,v)&=v-N_2(u,v).
\end{aligned} \tag{8.6}$$

As in the previous section, $E_{11}(.,v)$, $E_{22}(u,.)$ denote the Fréchet derivatives of $E_1(.,v)$ and $E_2(u,.)$, respectively.

We say that the operator $N : X \to X$, $N(u,v)=(N_1(u,v), N_2(u,v))$ is a *Perov contraction* if there exists a matrix $M=\left[m_{ij}\right]\in\mathcal{M}_{2,2}(\mathbb{R}_+)$ such that M^n tends to the zero matrix 0, and the following matricial Lipschitz condition is satisfied

$$\begin{bmatrix} |N_1(u,v)-N_1(\overline{u},\overline{v})|_1 \\ |N_2(u,v)-N_2(\overline{u},\overline{v})|_2 \end{bmatrix} \leq M \begin{bmatrix} |u-\overline{u}|_1 \\ |v-\overline{v}|_2 \end{bmatrix} \tag{8.7}$$

for every $u,\overline{u}\in X_1$ and $v,\overline{v}\in X_2$.

Notice that the property $M^n \to 0$ is equivalent to $\rho(M) < 1$, where $\rho(M)$ is the spectral radius of matrix M, and also to the fact that $I - M$ is nonsingular and all the elements of the matrix $(I-M)^{-1}$ are nonnegative (see Precup [143], [144]).

Theorem 8.5. *Assume that the above conditions are satisfied. In addition, we assume that $E_1(., v)$ and $E_2(u, .)$ are bounded from below for every $u \in X_1$, $v \in X_2$, and that there exist positive numbers R and a such that one of the following conditions holds:*

$$\begin{aligned} &\text{either } E_1(u,v) \geq \inf_{X_1} E_1(., v) + a \text{ for } |u|_1 \geq R \text{ and all } v \in X_2, \\ &\text{or } \ E_2(u,v) \geq \inf_{X_2} E_2(u, .) + a \text{ for } |v|_2 \geq R \text{ and all } u \in X_1. \end{aligned} \tag{8.8}$$

Then the unique fixed point (u^*, v^*) of N (guaranteed by Perov's fixed point theorem) is a Nash-type equilibrium of the pair of functionals (E_1, E_2), that is,

$$\begin{aligned} E_1\left(u^*, v^*\right) &= \inf_{X_1} E_1\left(., v^*\right) \\ E_2\left(u^*, v^*\right) &= \inf_{X_2} E_2\left(u^*, .\right). \end{aligned}$$

Proof. Assume that relation (8.8) holds for E_2. We shall construct recursively two sequences (u_n), (v_n), based on the Bishop-Phelps theorem. Let v_0 be any element of X_2. At any step n $(n \geq 1)$ we may find a $u_n \in X_1$ and a $v_n \in X_2$ such that

$$E_1(u_n, v_{n-1}) \leq \inf_{X_1} E_1(., v_{n-1}) + \frac{1}{n}, \quad |E_{11}(u_n, v_{n-1})|_1 \leq \frac{1}{n} \tag{8.9}$$

and

$$E_2(u_n, v_n) \leq \inf_{X_2} E_2(u_n, .) + \frac{1}{n}, \quad |E_{22}(u_n, v_n)|_2 \leq \frac{1}{n}. \tag{8.10}$$

For $\frac{1}{n} < a$, from (8.8) and (8.10) we have $|v_n|_2 < R$. Hence the sequence (v_n) is bounded. Let $\alpha_n := E_{11}(u_n, v_{n-1})$ and $\beta_n := E_{22}(u_n, v_n)$. Clearly $\alpha_n, \beta_n \to 0$. Also

$$\begin{aligned} u_n - N_1(u_n, v_{n-1}) &= \alpha_n \\ v_n - N_2(u_n, v_n) &= \beta_n. \end{aligned}$$

It follows that

$$\begin{aligned} \left|u_{n+p} - u_n\right|_1 &\leq \left|N_1\left(u_{n+p}, v_{n+p-1}\right) - N_1(u_n, v_{n-1})\right|_1 + \left|\alpha_{n+p} - \alpha_n\right|_1 \\ &\leq m_{11}\left|u_{n+p} - u_n\right|_1 + m_{12}\left|v_{n+p-1} - v_{n-1}\right|_2 + \left|\alpha_{n+p} - \alpha_n\right|_1 \end{aligned}$$

$$\leq m_{11}\left|u_{n+p}-u_n\right|_1+m_{12}\left|v_{n+p}-v_n\right|_2+\left|\alpha_{n+p}-\alpha_n\right|_1$$
$$+m_{12}\left(\left|v_{n+p-1}-v_{n-1}\right|_2-\left|v_{n+p}-v_n\right|_2\right).$$

Denote $a_{n,p}=\left|u_{n+p}-u_n\right|_1$, $b_{n,p}=\left|v_{n+p}-v_n\right|_2$, $c_{n,p}=\left|\alpha_{n+p}-\alpha_n\right|_1$, $d_{n,p}=\left|\beta_{n+p}-\beta_n\right|_2$. Then

$$a_{n,p}\leq m_{11}a_{n,p}+m_{12}b_{n,p}+c_{n,p}+m_{12}\left(b_{n-1,p}-b_{n,p}\right). \tag{8.11}$$

Similarly, we deduce that

$$b_{n,p}\leq m_{21}a_{n,p}+m_{22}b_{n,p}+d_{n,p}.$$

Hence

$$\begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix}\leq M\begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix}+\begin{bmatrix} c_{n,p}+m_{12}\left(b_{n-1,p}-b_{n,p}\right) \\ d_{n,p} \end{bmatrix}.$$

Consequently, since $I-M$ is invertible and its inverse contains only nonnegative elements, we may write

$$\begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix}\leq (I-M)^{-1}\begin{bmatrix} c_{n,p}+m_{12}\left(b_{n-1,p}-b_{n,p}\right) \\ d_{n,p} \end{bmatrix}.$$

Let $(I-M)^{-1}=\left[\gamma_{ij}\right]$. Then

$$\begin{aligned} a_{n,p}&\leq \gamma_{11}\left(c_{n,p}+m_{12}\left(b_{n-1,p}-b_{n,p}\right)\right)+\gamma_{12}d_{n,p} \\ b_{n,p}&\leq \gamma_{21}\left(c_{n,p}+m_{12}\left(b_{n-1,p}-b_{n,p}\right)\right)+\gamma_{22}d_{n,p}. \end{aligned} \tag{8.12}$$

From the second inequality, we deduce that

$$b_{n,p}\leq \frac{\gamma_{21}m_{12}}{1+\gamma_{21}m_{12}}b_{n-1,p}+\frac{\gamma_{21}c_{n,p}+\gamma_{22}d_{n,p}}{1+\gamma_{21}m_{12}}. \tag{8.13}$$

We observe that $\left(b_{n,p}\right)$ is bounded uniformly with respect to p. Lemma 8.1 shows that $b_{n,p}\to 0$ uniformly for $p\in\mathbb{N}$, and hence (v_n) is a Cauchy sequence. Next, the first inequality in (8.12) implies that (u_n) is also Cauchy. Let u^*, v^* be the limits of (u_n), (v_n), respectively. The conclusion now follows if we pass to the limit in (8.9) and (8.10).

In case that E_1 satisfies (8.8), we interchange E_1, E_2 in the construction of the two sequences, more exactly we obtain

$$E_2\left(u_{n-1},v_n\right)\leq \inf_{X_2}E_2\left(u_{n-1},.\right)+\frac{1}{n},\quad \left|E_{22}\left(u_{n-1},v_n\right)\right|_2\leq\frac{1}{n} \tag{8.14}$$

and

$$E_1(u_n, v_n) \leq \inf_{X_1} E_1(., v_n) + \frac{1}{n}, \quad |E_{11}(u_n, v_n)|_1 \leq \frac{1}{n}. \tag{8.15}$$

This completes the proof. □

The following elementary result is frequently used to argue the convergence of the iterative schemas.

Lemma 8.1. *Let $(x_{n,p})$, $(y_{n,p})$ be two sequences of real numbers depending on a parameter p, such that*

$$(x_{n,p}) \text{ is bounded uniformly with respect to } p,$$

and

$$0 \leq x_{n,p} \leq \lambda x_{n-1,p} + y_{n,p} \text{ for all } n, p \text{ and some } \lambda \in [0, 1). \tag{8.16}$$

If $y_{n,p} \to 0$ uniformly with respect to p, then $x_{n,p} \to 0$ uniformly with respect to p too.

Proof. Let $\varepsilon > 0$ be any number. Since $y_{n,p} \to 0$ uniformly with respect to p, there exists n_1 (not depending on p) such that $y_{n,p} \leq \varepsilon$ for all $n \geq n_1$. From $x_{n,p} \leq \lambda x_{n-1,p} + \varepsilon$ $(n \geq n_1)$, we deduce that

$$x_{n,p} \leq \lambda^{n-n_1} x_{n_1} + \varepsilon \left(\lambda + \lambda^2 + ... + \lambda^{n-n_1} \right) \leq \lambda^{n-n_1} c + \varepsilon \frac{\lambda}{1-\lambda},$$

where c is a bound for $x_{n,p}$. This yields $x_{n,p} \to 0$, uniformly in p. □

Remark 8.1. If instead of condition (8.8) we assume that there exist convergent subsequences (u_{n_j}), $(v_{n_j})$of the sequences (u_n), (v_n) given by (8.9) and (8.10), then the conclusion of Theorem 8.5 remains true. To prove this, we first show that the sequence $(b_{n_j-1,1})$ defined by $b_{n_j-1,1} = \left| v_{n_j} - v_{n_j-1} \right|_2$ is bounded. Indeed, from (8.13), we obtain that

$$b_{n_j-1,1} \leq \lambda b_{n_j-2,1} + (1-\lambda)^2 \quad (j \geq j_1).$$

This yields

$$b_{n_j-1,1} \leq \lambda^{n_j - n_{j-1}} b_{n_{j-1}-1,1} + 1 - \lambda \quad (j \geq j_2),$$

whence

$$b_{n_j-1,1} - 1 \leq \lambda \left(b_{n_{j-1}-1,1} - 1 \right) \quad (j \geq j_2). \tag{8.17}$$

Denote $z_j = b_{n_j-1,1}$. Notice that the case $z_j > 1$ for all $j \geq j_2$ is not possible. Since otherwise $0 \leq z_j - 1 \leq \lambda^{j-j_2} \left(z_{j_2} - 1 \right)$, whence $z_j \to 1$. However, by (8.16), this would imply the contradiction $1 \leq 1 - \lambda$. Therefore, there exists

$j_3 \geq j_2$ with $z_{j_3} \leq 1$. Then (8.17) implies that $z_j \leq 1$ for all $j \geq j_3$, hence $\left(z_j\right)$ is bounded, as claimed.

Next, from Lemma 8.1, applied for $p = 1$ and $x_{j,1} := b_{n_j - 1,1}$, we find that $b_{n_j-1,1} \to 0$ as $j \to \infty$. Hence the sequence $\left(v_{n_j - 1}\right)$ is convergent to the limit v^* of $\left(v_{n_j}\right)$. The conclusion follows if we let $j \to \infty$ in (8.9), (8.10) with $n = n_j$.

An analogue result holds for Perov contractions on the Cartesian product $\overline{B}_{R_1} \times \overline{B}_{R_2}$ of two balls of X_1 and X_2.

Theorem 8.6. *Let $N : \overline{B}_{R_1} \times \overline{B}_{R_2} \to X$, $N = (N_1, N_2)$ be a Perov generalized contraction, that is, relation (8.7) is satisfied for $u, \overline{u} \in \overline{B}_{R_1}$ and $v, \overline{v} \in \overline{B}_{R_2}$. Assume that for every $\lambda \in (0,1)$,*

$$
\begin{aligned}
u &\neq \lambda N_1(u, v) \quad \text{if} \quad |u|_1 = R_1, \; v \in \overline{B}_{R_2}, \\
v &\neq \lambda N_2(u, v) \quad \text{if} \quad |v|_2 = R_2, \; u \in \overline{B}_{R_1}.
\end{aligned}
$$

In addition, we assume that the representation (8.6) holds on $\overline{B}_{R_1} \times \overline{B}_{R_2}$ for two continuous functionals $E_1, E_2 : X \to \mathbb{R}$ such that $E_1(., v)$ is Fréchet differentiable for every $v \in X_2$, $E_2(u, .)$ is Fréchet differentiable for every $u \in X_1$, and that $E_1(., v)$, $E_2(u, .)$ are bounded from below on $\overline{B}_{R_1}$ and $\overline{B}_{R_2}$ respectively, for every $u \in \overline{B}_{R_1}$, $v \in \overline{B}_{R_2}$. Then the unique fixed point $(u^*, v^*) \in \overline{B}_{R_1} \times \overline{B}_{R_2}$ of N (guaranteed by the nonlinear alternative for Perov contractions, see Precup [143]) is a Nash-type equilibrium in $\overline{B}_{R_1} \times \overline{B}_{R_2}$ of the pair of functionals (E_1, E_2), that is,

$$
\begin{aligned}
E_1\left(u^*, v^*\right) &= \inf_{\overline{B}_{R_1}} E_1\left(., v^*\right) \\
E_2\left(u^*, v^*\right) &= \inf_{\overline{B}_{R_2}} E_2\left(u^*, .\right).
\end{aligned}
$$

The proof combines arguments from the proofs of Theorems 8.2 and 8.5.

8.2.1 Application: Oscillations of Two Pendulums

Consider the following periodic problem

$$
\begin{aligned}
u''(t) &= \nabla_x F(t, u(t), v(t)) \quad \text{a.e. on } (0,T) \\
v''(t) &= \nabla_y G(t, u(t), v(t)) \quad \text{a.e. on } (0,T) \\
u(0) - u(T) &= u'(0) - u'(T) = 0 \\
v(0) - v(T) &= v'(0) - v'(T) = 0,
\end{aligned}
\tag{8.18}
$$

where $F, G : (0,T) \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}$.

All functions of the type $H : (0,T) \times \mathbb{R}^n \to \mathbb{R}^m$, $H = H(t, x)$ $(n, m \geq 1)$, including F, G, $\nabla_x F$, and $\nabla_y G$, will be assumed to be L^1-Carathéodory, namely $H(., x)$ is measurable for each $x \in \mathbb{R}^n$, $H(t, .)$ is continuous for a.e. $t \in (0,T)$, and such that for each $R > 0$, there exists $b_R \in L^1(0, T; \mathbb{R}_+)$ with $|H(t, x)| \leq b_R(t)$ for a.e. $t \in (0, T)$ and all $x \in \mathbb{R}^n$, $|x| \leq R$.

In our case, the system does not have a variational structure, but it splits into two subsystems having each one a variational form.

A pair $(u, v) \in H^1_p(0, T; \mathbb{R}^{k_1}) \times H^1_p(0, T; \mathbb{R}^{k_2})$ is a solution of problem (8.18) if and only if

$$E_{11}(u, v) = 0, \quad E_{22}(u, v) = 0,$$

where $E_1, E_2 : H^1_p(0, T; \mathbb{R}^{k_1}) \times H^1_p(0, T; \mathbb{R}^{k_2}) \to \mathbb{R}$,

$$\begin{aligned} E_1(u, v) &= \int_0^T \left(\frac{1}{2} |u'|^2 + F(t, u(t), v(t)) \right) dt \\ E_2(u, v) &= \int_0^T \left(\frac{1}{2} |v'|^2 + G(t, u(t), v(t)) \right) dt. \end{aligned} \tag{8.19}$$

Here, $H^1_p(0, T; \mathbb{R}^k)$ is the space of functions of the form

$$u(t) = \int_0^t v(s)ds + c,$$

with $u(0) = u(T)$, $c \in \mathbb{R}^k$, and $v \in L^2(0, T; \mathbb{R}^k)$.

We define a scalar product in $H^1_p(0, T; \mathbb{R}^{k_i})$ $(i = 1, 2)$ by

$$(u, v)_i = \int_0^T \left[(u'(t), v'(t)) + m_i^2 (u(t), v(t)) \right] dt,$$

where $m_i \neq 0$. The corresponding norm is

$$|u|_i = \left(\int_0^T \left(|u'(t)|^2 + m_i^2 |u(t)|^2 \right) dt \right)^{1/2}.$$

We identify the dual $\left(H^1_p(0, T; \mathbb{R}^{k_i})\right)'$ to $H^1_p(0, T; \mathbb{R}^{k_i})$ via the mapping J_i defined by

$$\left(H^1_p\left(0, T; \mathbb{R}^{k_i}\right)\right)' \ni h \mapsto J_i h = u,$$

the unique weak solution of the problem

$$\begin{gathered} -u'' + m_i^2 u = h \quad \text{a.e. on } (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{gathered}$$

Then

$$E_{11}(u, v) = u - J_1\left(m_1^2 u - \nabla_x F(., u, v)\right),$$

$$E_{22}(u,v) = v - J_2\left(m_2^2 v - \nabla_y G(.,u,v)\right).$$

Hence

$$N_1(u,v) = J_1\left(m_1^2 u - \nabla_x F(.,u,v)\right), \quad N_2(u,v) = J_2\left(m_2^2 v - \nabla_y G(.,u,v)\right).$$

In what follows, we use the obvious inequality

$$|u|_{L^2} \le \frac{1}{m_i}|u|_i \quad \left(u \in H_p^1\left(0,T;\mathbb{R}^{k_i}\right)\right) \tag{8.20}$$

and the estimation of the norm of J_i, as linear operator from $L^2\left(0,T;\mathbb{R}^{k_i}\right)$ to $H_p^1\left(0,T;\mathbb{R}^{k_i}\right)$. To obtain this, we start with the definition of the operator J_i, which gives

$$|J_i h|_i^2 = (J_i h, J_i h)_i = (h, J_i h)_{L^2} \le |h|_{L^2}|J_i h|_{L^2} \le \frac{1}{m_i}|h|_{L^2}|J_i h|_i.$$

Hence

$$|J_i h|_i \le \frac{1}{m_i}|h|_{L^2} \quad \left(h \in L^2\left(0,T;\mathbb{R}^{k_i}\right)\right). \tag{8.21}$$

We say that a function $H:(0,T)\times\mathbb{R}^k \to R$ is of *coercive type* if the functional $E: H_p^1\left(0,T;\mathbb{R}^k\right) \to \mathbb{R}$,

$$E(u) = \int_0^T \left(\frac{1}{2}\left|u'(t)\right|^2 + H(t,u(t))\right)dt \tag{8.22}$$

is coercive, that is, $E(u) \to +\infty$ as $|u| \to \infty$. Here we have denoted

$$|u| = \left(\int_0^T \left(\left|u'(t)\right|^2 + |u(t)|^2\right)dt\right)^{1/2}.$$

Lemma 8.2. *Assume that for some $\gamma \in \mathbb{R}\setminus\{0\}$, $\nabla\left(H - \gamma^2|x|^2\right)$ is bounded by an L^1-function for all $x \in \mathbb{R}^k$ and the average of $H(t,x) - \gamma^2|x|^2$ with respect to t is bounded from below, more exactly:*

$$\left|\nabla\left(H(t,x) - \gamma^2|x|^2\right)\right| \le a(t)$$

for a.e. $t \in (0,T)$, all $x \in \mathbb{R}^k$, some $a \in L^1(0,T;\mathbb{R}_+)$, and

$$\int_0^T H(t,x)\,dt - T\gamma^2|x|^2 \ge C > -\infty$$

for all $x \in \mathbb{R}^k$ and some constant C. Then the functional (8.22) is coercive.

Proof. Denote $H_\gamma(t,x) = H(t,x) - \gamma^2|x|^2$. For $u \in H_p^1\left(0,T;\mathbb{R}^k\right)$, we have $u = \overline{u} + \widehat{u}$ where $\overline{u} = \int_0^T u(t)\,dt$ and $\widehat{u} = u - \overline{u}$. Then

$$\begin{aligned}
E(u) &= \int_0^T \left(\frac{1}{2}\left|u'(t)\right|^2 + \gamma^2 |u(t)|^2\right) dt + \int_0^T H_\gamma(t,\overline{u})\,dt \\
&\quad + \int_0^T \left[H_\gamma(t,u(t)) - H_\gamma(t,\overline{u})\right] dt \\
&\geq \min\left\{1, 2\gamma^2\right\}|u|^2 + C + \int_0^T \int_0^1 \left(\nabla H_\gamma(t,\overline{u} + s\widehat{u}(t)), \widehat{u}(t)\right) ds dt \\
&\geq \min\left\{1, 2\gamma^2\right\}|u|^2 + C - |a|_{L^1}|\widehat{u}|_\infty .
\end{aligned}$$

Since $|\widehat{u}|_\infty \leq c\,|\widehat{u}| \leq c\,|u|$, we deduce that

$$E(u) \geq \min\left\{1, 2\gamma^2\right\}|u|^2 + C - c\,|a|_{L^1}|u| \to \infty \ \text{ as } \ |u| \to \infty.$$

The proof is now complete. □

Notice that if H is of coercive type, then the functional (8.22) is bounded from below. Indeed, the coercivity property implies that there exists a positive number R such that $E(u) \geq 0$ if $|u| > R$. Since the injection of $H_p^1\left(0,T;\mathbb{R}^k\right)$ into $C\left(0,T;\mathbb{R}^k\right)$ is continuous, there exists $c > 0$ such that $|u|_\infty \leq c\,|u|$ for every $u \in H_p^1\left(0,T;\mathbb{R}^k\right)$. Then, for $|u| \leq R$, $|u|_\infty \leq cR$ and since H is L^1-Carathéodory, $H(t,u(t)) \geq -b(t)$ for a.e. $t \in (0,T)$. As a result, for $|u| \leq R$, one has $E(u) \geq -|b|_{L^1}$. Hence $E(u) \geq -|b|_{L^1}$ for all $u \in H_p^1\left(0,T;\mathbb{R}^k\right)$ as we claimed.

Our hypotheses are as follows:

(H1) for each $R > 0$, there exist $\sigma_1, \sigma_2 \in L^1(0,T;\mathbb{R}_+)$ and $\gamma \neq 0$ such that

$$F(t,x,y) \geq \gamma^2|x|^2 - \sigma_1(t)|x| - \sigma_2(t)$$

for a.e. $t \in (0,T)$, all $x \in \mathbb{R}^{k_1}$ and $y \in \mathbb{R}^{k_2}$ with $|y| \leq R$;

(H2) there exist $g, g_1 : (0,T) \times \mathbb{R}^{k_2} \to \mathbb{R}$ of coercive type with

$$g(t,y) \leq G(t,x,y) \leq g_1(t,y) \tag{8.23}$$

for all $x \in \mathbb{R}^{k_1}$, $y \in \mathbb{R}^{k_2}$, and a.e. $t \in (0,T)$;

(H3) there exist $m_{ij} \in \mathbb{R}_+$ $(i,j = 1,2)$ with

$$\begin{aligned}
\left|m_1^2(x-\overline{x}) - \nabla_x\left(F(t,x,y) - F(t,\overline{x},\overline{y})\right)\right| &\leq m_{11}|x-\overline{x}| + m_{12}|y-\overline{y}| \\
\left|m_2^2(y-\overline{y}) - \nabla_y\left(G(t,x,y) - G(t,\overline{x},\overline{y})\right)\right| &\leq m_{21}|x-\overline{x}| + m_{22}|y-\overline{y}|
\end{aligned} \tag{8.24}$$

for all $x, \overline{x} \in \mathbb{R}^{k_1}$, $y, \overline{y} \in \mathbb{R}^{k_2}$ and a.e. $t \in (0, T)$, such that the spectral radius of the matrix

$$M = \begin{bmatrix} \frac{m_{11}}{m_1^2} & \frac{m_{12}}{m_1 m_2} \\ \frac{m_{21}}{m_1 m_2} & \frac{m_{22}}{m_2^2} \end{bmatrix} \tag{8.25}$$

is strictly less than one.

Theorem 8.7. *Under hypotheses (H1)–(H3), problem (8.18) has a unique solution*

$$(u, v) \in H_p^1\left(0, 1; \mathbb{R}^{k_1}\right) \times H_p^1\left(0, 1; \mathbb{R}^{k_2}\right),$$

which is a Nash-type equilibrium of the pair of energy functionals (E_1, E_2) *given by (8.19).*

Proof. From (H1) we have that $E_1(., v)$ is bounded from below for each $v \in H_p^1\left(0, 1; \mathbb{R}^{k_2}\right)$. Indeed, if $R = |v|_\infty$, then

$$\begin{aligned} E_1(u, v) &\geq \int_0^T \left(\frac{1}{2}\left|u'(t)\right|^2 + \gamma^2 |u(t)|^2 - \sigma_1(t)|u(t)| - \sigma_2(t)\right) dt \\ &\geq C_1 |u|_1^2 - C_2 |u|_1 - C_3 \end{aligned}$$

whence the desired conclusion.

Next, using the first inequality in (8.23), we have

$$E_2(u, v) \geq \phi(v) := \int_0^T \left(\frac{1}{2}\left|v'(t)\right|^2 + g(t, v(t))\right) dt.$$

Since g is of coercive type, ϕ is bounded from below. Thus, $E_2(u, .)$ is bounded from below even uniformly with respect to u.

Furthermore, if we denote

$$\phi_1(v) = \int_0^T \left(\frac{1}{2}\left|v'(t)\right|^2 + g_1(t, v(t))\right) dt,$$

we fix any number $a > 0$, we use (8.23) and the coercivity of ϕ, then we may find a number $R > 0$ such that

$$\inf E_2(u, .) + a \leq \inf \phi_1 + a \leq \phi(v)$$

for $|v|_2 \geq R$. Since $E_2(u, v) \geq \phi(v)$, this shows that condition (8.8) is satisfied by E_2.

Finally, using (8.20), (8.21) and (8.24) we obtain

$$\begin{aligned}|N_1(u,v)-N_1(\overline{u},\overline{v})|_1 &= \left|J_1\left(m_1^2(u-\overline{u})-\nabla_x(F(.,u,v)-F(.,\overline{u},\overline{v}))\right)\right|_1\\ &\le \frac{1}{m_1}\left|m_1^2(u-\overline{u})-\nabla_x(F(.,u,v)-F(.,\overline{u},\overline{v}))\right|_{L^2}\\ &\le \frac{m_{11}}{m_1}|u-\overline{u}|_{L^2}+\frac{m_{12}}{m_1}|v-\overline{v}|_{L^2}\\ &\le \frac{m_{11}}{m_1^2}|u-\overline{u}|_1+\frac{m_{12}}{m_1m_2}|v-\overline{v}|_2\,.\end{aligned}$$

A similar inequality holds for N_2 and so condition (8.7) is satisfied with the matrix M given by (8.25). Therefore all the hypotheses of Theorem 8.5 are fulfilled. □

Example 8.1. Consider the system of two scalar equations

$$\begin{aligned}u'' &= 2\gamma_1^2u+a(t)\sin u(t)+b(t)\cos u(t)\cos v(t)+c(t)\\ v'' &= 2\gamma_2^2v+A(t)\sin u(t)\cos v(t)+B(t)\cos v(t)\,,\end{aligned}\tag{8.26}$$

where $\gamma_1,\gamma_2\neq 0$ and $a,b,A,B\in L^\infty(0,T)$, $c\in L^1(0,T)$. In this case,

$$\begin{aligned}F(t,x,y) &= \gamma_1^2x^2-a(t)\cos x+b(t)\sin x\cos y+c(t)x\\ G(t,x,y) &= \gamma_2^2y^2+A(t)\sin x\sin y+B(t)\sin y\end{aligned}$$

and we let $m_i=\gamma_i\sqrt{2}$ $(i=1,2)$. If the spectral radius of the matrix

$$M=\begin{bmatrix}\frac{1}{2\gamma_1^2}(|a|_\infty+|b|_\infty) & \frac{|b|_\infty}{2\gamma_1\gamma_2}\\ \frac{|A|_\infty}{2\gamma_1\gamma_2} & \frac{1}{2\gamma_2^2}(|A|_\infty+|B|_\infty)\end{bmatrix}$$

is strictly less than one, then the system (8.26) has a unique T-periodic solution, which is a Nash type equilibrium of the pair of energy functionals of the system.

8.3 NASH EQUILIBRIUM FOR SYSTEMS OF VARIATIONAL INEQUALITIES

In this section, the solutions of some systems of variational inequalities are obtained as Nash-type equilibria of the corresponding systems of Szulkin functionals. This is achieved by an iterative scheme based on Ekeland's variational principle, whose convergence is proved via the vector technique involving inverse-positive matrices. An application to periodic solutions for a system of two second order ordinary differential equations with singular ϕ-Laplace operators is included. In this section we follow the results developed by Precup [146, 147].

Consider the following system of variational inequalities: find $(x, y) \in X \times Y$ such that

$$\begin{cases} \left\langle \mathcal{F}'_x(x, y), u - x \right\rangle + \varphi(u) - \varphi(x) \geq 0 \\ \left\langle \mathcal{G}'_y(x, y), v - y \right\rangle + \psi(v) - \psi(y) \geq 0 \end{cases} \quad \text{for all } (u, v) \in X \times Y, \tag{8.27}$$

where X, Y are Banach spaces with norms $|.|_X, |.|_Y$ and $\mathcal{F}'_x, \mathcal{G}'_y$ are the Fréchet derivatives of $\mathcal{F}$ and $\mathcal{G}$ in the first and second variable, respectively.

We first assume that the following condition is fulfilled:

(H$_0$) $\mathcal{F}, \mathcal{G} : X \times Y \to \mathbb{R}$ are of class C^1 with respect to the first and the second variable, respectively, and $\varphi : X \to (-\infty, +\infty]$, $\psi : Y \to (-\infty, +\infty]$ are proper, lower semicontinuous and convex functionals.

Then a couple of elements $(x, y) \in D(\varphi) \times D(\psi)$ is a solution of the system if x is a critical point in Szulkin's sense of the functional $\mathcal{F}(., y) + \varphi$ and y is a critical point in Szulkin's sense of the functional $\mathcal{G}(x, .) + \psi$. We are interested in such a solution which is a *Nash-type equilibrium* of the pair of functionals (E_1, E_2), where $E_1, E_2 : X \times Y \to (-\infty, +\infty]$,

$$E_1 := \mathcal{F} + \varphi, \quad E_2 = \mathcal{G} + \psi,$$

that is

$$E_1(x, y) = \min_{u \in X} E_1(u, y), \quad E_2(x, y) = \min_{v \in Y} E_2(x, v).$$

From a physical point of view, a Nash-type equilibrium (x, y) for two interconnected mechanisms whose energies are E_1, E_2, is such that the motion of each mechanism is conformed to the minimum energy principle by taking into account the motion of the other.

To obtain such a solution of the system (8.27), an iterative scheme is introduced, a Palais-Smale type condition is defined, and the convergence of the iterative procedure is proved via a vector technique based on inverse-positive matrices. In such a way, the abstract part of this section represents a vectorization of the direct variational principle for Szulkin-type functionals [165].

The main abstract result of this section is illustrated with an application to the study of the periodic problem for a system of equations involving the singular ϕ-Laplace operator:

$$\begin{cases} \left(\phi_1\left(x'\right)\right)' = \nabla_x F_1(t, x, y) \\ \left(\phi_2\left(y'\right)\right)' = \nabla_y F_2(t, x, y). \end{cases} \tag{8.28}$$

We point out that this system is composed by two equations having a variational form each, but without a variational structure in its whole. Another feature of the analysis we will develop is that we work in the Lebesgue space L^2 instead of the standard space of continuous functions.

A function $H:(0,T)\times\mathbb{R}^n\to\mathbb{R}^m$, $H=H(t,x)$ $(n,m\geq 1)$ is said to be L^1-*Carathéodory*, if it satisfies the *Carathéodory conditions*, that is, $H(.,x)$ is measurable for each $x\in\mathbb{R}^n$ and $H(t,.)$ is continuous for a.e. $t\in(0,T)$; and for each $r>0$, there is $b_r\in L^1(0,T;\mathbb{R}_+)$ such that $|H(t,x)|\leq b_r(t)$ for a.e. $t\in(0,T)$ and all $x\in\overline{B}_r(\mathbb{R}^n)$.

The function H is said to be (p,q)-*Carathéodory* $(1\leq p,q<\infty)$ if it satisfies the Carathéodory conditions and $|H(t,x)|\leq a|x|^{p/q}+b(t)$ for all $x\in\mathbb{R}^n$, a.e. $t\in(0,T)$ and some $a\in\mathbb{R}_+$, $b\in L^q(0,T;\mathbb{R}_+)$.

The superposition operator $x\mapsto H(.,x(.))$ is well-defined and continuous from $C([0,T];\mathbb{R}^n)$ to $L^1(0,T;\mathbb{R}^m)$, provided that H is L^1-Carathéodory. The superposition operator is from $L^p(0,T;\mathbb{R}^n)$ to $L^q(0,T;\mathbb{R}^m)$ if H is (p,q)-Carathéodory.

A square matrix of real numbers is said to be *inverse-positive* if it is nonsingular and all the elements of its inverse are nonnegative. A class of such kind of matrices is given by the matrices of the form $I-A$, where I is the unit matrix, the elements of A are nonnegative, and the spectral radius of A is strictly less than one. However, there are matrices A with not all elements nonnegative and spectral radius bigger than one, such that $I-A$ is inverse-positive. An example is the matrix $A=\begin{bmatrix}-2 & a\\ 0 & -1\end{bmatrix}$, where $a>0$. Also note that a matrix of the form $\begin{bmatrix}a & -b\\ -c & d\end{bmatrix}$, with $a,b,c,d\geq 0$ is inverse-positive if and only if its determinant is positive, that is, $ad-bc>0$.

Let us now assume the following hypothesis.

(H$_1$) The functionals $E_1(.,y)$ and $E_2(x,.)$ are bounded from below for every $x\in D(\varphi)$ and $y\in D(\psi)$.

Theorem 8.8. *Assume that conditions* (H_0), (H_1) *hold. Then for every* $y_0\in D(\psi)$, *there exist sequences* (x_n) *and* (y_n) *such that* $x_n\in D(\varphi)$, $y_n\in D(\psi)$,

$$E_1(x_n,y_{n-1})\leq\inf_X E_1(.,y_{n-1})+\frac{1}{n},\tag{8.29}$$

$$\left\langle\mathcal{F}'_x(x_n,y_{n-1}),u-x_n\right\rangle+\varphi(u)-\varphi(x_n)\geq-\frac{1}{n}|u-x_n|_X,\quad \textit{for every}\ u\in D(\varphi),\tag{8.30}$$

$$E_2(x_n,y_n)\leq\inf_Y E_2(x_n,.)+\frac{1}{n},\tag{8.31}$$

$$\left\langle\mathcal{G}'_y(x_n,y_n),v-y_n\right\rangle+\psi(v)-\psi(y_n)\geq-\frac{1}{n}|v-y_n|_Y,\quad \textit{for every}\ v\in D(\psi).\tag{8.32}$$

Proof. For $n=1$, we first obtain x_1 by applying Theorem A.6 to the functional $E_1(.,y_0)$. Then y_1 is obtained similarly for the functional $E_2(x_1,.)$. Furthermore, at any step n, we obtain x_n and then y_n by applying Theorem A.6 to $E_1(., y_{n-1})$ and $E_2(x_n, .)$, respectively. $\square$

We now require a stronger continuity property for $\mathcal{F}, \mathcal{G}$:

($\mathbf{H}_0^*$) Condition (H_0) is satisfied and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}'_x$, $\mathcal{G}'_y$ are continuous on $X \times Y$.

We also define the following Palais-Smale compactness condition for the pair of functionals (E_1, E_2).

($\mathbf{PS}^*$) If $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 0}$ are any sequences such that the conditions (8.29)–(8.32) are satisfied, then (x_n), (y_n) possess convergent subsequences $\left(x_{n_j}\right)$, $\left(y_{n_j}\right)$ with the additional property

$$y_{n_j} - y_{n_j - 1} \to 0 \quad \text{as } j \to \infty. \tag{8.33}$$

Theorem 8.9. *Assume that the conditions (H_0^*) and (PS^*) are satisfied. Then the system (8.27) has at least one solution which is a Nash-type equilibrium of the pair of functionals (E_1, E_2).*

Proof. By Theorem 8.8, if y_0 is any fixed element of $D(\psi)$, then there are sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 0}$ satisfying the conditions (8.29)–(8.32). The (PS^*) condition guarantees the existence of the convergent subsequences $\left(x_{n_j}\right)$, $\left(y_{n_j}\right)$ with the additional property (8.33). Let x, y be the limits of the corresponding subsequences. Then

$$x_{n_j} \to x, \quad y_{n_j} \to y \quad \text{and} \quad y_{n_j - 1} \to y \quad \text{as} \quad j \to \infty.$$

The conclusion now follows from the inequalities (8.29)–(8.32) written for $n = n_j$, if we pass to the limit with $j \to \infty$ by taking into account (H_0^*). $\square$

The next result gives us a sufficient condition for (PS^*) to hold, namely

($\mathbf{H}_2$) There exist constants $m_{ij} \in \mathbb{R}_+$ $(i, j = 1, 2)$ with

$$m_{11}m_{22} - m_{12}m_{21} > 0, \tag{8.34}$$

and exponents $\beta, \gamma > 1$ such that

$$\left\langle \mathcal{F}'_x(x, y) - \mathcal{F}'_x(u, v), x - u \right\rangle \geq m_{11}|x - u|_X^{\beta} - m_{12}|x - u|_X |y - v|_Y^{\gamma - 1}$$
$$\left\langle \mathcal{G}'_y(x, y) - \mathcal{G}'_y(u, v), y - v \right\rangle \geq -m_{21}|x - u|_X^{\beta - 1}|y - v|_Y + m_{22}|y - v|_Y^{\gamma}$$

for all $x, u \in X$; $y, v \in Y$.

Theorem 8.10. *Assume that (H_0), (H_1), and (H_2) hold. If (y_n) is bounded, then the sequences (x_n), (y_n) given by (8.30), (8.32) are convergent.*

Proof. From (8.30) we obtain

$$\begin{aligned}&\left\langle \mathcal{F}_x'\left(x_{n+p}, y_{n+p-1}\right) - \mathcal{F}_x'\left(x_n, y_{n-1}\right), x_n - x_{n+p}\right\rangle \\ &\geq -\left(\frac{1}{n} + \frac{1}{n+p}\right)\left|x_{n+p} - x_n\right|_X .\end{aligned}$$

Since

$$\begin{aligned}&\left\langle \mathcal{F}_x'\left(x_{n+p}, y_{n+p-1}\right) - \mathcal{F}_x'\left(x_n, y_{n-1}\right), x_n - x_{n+p}\right\rangle \\ &\leq -m_{11}\left|x_{n+p} - x_n\right|_X^{\beta} + m_{12}\left|x_{n+p} - x_n\right|_X \left|y_{n+p-1} - y_{n-1}\right|_Y^{\gamma-1}\end{aligned}$$

we deduce

$$m_{11}\left|x_{n+p} - x_n\right|_X^{\beta-1} - m_{12}\left|y_{n+p-1} - y_{n-1}\right|_Y^{\gamma-1} \leq \frac{2}{n}.$$

Similarly, from (8.32) we deduce that

$$-m_{21}\left|x_{n+p} - x_n\right|_X^{\beta-1} + m_{22}\left|y_{n+p} - y_n\right|_Y^{\gamma-1} \leq \frac{2}{n}.$$

Denote $a_{n,p} := \left|x_{n+p} - x_n\right|_X^{\beta-1}$, $b_{n,p} := \left|y_{n+p} - y_n\right|_Y^{\gamma-1}$. Then

$$\begin{aligned}&m_{11}a_{n,p} - m_{12}b_{n,p} \leq m_{12}\left(b_{n-1,p} - b_{n,p}\right) + \frac{2}{n}, \\ &-m_{21}a_{n,p} + m_{22}b_{n,p} \leq \frac{2}{n}.\end{aligned} \tag{8.35}$$

Therefore

$$M\begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix} \leq \begin{bmatrix} m_{12}\left(b_{n-1,p} - b_{n,p}\right) + \frac{2}{n} \\ \frac{2}{n} \end{bmatrix}, \tag{8.36}$$

where

$$M = \begin{bmatrix} m_{11} & -m_{12} \\ -m_{21} & m_{22} \end{bmatrix}.$$

From (8.34) it follows that the matrix M is inverse-positive. It follows that we can multiply relation (8.36) by M^{-1} to obtain

$$\begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix} \leq M^{-1}\begin{bmatrix} m_{12}\left(b_{n-1,p} - b_{n,p}\right) + \frac{2}{n} \\ \frac{2}{n} \end{bmatrix}.$$

If $M^{-1} = \left[d_{ij}\right]_{1\leq i,j\leq 2}$, then

$$a_{n,p} \leq d_{11}\left[m_{12}\left(b_{n-1,p} - b_{n,p}\right) + \frac{2}{n}\right] + d_{12}\frac{2}{n}, \tag{8.37}$$

$$b_{n,p} \leq d_{21}\left[m_{12}\left(b_{n-1,p} - b_{n,p}\right) + \frac{2}{n}\right] + d_{22}\frac{2}{n}.$$

The last inequality yields

$$b_{n,p} \leq \frac{d_{21}m_{12}}{1+d_{21}m_{12}} b_{n-1,p} + \frac{d_{21}+d_{22}}{1+d_{21}m_{12}}\frac{2}{n}.$$

Thus, by Lemma 8.1, we deduce that $b_{n,p} \to 0$ in Y, uniformly with respect to p, hence the sequence (y_n) is Cauchy in Y. Now the first inequality of (8.37) implies that the sequence (x_n) is also Cauchy in X. □

Remark 8.2. We notice that under the assumption (H_2), the system (8.27) has at most one solution. Indeed, if (x, y) and $(\overline{x}, \overline{y})$ are two solutions of (8.27), then

$$\left\langle \mathcal{F}'_x (x, y), \overline{x} - x \right\rangle + \varphi(\overline{x}) - \varphi(x) \geq 0$$
$$\left\langle \mathcal{F}'_x (\overline{x}, \overline{y}), x - \overline{x} \right\rangle + \varphi(\overline{y}) - \varphi(\overline{y}) \geq 0$$

whence

$$\left\langle \mathcal{F}'_x (x, y) - \mathcal{F}'_x (\overline{x}, \overline{y}), \overline{x} - x \right\rangle \geq 0.$$

It follows that

$$m_{11}\left|x - \overline{x}\right|_X^{\beta-1} - m_{12}\left|y - \overline{y}\right|_Y^{\gamma-1} \leq 0.$$

Similarly we have

$$-m_{21}\left|x - \overline{x}\right|_X^{\beta-1} + m_{22}\left|y - \overline{y}\right|_Y^{\gamma-1} \leq 0.$$

These two inequalities yield

$$M\begin{bmatrix} |x-\overline{x}|_X^{\beta-1} \\ |y-\overline{y}|_Y^{\gamma-1} \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

whence

$$\begin{bmatrix} |x-\overline{x}|_X^{\beta-1} \\ |y-\overline{y}|_Y^{\gamma-1} \end{bmatrix} \leq M^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Consequently, $|x-\overline{x}|_X^{\beta-1} = |y-\overline{y}|_Y^{\gamma-1} = 0$. It follows that $(x, y) = (\overline{x}, \overline{y})$.

The next condition will guarantee the boundedness of the sequence (y_n).

(H_3) There exist $a, R > 0$ such that

$$E_2(x, y) \geq \inf_{D(\psi)} E_2(x, .) + a \quad \text{for } x \in D(\varphi),\ y \in D(\psi),\ |y|_Y \geq R. \tag{8.38}$$

Theorem 8.11. *If the conditions* $\left(H_0^*\right)$, (H_1), (H_2), *and* (H_3) *hold, then the system (8.27) has a unique solution* (x, y) *which is a Nash-type equilibrium of the pair of functionals* (E_1, E_2).

Proof. For $\frac{1}{n} < a$, from (8.31) and (8.38), we have $|y_n|_Y < R$. Hence the sequence (y_n) is bounded and the conclusion follows from Theorems 8.9, 8.10 and Remark 8.2. □

8.3.1 Application to Periodic Solutions of Second-Order Systems

To illustrate the theory let us consider the periodic problem

$$\begin{aligned} \left(\phi_1\left(x'\right)\right)' &= \nabla_x F_1\left(t, x, y\right) \\ \left(\phi_2\left(y'\right)\right)' &= \nabla_y F_2\left(t, x, y\right) \\ x\left(0\right) - x\left(T\right) &= x'\left(0\right) - x'\left(T\right) = 0 \\ y\left(0\right) - y\left(T\right) &= y'\left(0\right) - y'\left(T\right) = 0 \end{aligned} \tag{8.39}$$

where $F_1, F_2 : (0, T) \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}$.

We look for solutions (x, y) with $x \in C^1\left([0, T], \mathbb{R}^{k_1}\right)$, $y \in C^1\left([0, T], \mathbb{R}^{k_2}\right)$, such that $\left|x'\right|_{L^\infty} < r_1$, $\left|y'\right|_{L^\infty} < r_2$, $\phi_1\left(x'\right)$ and $\phi_2\left(y'\right)$ are differentiable a.e., and (8.39) is satisfied a.e. on $(0, T)$.

We shall assume that F_1, F_2 are $(2, 1)$-Carathéodory and $\nabla_x F_1$ and $\nabla_y F_2$ are $(2, 2)$-Carathéodory functions. As concerns the functions ϕ_i, $i = 1, 2$, we shall assume the following condition:

(h$_\phi$) $\phi_i : B_{r_i}\left(\mathbb{R}^{k_i}\right) \to \mathbb{R}^{k_i}$ is a homeomorphism such that $\phi_i\left(0\right) = 0$, $\phi_i = \nabla \Phi_i$ with $\Phi_i : \overline{B}_{r_i}\left(\mathbb{R}^{k_i}\right) \to \mathbb{R}$ of class C^1 on $B_{r_i}\left(\mathbb{R}^{k_i}\right)$, continuous and strictly convex on $\overline{B}_{r_i}\left(\mathbb{R}^{k_i}\right)$ and with $\Phi_i\left(0\right) = 0$.

The typical example of such a homeomorphism is the following function arising from special relativity, $\phi : B_r\left(\mathbb{R}^k\right) \to \mathbb{R}^k$,

$$\phi\left(z\right) = \frac{z}{\sqrt{r^2 - |z|^2}},$$

for which $\phi\left(z\right) = \nabla \Phi\left(z\right)$, $\Phi\left(z\right) = r - \sqrt{r^2 - |z|^2}$.

Finding solutions of the system (8.39) it suffices to obtain pairs (x, y) of functions such that x, y are critical points in Szulkin's sense of the functionals $E_1\left(., y\right)$ and $E_2\left(x, .\right)$, respectively, where $E_1, E_2 : L^2\left(0, T; \mathbb{R}^{k_1}\right) \times L^2\left(0, T; \mathbb{R}^{k_2}\right) \to (-\infty, +\infty]$,

$$E_1 = \mathcal{F} + \varphi, \quad E_2 = \mathcal{G} + \psi \tag{8.40}$$

$$\mathcal{F}\left(x, y\right) = \int_0^T F_1\left(t, x\left(t\right), y\left(t\right)\right) dt, \quad \mathcal{G}\left(x, y\right) = \int_0^T F_2\left(t, x\left(t\right), y\left(t\right)\right) dt,$$

$$\varphi(x) = \begin{cases} \int_0^T \Phi_1\left(x'(t)\right) dt, & \text{if } x \in K_1 \\ +\infty, & \text{otherwise,} \end{cases}$$

$$\psi(y) = \begin{cases} \int_0^T \Phi_2\left(y'(t)\right) dt, & \text{if } y \in K_2 \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$K_i := \left\{ z \in W^{1,\infty}\left(0, T; \mathbb{R}^{k_i}\right) : z(0) = z(T) \text{ and } \left|z'\right|_\infty \le r_i \right\} \quad (i = 1, 2).$$

Thus

$$D(\varphi) = K_1 \quad \text{and} \quad D(\psi) = K_2.$$

By straightforward computation we obtain that the functionals $\mathcal{F}$, $\mathcal{G}$, φ, and ψ satisfy (H_0^*), where

$$X = L^2\left(0, T; \mathbb{R}^{k_1}\right) \quad \text{and} \quad Y = L^2\left(0, T; \mathbb{R}^{k_2}\right).$$

In what follows, we denote by $|.|_X$, $|.|_Y$ the L^2-norms of $L^2\left(0, T; \mathbb{R}^{k_1}\right)$ and $L^2\left(0, T; \mathbb{R}^{k_2}\right)$, respectively.

In order to guarantee condition (H_2) we require the following condition:

($\mathbf{h}_F$) There exist constants $m_{ij} \in \mathbb{R}_+$ ($i, j = 1, 2$), with

$$m_{11}m_{22} - m_{12}m_{21} > 0,$$

such that

$$\begin{aligned} &\langle \nabla_x F_1(t, x, y) - \nabla_x F_1(t, u, v),\ x - u \rangle \\ &\ge m_{11} |x - u|^2 - m_{12} |x - u| |y - v|, \\ &\left\langle \nabla_y F_2(t, x, y) - \nabla_y F_2(t, u, v),\ y - v \right\rangle \\ &\ge -m_{21} |x - u| |y - v| + m_{22} |y - v|^2 \end{aligned} \tag{8.41}$$

for all $x, u \in \mathbb{R}^{k_1}$; $y, v \in \mathbb{R}^{k_2}$.

Indeed, if $x, u \in L^2\left(0, T; \mathbb{R}^{k_1}\right)$ and $y, v \in L^2\left(0, T; \mathbb{R}^{k_2}\right)$, then

$$\begin{aligned} &\left\langle \mathcal{F}'_x(x, y) - \mathcal{F}'_x(u, v), x - u \right\rangle \\ &= \int_0^T \langle \nabla_x F_1(t, x(t), y(t)) - \nabla_x F_1(t, u(t), v(t)),\ x(t) - u(t) \rangle \, dt \end{aligned}$$

$$\geq m_{11} \int_0^T |x(t) - u(t)|^2 \, dt - m_{12} \int_0^T |x(t) - u(t)| \, |y(t) - v(t)| \, dt$$
$$\geq m_{11} |x - u|_X^2 - m_{12} |x - u|_X \, |y - v|_Y \, .$$

Hence the first inequality in (H_2) is fulfilled with $\beta = \gamma = 2$. The second inequality follows similarly.

In order to satisfy (H_3) we assume the following coercivity condition on F_2:

($\mathbf{h}_c$) There exist L^1-Carathéodory functions $g, h : (0, T) \times \mathbb{R}^{k_2} \to \mathbb{R}$ such that

$$g(t, y) \leq F_2(t, x, y) \leq h(t, y) \tag{8.42}$$

for a.e. $t \in (0, T)$, all $x \in \mathbb{R}^{k_1}$, $y \in \mathbb{R}^{k_2}$, and

$$\alpha(y) := \int_0^T g(t, y(t)) \, dt \to \infty \quad \text{as} \quad y \in K_2, \; |y|_Y \to \infty. \tag{8.43}$$

We first notice that α is bounded from below on K_2. Indeed, from (8.43), there exists $R > 0$ such that

$$\alpha(y) \geq 0 \quad \text{for all} \quad y \in K_2 \quad \text{with} \quad |y|_Y > R. \tag{8.44}$$

On the other hand, if $|y|_Y \leq R$ and $t_0 \in [0, T]$ is such that $|y(t_0)| = |y|_{L^\infty}$, from

$$|y(t)| \geq |y(t_0)| - |y(t) - y(t_0)| = |y|_{L^\infty} - \left| \int_{t_0}^t y'(s) \, ds \right| \geq |y|_{L^\infty} - T r_2$$

we have

$$|y|_{L^\infty} \leq |y(t)| + T r_2 \quad \text{for all } t \in [0, T],$$

whence, by passing to the L^2-norm, we deduce

$$|y|_{L^\infty} \leq \frac{1}{\sqrt{T}} |y|_Y + T r_2 \leq \frac{1}{\sqrt{T}} R + T r_2 =: R_1. \tag{8.45}$$

Since g is L^1-Carathéodory, there exists a function $b_{R_1} \in L^1(0, T; \mathbb{R}_+)$ such that $|g(t, y)| \leq b_{R_1}(t)$ for a.e. $t \in (0, T)$ and $y \in \overline{B}_{R_1}\left(\mathbb{R}^{k_2}\right)$. Then

$$\alpha(y) \geq - \left| b_{R_1} \right|_{L^1} \quad \text{for all } y \in K_2 \text{ with } |y|_Y \leq R. \tag{8.46}$$

Relations (8.44) and (8.46) show that α is bounded from below on K_2 as claimed. Next, let

$$\beta(y) := \int_0^T h(t, y(t)) \, dt.$$

If we fix any number $a > 0$, we use (8.42) and the coercivity of α we may find $R > 0$ such that

$$\inf_{D(\psi)} \mathcal{G}(x, .) + a \leq \inf_{D(\psi)} \beta + a \leq \alpha(y)$$

for all $y \in D(\psi)$ with $|y|_Y \geq R$. Since $\mathcal{G}(x, y) \geq \alpha(y)$, this proves that condition (H_2) is satisfied.

Finally, (H_1) will be guaranteed by using the following condition:

($\mathbf{h}_b$) For each $R > 0$, there exists $\sigma_R, \eta_R \in L^1(0, T; \mathbb{R}_+)$ and $\gamma_R > 0$ such that

$$F_1(t, x, y) \geq \gamma_R |x|^2 - \sigma_R(t)|x| - \eta_R(t)$$

for a.e. $t \in (0, T)$, all $x \in \mathbb{R}^{k_1}$ and $y \in \overline{B}_R\left(\mathbb{R}^{k_2}\right)$.

To check (H_1), let $x \in K_1,\ y \in K_2$. According to (8.45) we have

$$|y|_{L^\infty} \leq \frac{1}{\sqrt{T}}|y|_Y + Tr_2 =: R.$$

Then

$$\mathcal{F}(x, y) \geq \gamma_R |x|_X^2 - |\sigma_R|_{L^1}|x|_{L^\infty} - |\eta_R|_{L^1}.$$

Furthermore, from the similar inequality to (8.45) for K_1, we obtain

$$\mathcal{F}(x, y) \geq \gamma_R |x|_X^2 - |\sigma_R|_{L^1}\left(\frac{1}{\sqrt{T}}|x|_X + Tr_1\right) - |\eta_R|_{L^1},$$

where the right-hand side is bounded from below as a quadratic function. Thus, both functionals $\mathcal{F}(., y)$, $\mathcal{G}(x, .)$ are bounded from below on K_1 and K_2, respectively. On the other hand, the functionals φ and ψ are bounded from below as follows from their definition and the continuity of Φ_1, Φ_2 on their compact domains. Therefore condition (H_1) is satisfied.

As a conclusion we have the following theorem.

Theorem 8.12. *Under hypotheses $(h)_\phi$, $(h)_F$, $(h)_c$, and $(h)_b$, the problem (8.39) has a solution (x, y) which is a Nash-type equilibrium of the pair of energy functionals (E_1, E_2) given by (8.40).*

Example 8.2. Consider the coupled system of two scalar equations

$$\begin{aligned} \left(\phi_1\left(x'\right)\right)' &= m_1^2 x + a_1(t)\sin y + b_1(t)\cos x \cos y + c_1(t) \\ \left(\phi_2\left(y'\right)\right)' &= m_2^2 y + a_2(t)\sin x + b_2(t)\sin x \cos y + c_2(t) \end{aligned} \tag{8.47}$$

where ϕ_1, ϕ_2 satisfy $(h)_\Phi$; $m_1, m_2 \neq 0$; $a_1, a_2, b_1, b_2 \in L^\infty(0, T)$ and $c_1, c_2 \in L^2(0, T)$. In this case,

$$F_1(t, x, y) = m_1^2 x^2/2 + a_1(t)\, x \sin y + b_1(t)\sin x \cos y + c_1(t)\, x$$

$$F_2(t,x,y) = m_2^2 y^2/2 + a_2(t)\, y \sin x + b_2(t) \sin x \sin y + c_2(t)\, y.$$

If

$$m_i^2 > |b_i|_{L^\infty}, \quad i = 1, 2$$

and

$$\left(m_1^2 - |b_1|_{L^\infty}\right)\left(m_2^2 - |b_2|_{L^\infty}\right) > \left(|a_1|_{L^\infty} + |b_1|_{L^\infty}\right)\left(|a_2|_{L^\infty} + |b_2|_{L^\infty}\right), \tag{8.48}$$

then the system (8.47) has a T-periodic solution, which is a Nash type equilibrium of the pair of corresponding energy functionals of the system. Indeed, we have

$$\begin{aligned}
&\langle \nabla_x F_1(t,x,y) - \nabla_x F_1(t,u,v),\ x-u\rangle \\
&= m_1^2 (x-u)^2 + a_1(t)(\sin y - \sin v)(x-u) \\
&\quad + b_1(t)(\cos x \cos y - \cos u \cos v)(x-u) \\
&\geq m_1^2 (x-u)^2 - |a_1|_{L^\infty} |x-u|\,|y-v| - |b_1|_{L^\infty} (x-u)^2 \\
&\quad - |b_1|_{L^\infty} |x-u|\,|y-v| \\
&\geq \left(m_1^2 - |b_1|_{L^\infty}\right)(x-u)^2 - \left(|a_1|_{L^\infty} + |b_1|_{L^\infty}\right)|x-u|\,|y-v|.
\end{aligned}$$

Hence the first inequality in (8.41) holds with

$$m_{11} = m_1^2 - |b_1|_{L^\infty}, \quad m_{12} = |a_1|_{L^\infty} + |b_1|_{L^\infty}.$$

Similarly, the second inequality in (8.41) holds with

$$m_{21} = |a_2|_{L^\infty} + |b_2|_{L^\infty}, \quad m_{22} = m_2^2 - |b_2|_{L^\infty}.$$

Thus, (8.48) is equivalent to (8.34). The condition $(\mathrm{h})_c$ also holds with

$$\begin{aligned}
g(t,y) &= m_2^2 y^2/2 - \left(|a_2(t)| + |c_2(t)|\right)|y| - |b_2(t)| \\
h(t,y) &= m_2^2 y^2/2 + \left(|a_2(t)| + |c_2(t)|\right)|y| + |b_2(t)|
\end{aligned}$$

as follows using (8.45). Indeed

$$\begin{aligned}
\int_0^T g(t,y(t))\,dt &\geq \frac{m_1^2}{2}|y|_Y^2 - \left(|a_2|_{L^1} + |c_2|_{L^1}\right)|y|_{L^\infty} - |b_2|_{L^1} \\
&\geq \frac{m_1^2 T}{2}|y|_Y^2 - \left(|a_2|_{L^1} + |c_2|_{L^1}\right)\left(\frac{1}{\sqrt{T}}|y|_Y + Tr_2\right) - |b_2|_{L^1}
\end{aligned}$$

for every $y \in K_2$. This shows that $\alpha(y) \to \infty$ as $y \in K_2$, $|y|_Y \to \infty$.

8.4 NASH EQUILIBRIUM OF NONVARIATIONAL SYSTEMS

Many systems arising in mathematical modeling require positive solutions as acceptable states of the investigated real processes. Mathematically, finding positive solutions means to work in the positive cone of the space of all possible states. However, a cone is an unbounded set and, in many cases, nonlinear problems have several positive solutions. That is why it is important to localize solutions in bounded subsets of a cone. This problem becomes even more interesting in the case of nonlinear systems that do not have a variational structure, but each of its component equations has, namely there exist real "energy" functionals E_1 and E_2 such that the system is equivalent to the equations

$$\begin{cases} E_{11}(u,v) = 0 \\ E_{22}(u,v) = 0. \end{cases}$$

We recall that $E_{11}(u,v)$ is the partial derivative of E_1 with respect to u, and $E_{22}(u,v)$ is the partial derivative of E_2 with respect to v.

A problem of real interest is to see how the solutions (u,v) of this system are connected with the variational properties of the functionals E_1 and E_2. One possible situation, which fits to physical principles, is that a solution (u,v) is a Nash-type equilibrium of the pair of functionals (E_1, E_2), that is,

$$E_1(u,v) = \min_w E_1(w,v)$$
$$E_2(u,v) = \min_w E_2(u,w).$$

A result in this direction is given in Section 8.2 for the case when $\min_w$ is achieved, either on an entire Banach space and then, or on a ball. Nonsmooth analogues of those results, in the abstract framework of Szulkin functionals, have been developed in Section 8.3.

In this section, we are concerned with the localization of a Nash-type equilibrium in the Cartesian product of two conical sets, more exactly if $u \in K_1$ and $v \in K_2$, where K_i $(i = 1,2)$ is a cone of a Hilbert space X_i with norm $\|\cdot\|_i$, and

$$r_1 \leq l_1(u), \quad \|u\|_1 \leq R_1,$$
$$r_2 \leq l_2(v), \quad \|v\|_2 \leq R_2,$$

for some positive numbers r_i and R_i, $i = 1,2$. The main feature in this section is that $l_i : K_i \to \mathbb{R}_+$ are two *given functionals*. Usually, l_i are the corresponding norms, while here they are upper semicontinuous concave functionals. For instance, in applications, such a functional $l(u)$ can be $\inf u$. If in addition, due to some embedding result, the norm $\|u\|$ is comparable with $\sup u$ in the sense that $\sup u \leq c\|u\|$ for every nonnegative function u and some constant $c > 0$, then the values of any nonnegative function u satisfying $r \leq l(u)$ and $\|u\| \leq R$ belong to the interval $[r, cR]$, which is very convenient for finding multiple solutions located in disjoint annular conical sets.

In the first part of this section we are concerned with the localization of a critical point of minimum type in a convex conical set as above and we explain how this result can be used in order to obtain finitely or infinitely many solutions. The result can be seen as a variational analogue of some Krasnoselskii's type compression-expansion theorems from fixed point theory. Next, we obtain the vector version of this result for gradient type systems. In particular, this result allows to localize individually the components of a solution. The final part of this section is devoted to the existence and localization of Nash-type equilibria for nonvariational systems. An iterative algorithm is used and its convergence is established assuming a local matricial contraction condition. The main abstract result is illustrated with an application dealing with the periodic problem. The results of this section are due to Precup [148].

8.4.1 A Localization Critical Point Theorem

Let X be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, which is identified with its dual. Assume that $K \subset X$ is a wedge, and let $l : K \to \mathbb{R}_+$ be a concave upper semicontinuous function with $l(0) = 0$. Let $E \in C^1(X)$ be a functional and let $N : X \to X$ be the operator $N(u) := u - E'(u)$.

For any positive numbers r and R we consider the conical set

$$K_{rR} := \{u \in K : \ r \leq l(u) \ \text{ and } \ \|u\| \leq R\}.$$

Then K_{rR} is a convex set since l is concave, and it is closed since l is upper semicontinuous.

Assume that $K_{rR} \neq \emptyset$ and

$$N(K_{rR}) \subset K.$$

Lemma 8.3. *Let the following conditions be satisfied:*

$$m := \inf_{u \in K_{rR}} E(u) > -\infty; \tag{8.49}$$

there exists $\varepsilon > 0$ *such that* $E(u) \geq m + \varepsilon$ *for*

all $u \in K_{rR}$ *which simultaneously satisfy* $l(u) = r$ *and* $\|u\| = R$;
(8.50)

$$l(N(u)) \geq r \ \textit{ for every } u \in K_{rR}. \tag{8.51}$$

Then there exists a sequence $(u_n) \subset K_{rR}$ *such that*

$$E(u_n) \leq m + \frac{1}{n} \tag{8.52}$$

and

$$\left\| E'(u_n) + \lambda_n u_n \right\| \leq \frac{1}{n}, \tag{8.53}$$

where

$$\lambda_n = \begin{cases} -\frac{\langle E'(u_n),u_n\rangle}{R^2} & \textit{if } \|u_n\| = R \textit{ and } \left\langle E'(u_n), u_n \right\rangle < 0 \\ 0 & \textit{otherwise.} \end{cases} \tag{8.54}$$

Proof. By Ekeland's variational principle, there exists a sequence $(u_n) \subset K_{rR}$ such that

$$E(u_n) \le m + \frac{1}{n}, \tag{8.55}$$

$$E(u_n) \le E(v) + \frac{1}{n}\|v - u_n\| \tag{8.56}$$

for all $v \in K_{rR}$ and $n \ge 1$.

We distinguish the following distinct situations.

Case 1. There is a subsequence of (u_n) (also denoted by (u_n)) such that $r \le l(u_n)$ and $\|u_n\| < R$ for every n. For a fixed but arbitrary n and $t > 0$, consider the element

$$v = u_n - tE'(u_n).$$

Since $v = (1-t)u_n + tN(u_n)$ and both u_n and $N(u_n)$ belong to K, we deduce that $v \in K$ for every $t \in (0, 1)$. Next, the concavity of l and relation (8.51) yield

$$l(v) \ge (1-t)l(u_n) + tl(N(u_n)) \ge r$$

for all $t \in (0, 1)$. In addition, the continuity of the norm gives $\|v\| \le R$ for every $t \in (0, 1)$ small enough. It follows that $v \in K_{rR}$ for all sufficiently small $t > 0$. Replacing v in (8.56) we obtain

$$E\left(u_n - tE'(u_n)\right) - E(u_n) \ge -\frac{t}{n}\left\|E'(u_n)\right\|.$$

Dividing by t and letting t go to zero yields

$$-\left\langle E'(u_n), E'(u_n)\right\rangle \ge -\frac{1}{n}\left\|E'(u_n)\right\|.$$

Therefore

$$\left\|E'(u_n)\right\| \le \frac{1}{n}.$$

Thus, in this case, relation (8.53) holds with $\lambda_n = 0$.

Case 2. There is a subsequence of (u_n) (also denoted by (u_n)) such that $\|u_n\| = R$ for every n. Passing eventually to a new subsequence, in view of (8.50) and (8.55), we may assume that $l(u_n) > r$ for every n. Two subcases are possible:

(a) For a subsequence (still denoted by (u_n)), we have $\langle E'(u_n), u_n\rangle > 0$ for every n. Then for any fixed index n, the above choice of v in (8.56) is still possible since

$$\begin{aligned}\|v\|^2 &= \left\|u_n - tE'(u_n)\right\|^2 = \|u_n\|^2 + t^2\left\|E'(u_n)\right\|^2 - 2t\langle E'(u_n), u_n\rangle \\ &= R^2 + t^2\left\|E'(u_n)\right\|^2 - 2t\langle E'(u_n), u_n\rangle \le R^2\end{aligned}$$

for $0 < t \le 2\langle E'(u_n), u_n\rangle / \left\|E'(u_n)\right\|^2$.

(b) Assume that $\langle E'(u_n), u_n\rangle \le 0$ for every n. Then for any fixed index n, we use (8.56) with

$$v = u_n - t\left(E'(u_n) + \lambda_n u_n + \epsilon u_n\right),$$

where $t, \epsilon > 0$ and $\lambda_n = -\langle E'(u_n), u_n\rangle / R^2 \ge 0$. Since

$$v = (1-t)\,\frac{1 - t - t\lambda_n - t\epsilon}{1-t}\,u_n + tN(u_n),$$

we immediately see that $v \in K$ for every $t \in (0,1)$ small enough that $1 - t - t\lambda_n - t\epsilon > 0$. Also,

$$\langle E'(u_n) + \lambda_n u_n + \epsilon u_n,\ u_n\rangle = \epsilon R^2 > 0,$$

and as in case (a), we find that $\|v\| \le R$ for sufficiently small $t > 0$. On the other hand, from $l(u_n) > r$, we have $\delta l(u_n) = r$ for some number $\delta \in (0,1)$. Then, for any $\rho \in [\delta, 1]$, we have

$$\begin{aligned}l(\rho u_n) &= l(\rho u_n + (1-\rho)0) \ge \rho l(u_n) + (1-\rho)l(0) \\ &= \rho l(u_n) \ge \delta l(u_n) = r.\end{aligned}$$

In particular, we may take $\rho = (1 - t - t\lambda_n - t\epsilon)/(1-t)$ which belongs to $[\delta, 1]$ for sufficiently small t. Consequently,

$$\begin{aligned}l(v) &= l\left((1-t)\,\frac{1 - t - t\lambda_n - t\epsilon}{1-t}\,u_n + tN(u_n)\right) \\ &= l((1-t)\rho u_n + tN(u_n)) \ge (1-t)l(\rho u_n) + tl(N(u_n)) \ge r.\end{aligned}$$

Therefore $v \in K_{rR}$ for every sufficiently small $t > 0$. Replacing v in (8.56) and letting $t \to 0$ yields

$$\langle E'(u_n), -E'(u_n) - \lambda_n u_n - \epsilon u_n\rangle \ge -\frac{1}{n}\left\|E'(u_n) + \lambda_n u_n + \epsilon u_n\right\|.$$

Finally, let ϵ tend to zero and use $\langle u_n, E'(u_n) + \lambda_n u_n \rangle = 0$ to deduce

$$\| E'(u_n) + \lambda_n u_n \| \leq \frac{1}{n},$$

which is relation (8.53). □

Lemma 8.3 yields the following critical point theorem.

Theorem 8.13. *Assume that hypotheses of Lemma 8.3 are satisfied. In addition, assume that there is a number ν such that*

$$\langle E'(u), u \rangle \geq \nu \text{ for every } u \in K_{rR} \text{ with } \|u\| = R, \tag{8.57}$$

$$E'(u) + \lambda u \neq 0 \text{ for all } u \in K_{rR} \text{ with } \|u\| = R \text{ and } \lambda > 0, \tag{8.58}$$

and a Palais-Smale type condition holds, more exactly, any sequence as in the conclusion of Lemma 8.3 has a convergent subsequence. Then there exists $u \in K_{rR}$ such that

$$E(u) = m \text{ and } E'(u) = 0.$$

Proof. The sequence (λ_n) given by (8.54) is bounded as a consequence of (8.57). Hence, passing eventually to a subsequence we may suppose that (λ_n) converges to some number λ. Clearly $\lambda \geq 0$. Next, using the Palais-Smale type condition we may assume that the sequence (u_n) converges to some element $u \in K_{rR}$. Then letting $n \to \infty$ in (8.52) and (8.53) gives $E(u) = m$ and $E'(u) + \lambda u = 0$. From (8.54) we have that the case $\lambda > 0$ is possible only if $\|u\| = R$, which is excluded by assumption (8.58). Therefore $\lambda = 0$ and so $E'(u) = 0$. □

If the functional l is continuous on K_{rR}, then instead of hypothesis (8.51) we can take the weaker boundary condition

$$l(N(u)) \geq r \text{ for every } u \in K_{rR} \text{ with } l(u) = r.$$

Let us now assume that there exists $c > 0$ such that

$$l(u) \leq c\|u\| \tag{8.59}$$

for all $u \in K$. Then from the assumption $K_{rR} \neq \emptyset$, we deduce that $r \leq cR$. Indeed, if $u \in K_{rR}$, then $r \leq l(u) \leq c\|u\| \leq cR$.

Also, if

$$r_1 \leq cR_1, \ r_2 \leq cR_2 \text{ and } cR_1 < r_2,$$

then the sets $K_{r_1R_1}$ and $K_{r_2R_2}$ are disjoint. Indeed, if $u \in K_{r_1R_1}$, then

$$r_1 \leq l(u) \leq c\|u\| \leq cR_1 < r_2.$$

Hence $l(u) < r_2$ which shows that $u \notin K_{r_2 R_2}$. The same conclusion holds if

$$r_1 \leq cR_1, \ r_2 \leq cR_2 \ \text{ and } \ r_1 > cR_2.$$

These remarks allow us to state the following multiplicity results.

Theorem 8.14. *Assume that condition (8.59) holds.*

(i) *If there are finite or infinite sequences of numbers $(r_j)_{1\leq j\leq n}$, $(R_j)_{1\leq j\leq n}$ (for $1 \leq n \leq +\infty$) with $r_j \leq cR_j$ for $1 \leq j \leq n$ and $cR_j < r_{j+1}$ for $1 \leq j < n$, such that the assumptions of Theorem 8.13 are satisfied for each of the sets $K_{r_j R_j}$, then for every j, there exists $u_j \in K_{r_j R_j}$ with*

$$E(u_j) = \inf_{K_{r_j R_j}} E \quad \textit{and} \quad E'(u_j) = 0. \tag{8.60}$$

(ii) *If there are infinite sequences of numbers $(r_j)_{j\geq 1}$, $(R_j)_{j\geq 1}$ with $cR_{j+1} < r_j \leq cR_j$ for all j, such that the assumptions of Theorem 8.13 hold for each of the sets $K_{r_j R_j}$, then for every j, there exists $u_j \in K_{r_j R_j}$ which satisfies (8.60).*

8.4.2 Localization of Nash-Type Equilibria of Nonvariational Systems

We first establish a vector version of the localization critical point theorem. For this purpose, we consider two Hilbert spaces X_1 and X_2 with scalar products $\langle .,. \rangle_i$ and norms $\|.\|_i$ $(i = 1, 2)$. Let $K_i \subset X_i$ denote two wedges and let $l_i : K_i \to \mathbb{R}_+$ be upper semicontinuous functionals with $l_i(0) = 0$. We assume that E is a C^1-functional on the product space $X_1 \times X_2$. We have $E'(u,v) = \left(E'_u(u,v), E'_v(u,v)\right)$, for $u \in X_1$, $v \in X_2$, and we denote by N_1, N_2 the operators

$$N_1(u,v) = u - E'_u(u,v), \quad N_2(u,v) = v - E'_v(u,v). \tag{8.61}$$

In what follows, we are interested to find a solution (u, v) of the system

$$\begin{cases} u = N_1(u,v) \\ v = N_2(u,v), \end{cases} \tag{8.62}$$

or equivalently, a critical point of E, that is,

$$\begin{cases} E'_u(u,v) = 0 \\ E'_v(u,v) = 0, \end{cases}$$

which minimizes E in a set of the form $K_{rR} := (K_1)_{r_1 R_1} \times (K_2)_{r_2 R_2}$, where $r = (r_1, r_2)$, $R = (R_1, R_2)$ and

$$(K_i)_{r_i R_i} = \left\{ w \in K_i : \ r_i \leq l_i(w) \ \text{ and } \ \|w\|_i \leq R_i \right\}.$$

Applying the Ekeland variational principle to the functional E and to the closed subset K_{rR} of $X_1 \times X_2$ we obtain the following vector versions of Lemma 8.3 and Theorem 8.13.

Lemma 8.4. *Let the following conditions be satisfied:*

$$m := \inf_{(u,v)\in K_{rR}} E(u,v) > -\infty;$$

there exists $\varepsilon > 0$ *such that* $E(u,v) \geq m + \varepsilon$ *if*

$$l_1(u) = r_1 \text{ and } \|u\|_1 = R_1, \text{ or } l_2(v) = r_2 \text{ and } \|v\|_2 = R_2;$$

$$l_1(N_1(u,v)) \geq r_1 \text{ and } l_2(N_2(u,v)) \geq r_2 \text{ for every } (u,v) \in K_{rR}.$$

Then there exists a minimizing sequence $(u_n, v_n) \subset K_{rR}$*, i.e.,* $E(u_n, v_n) \to m$ *as* $n \to \infty$*, such that*

$$E(u_n, v_n) \leq m + \frac{1}{n},$$

$$\left\| E'_u(u_n, v_n) + \lambda_n u_n \right\|_1 \leq \frac{1}{n} \quad \text{and} \quad \left\| E'_v(u_n, v_n) + \mu_n v_n \right\|_2 \leq \frac{1}{n},$$

where

$$\lambda_n = \begin{cases} -\dfrac{\left\langle E'_u(u_n, v_n), u_n \right\rangle_1}{R_1^2} & \text{if } \|u_n\|_1 = R_1 \text{ and } \left\langle E'_u(u_n, v_n), u_n \right\rangle_1 < 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_n = \begin{cases} -\dfrac{\left\langle E'_v(u_n, v_n), v_n \right\rangle_1}{R_2^2} & \text{if } \|v_n\|_2 = R_2 \text{ and } \left\langle E'_v(u_n, v_n), v_n \right\rangle_2 < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 8.15. *Assume that the assumptions of Lemma 8.4 are satisfied. In addition, we assume that there exists a real number* ν *such that*

$$\left\langle E'_u(u,v), u \right\rangle_1 \geq \nu \text{ for every } (u,v) \in K_{rR} \text{ with } \|u\|_1 = R_1,$$

$$\left\langle E'_v(u,v), v \right\rangle_2 \geq \nu \text{ for every } (u,v) \in K_{rR} \text{ with } \|v\|_2 = R_2,$$

$$E'_u(u,v) + \lambda u \neq 0 \text{ for all } (u,v) \in K_{rR} \text{ with } \|u\|_1 = R_1 \text{ and } \lambda > 0,$$

$$E'_v(u,v) + \lambda v \neq 0 \text{ for all } (u,v) \in K_{rR} \text{ with } \|v\|_2 = R_2 \text{ and } \lambda > 0,$$

and the Palais-Smale type condition holds. Then there exists $(u,v) \in K_{rR}$ *such that*

$$E(u,v) = m \text{ and } E'(u,v) = 0.$$

Our main purpose in what follows is to deal with system (8.62) but without assuming the existence of a functional E with property (8.61). Instead, we assume that each equation of the system has a variational structure, that is, there are two C^1 functionals $E_i : X := X_1 \times X_2 \to \mathbb{R}$, such that

$$N_1(u,v) = u - E_{11}(u,v), \quad N_2(u,v) = v - E_{22}(u,v),$$

where by E_{11}, E_{22} we mean the partial derivatives of E_1, E_2 with respect to u and v, respectively.

We look for a point (u,v) in a set of the form $K_{rR} := (K_1)_{r_1R_1} \times (K_2)_{r_2R_2}$, which solves problem (8.62) and is a *Nash-type equilibrium in* K_{rR} of the pair of functionals (E_1, E_2), more exactly

$$E_1(u,v) = \min_{w \in (K_1)_{r_1R_1}} E_1(w,v),$$
$$E_2(u,v) = \min_{w \in (K_2)_{r_2R_2}} E_2(u,w).$$

We say that the operator $N : X \to X$, $N(u,v) = (N_1(u,v), N_2(u,v))$ is a *Perov contraction on* K_{rR} if there exists a matrix $M = [m_{ij}] \in \mathcal{M}_{2,2}(\mathbb{R}_+)$ such that M^n tends to the zero matrix as $n \to \infty$, and the following matricial Lipschitz condition is satisfied

$$\begin{bmatrix} \|N_1(u,v) - N_1(\overline{u},\overline{v})\|_1 \\ \|N_2(u,v) - N_2(\overline{u},\overline{v})\|_2 \end{bmatrix} \leq M \begin{bmatrix} \|u - \overline{u}\|_1 \\ \|v - \overline{v}\|_2 \end{bmatrix} \tag{8.63}$$

for every $u, \overline{u} \in (K_1)_{r_1R_1}$ and $v, \overline{v} \in (K_2)_{r_2R_2}$.

Notice that for a square matrix of nonnegative elements, the property $M^n \to 0$ is equivalent to $\rho(M) < 1$, where $\rho(M)$ is the spectral radius of matrix M, and also to the fact that $I - M$ is nonsingular and all the elements of the matrix $(I - M)^{-1}$ are nonnegative (see [144]). In case of square matrices M of order 2, the above property is characterized by the inequality

$$\operatorname{tr}(M) < \min\{2,\ 1 + \det(M)\}.$$

We impose the following hypotheses:

(H1) For each $v \in (K_2)_{r_2R_2}$, the functional $E_1(.,v)$ is bounded from below on $(K_1)_{r_1R_1}$;
for each $u \in (K_1)_{r_1R_1}$, the functional $E_2(u,.)$ is bounded from below on $(K_2)_{r_2R_2}$.

(H2) $l_1(N_1(u,v)) \geq r_1$ for every $(u,v) \in K_{rR}$; $N_1(u,v) \neq (1+\lambda)u$ for all $(u,v) \in K_{rR}$ with $\|u\|_1 = R_1$ and $\lambda > 0$;
$l_2(N_2(u,v)) \geq r_2$ for every $(u,v) \in K_{rR}$; $N_2(u,v) \neq (1+\lambda)v$ for all $(u,v) \in K_{rR}$ with $\|v\|_2 = R_2$ and $\lambda > 0$.

(H3) For each $v \in (K_2)_{r_2R_2}$, there exists $\varepsilon > 0$ such that $E_1(u,v) \geq \inf_{(K_1)_{r_1R_1}} E_1(.,v) + \varepsilon$ whenever u simultaneously satisfies $l_1(u) = r_1$ and $\|u\|_1 = R_1$;

for each $u \in (K_1)_{r_1 R_1}$, there exists $\varepsilon > 0$ such that $E_2(u, v) \geq \inf_{(K_2)_{r_2 R_2}} E_2(u, .) + \varepsilon$ whenever v simultaneously satisfies $l_2(v) = r_2$ and $\|v\|_2 = R_2$.

(H4) N is a Perov contraction on K_{rR}.

Let us underline the local character of the contraction condition (H4). This is essential for multiple Nash-type equilibria when (H4) is required to hold on disjoint bounded sets of the type K_{rR} but not on the entire K. Thus the 'slope' of N has to be 'small' on the sets K_{rR} but can be unlimited large between these sets, which makes possible to fulfill the boundary conditions (H2).

Theorem 8.16. *Assume that conditions* (H1)–(H4) *hold. Then there exists a solution* $(u, v) \in K_{rR}$ *of system (8.62) which is a Nash-type equilibrium on* K_{rR} *of the pair of functionals* (E_1, E_2).

Proof. The main idea is to construct recursively two sequences (u_n), (v_n), based on Lemma 8.3. Let v_0 be any element of $(K_2)_{r_2 R_2}$. At any step n $(n \geq 1)$ we may find a $u_n \in (K_1)_{r_1 R_1}$ and a $v_n \in (K_2)_{r_2 R_2}$ such that

$$E_1(u_n, v_{n-1}) \leq \inf_{(K_1)_{r_1 R_1}} E_1(., v_{n-1}) + \frac{1}{n}, \quad \|E_{11}(u_n, v_{n-1}) + \lambda_n u_n\|_1 \leq \frac{1}{n} \tag{8.64}$$

and

$$E_2(u_n, v_n) \leq \inf_{(K_2)_{r_2 R_2}} E_2(u_n, .) + \frac{1}{n}, \quad \|E_{22}(u_n, v_n) + \mu_n v_n\|_2 \leq \frac{1}{n}, \tag{8.65}$$

where

$$\lambda_n = \begin{cases} -\dfrac{\langle E_{11}(u_n, v_{n-1}), u_n \rangle_1}{R_1^2} & \text{if } \|u_n\|_1 = R_1 \\ & \text{and } \langle E_{11}(u_n, v_{n-1}), u_n \rangle_1 < 0 \\ 0 & \text{otherwise,} \end{cases}$$

and the expression of μ_n is analogous.

Condition (H4) guarantees that the operators N_i are bounded, so the boundedness of the sequences of real numbers (λ_n) and (μ_n). Therefore, passing to subsequences, we may assume that the sequences (λ_n) and (μ_n) are convergent.

Let

$$\alpha_n := E_{11}(u_n, v_{n-1}) + \lambda_n u_n \quad \text{and} \quad \beta_n := E_{22}(u_n, v_n) + \mu_n v_n.$$

We observe that $\alpha_n, \beta_n \to 0$. Also

$$\begin{aligned} (1 + \lambda_n) u_n - N_1(u_n, v_{n-1}) &= \alpha_n \\ (1 + \mu_n) v_n - N_2(u_n, v_n) &= \beta_n. \end{aligned} \tag{8.66}$$

Since $\lambda_n > 0$, the first equality if (8.66) written for n and $n+p$ yields

$$\begin{aligned}
&\left\|u_{n+p}-u_n\right\|_1 \\
&\le (1+\lambda_n)\left\|u_{n+p}-u_n\right\|_1 \\
&= \left\|(1+\lambda_n)\,u_{n+p}-(1+\lambda_n)\,u_n\right\|_1 \\
&= \left\|\left(1+\lambda_{n+p}\right)u_{n+p}-(1+\lambda_n)\,u_n-\left(\lambda_{n+p}-\lambda_n\right)u_{n+p}\right\|_1 \\
&\le \left\|N_1\left(u_{n+p},v_{n+p-1}\right)-N_1\left(u_n,v_{n-1}\right)\right\|_1+\left\|\alpha_{n+p}-\alpha_n\right\|_1 \\
&\quad +\left|\lambda_{n+p}-\lambda_n\right|\left\|u_{n+p}\right\|_1 .
\end{aligned}$$

Furthermore, using $\left\|u_{n+p}\right\|_1 \le R_1$ and relation (8.63) we deduce that

$$\begin{aligned}
&\left\|u_{n+p}-u_n\right\|_1 \\
&\le m_{11}\left\|u_{n+p}-u_n\right\|_1+m_{12}\left\|v_{n+p-1}-v_{n-1}\right\|_2+\left\|\alpha_{n+p}-\alpha_n\right\|_1 \\
&\quad + R_1\left|\lambda_{n+p}-\lambda_n\right| \\
&= m_{11}\left\|u_{n+p}-u_n\right\|_1+m_{12}\left\|v_{n+p}-v_n\right\|_2+\left\|\alpha_{n+p}-\alpha_n\right\|_1 \\
&\quad + R_1\left|\lambda_{n+p}-\lambda_n\right|+m_{12}\left(\left\|v_{n+p-1}-v_{n-1}\right\|_2-\left\|v_{n+p}-v_n\right\|_2\right).
\end{aligned}$$

Denote

$$\begin{aligned}
a_{n,p} &= \left\|u_{n+p}-u_n\right\|_1, \quad b_{n,p}=\left\|v_{n+p}-v_n\right\|_2, \\
c_{n,p} &= \left\|\alpha_{n+p}-\alpha_n\right\|_1+R_1\left|\lambda_{n+p}-\lambda_n\right|, \\
d_{n,p} &= \left\|\beta_{n+p}-\beta_n\right\|_2+R_2\left|\mu_{n+p}-\mu_n\right|.
\end{aligned}$$

Clearly, $c_{n,p}\to 0$ and $d_{n,p}\to 0$ uniformly with respect to p. It follows that

$$a_{n,p} \le m_{11}a_{n,p}+m_{12}b_{n,p}+c_{n,p}+m_{12}\left(b_{n-1,p}-b_{n,p}\right). \qquad (8.67)$$

Similarly, from the second equality in (8.66), we find

$$b_{n,p} \le m_{21}a_{n,p}+m_{22}b_{n,p}+d_{n,p}.$$

Hence

$$\begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix} \le M \begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix} + \begin{bmatrix} c_{n,p}+m_{12}\left(b_{n-1,p}-b_{n,p}\right) \\ d_{n,p} \end{bmatrix}.$$

Consequently, since $I-M$ is invertible and its inverse contains only nonnegative elements, we may write

$$\begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix} \le (I-M)^{-1} \begin{bmatrix} c_{n,p}+m_{12}\left(b_{n-1,p}-b_{n,p}\right) \\ d_{n,p} \end{bmatrix}.$$

Let $(I-M)^{-1}=\left[\gamma_{ij}\right]$. Then

$$a_{n,p}\le \gamma_{11}\left(c_{n,p}+m_{12}\left(b_{n-1,p}-b_{n,p}\right)\right)+\gamma_{12}d_{n,p} \qquad (8.68)$$
$$b_{n,p}\le \gamma_{21}\left(c_{n,p}+m_{12}\left(b_{n-1,p}-b_{n,p}\right)\right)+\gamma_{22}d_{n,p}.$$

From the second inequality, one has

$$b_{n,p}\le \frac{\gamma_{21}m_{12}}{1+\gamma_{21}m_{12}}b_{n-1,p}+\frac{\gamma_{21}c_{n,p}+\gamma_{22}d_{n,p}}{1+\gamma_{21}m_{12}}.$$

Clearly $\left(b_{n,p}\right)$ is bounded uniformly with respect to p. Next, we apply Lemma 8.1. It follows that $b_{n,p}\to 0$ uniformly for $p\in\mathbb{N}$, and hence (v_n) is a Cauchy sequence. Next, the first inequality in (8.68) implies that (u_n) is also a Cauchy sequence. Let u^*, v^* be the limits of the sequences (u_n), (v_n), respectively. The conclusion of Theorem 8.16 now follows if we pass to the limit in (8.64), (8.65) and we use (H2). □

8.5 APPLICATIONS TO PERIODIC PROBLEMS

8.5.1 Case of a Single Equation

Consider the following periodic problem:

$$-u''(t)+a^2u(t)=f(u(t)) \quad \text{on } (0,T) \qquad (8.69)$$
$$u(0)-u(T)=u'(0)-u'(T)=0$$

where $a\neq 0$ and $f:\mathbb{R}\to\mathbb{R}$ is a continuous function with $f(\mathbb{R}_+)\subset\mathbb{R}_+$.

Let $X:=H_p^1(0,T)$ be the space of functions of the form

$$u(t)=\int_0^t v(s)\,ds+C,$$

with $u(0)=u(T)$, $C\in\mathbb{R}$ and $v\in L^2(0,T)$, endowed with the inner product

$$\langle u,v\rangle=\int_0^T\left(u'v'+a^2uv\right)dt$$

and the corresponding norm

$$\|u\|=\left(\int_0^T\left(u'^2+a^2u^2\right)dt\right)^{\frac{1}{2}}.$$

Let K be the positive cone of X, that is, $K=\{u\in H_p^1(0,T):\ u\ge 0 \text{ on } [0,T]\}$, and let $l:K\to\mathbb{R}_+$ be given by

$$l(u)=\min_{t\in[0,T]}u(t).$$

The energy functional associated to the problem is $E : H_p^1(0,T) \to \mathbb{R}$,

$$E(u) = \frac{1}{2}\|u\|^2 - \int_0^T F(u(t))\,dt,$$

where

$$F(\tau) = \int_0^\tau f(s)\,ds.$$

The identification of the dual $\left(H_p^1(0,T)\right)'$ to the space $H_p^1(0,T)$ via the mapping $J : \left(H_p^1(0,T)\right)' \to H_p^1(0,T)$, $J(v) = w$, where w is the weak solution of the problem

$$\begin{gathered} -w'' + a^2 w = v \ \text{ on } \ (0,T), \\ w(0) - w(T) = w'(0) - w'(T) = 0 \end{gathered}$$

yields to the representation

$$E'(u) = u - N(u),$$

where

$$N(u) = J(f(u(\cdot))).$$

Note that the condition $f(\mathbb{R}_+) \subset \mathbb{R}_+$ guarantees that $N(K) \subset K$.

Let $c > 0$ be the embedding constant of the inclusion $H_p^1(0,T) \subset C[0,T]$, that is, $\|u\|_{C[0,T]} \le c\|u\|$ for all $u \in H_p^1(0,T)$.

Note that for $u \equiv 1$, the above inequality gives $1 \le ac\sqrt{T}$, whence $a^2 \ge 1/\left(c^2 T\right)$. Also, if r and R are positive numbers and $a\sqrt{T}r \le R$, then the set K_{rR} is nonempty. Indeed, any constant $\lambda \in \left[r, R/\left(a\sqrt{T}\right)\right]$ belongs to K_{rR}, since $l(\lambda) = \lambda \ge r$ and $\|\lambda\| = \left(\int_0^T a^2\lambda^2 ds\right)^{1/2} = a\lambda\sqrt{T} \le R$.

Theorem 8.17. *Let r, R be positive constants such that $a\sqrt{T}r \le R$. Assume that f is nondecreasing on the interval $[r, cR]$ and that the following conditions hold:*

$$E(r) < \frac{R^2}{2} - TF(cR), \tag{8.70}$$

and

$$f(r) \ge a^2 r, \quad f(cR) \le \frac{R}{cT}. \tag{8.71}$$

Then problem (8.69) has a positive solution u with $r \le u(t) \le cR$ for all $t \in [0,T]$, which minimizes E in the set K_{rR}.

Proof. **(i)** Check of condition (8.49). Let $u \in K_{rR}$. One has $r \leq u(t) \leq cR$ for all $t \in [0, T]$. Then, since F is nondecreasing on $\mathbb{R}_+$,

$$E(u) \geq -\int_0^T F(u(s))\,ds \geq -TF(cR) > -\infty.$$

(ii) Check of condition (8.50). Take any u with $l(u) = r$ and $\|u\| = R$. Then

$$E(u) = \frac{R^2}{2} - \int_0^T F(u(s))\,ds \geq \frac{R^2}{2} - TF(cR).$$

Thus our claim holds in view of the strict inequality (8.70) and the obvious inequality $m \leq E(r)$ (note that the constant function r belongs to K_{rR}).

(iii) Check of condition (8.51). Let $u \in K_{rR}$. Then

$$\begin{aligned} l(N(u)) &= l(J(f(u))) \geq l(J(f(r))) = f(r)\, l(J(1)) \\ &= \frac{f(r)}{a^2} \geq r, \end{aligned}$$

in virtue of the first inequality in (8.71).

(iv) Check of condition (8.58). Assume that $E'(u) + \lambda u = 0$ for some $u \in K_{rR}$ with $\|u\| = R$ and $\lambda > 0$. Then

$$(1+\lambda)\left(-u'' + a^2 u\right) = f(u),$$

whence

$$R^2 < (1+\lambda) R^2 = \langle f(u), u\rangle_{L^2} \leq Tf(cR)\,cR,$$

that is

$$\frac{R}{cT} < f(cR),$$

which contradicts the second inequality in (8.71).

(v) Condition (8.57) being immediate and the required Palais-Smale type condition being a consequence of the compact embedding of $H_p^1(0, T)$ into $C[0, T]$, Theorem 8.13 yields the conclusion. □

Example 8.3. For each $\lambda > 0$, the equation $-u'' + a^2 u = \lambda\sqrt{u}$ has a T-periodic solution satisfying $u(t) \geq \lambda^2/a^4$ for all $t \in [0, T]$.

Indeed, if we take $r = \lambda^2/a^4$, then the first condition from (8.71) is satisfied with equality. Next, we choose R large enough that (8.70) and the second inequality (8.71) hold, that is,

$$E(r) < \frac{R^2}{2} - \lambda\frac{2}{3}T(cR)^{\frac{3}{2}} \quad \text{and} \quad \lambda\sqrt{cR} \leq \frac{R}{cT}.$$

8.5.2 Case of a Variational System

We now consider the periodic problem for the following system:

$$-u''(t) + a_1^2 u(t) = f_1(u(t), v(t)) \quad \text{on } (0,T) \tag{8.72}$$
$$-v''(t) + a_2^2 v(t) = f_2(u(t), v(t)) \quad \text{on } (0,T)$$

in the case when f_1, f_2 are the partial derivatives of a function $F : \mathbb{R}^2 \to \mathbb{R}$ with respect to the first and the second variable, respectively. We assume that $a_i \neq 0$ and $f_i(\mathbb{R}_+ \times \mathbb{R}_+) \subset \mathbb{R}_+$, for $i = 1, 2$.

Let X_1 denote the space $H_p^1(0,T)$ endowed with the scalar product

$$\langle u, v \rangle_1 = \int_0^T \left(u'v' + a_1^2 uv \right) ds$$

and the induced norm $\|.\|_1$. Assume that X_2 is the same space endowed with the analogue scalar product and norm $\langle ., . \rangle_2$, $\|.\|_2$. Also $K_1 = K_2$ is the cone of nonnegative functions in $H_p^1(0,T)$, and $l_1(w) = l_2(w) = \min_{t\in[0,T]} w(t)$ for $w \in H_p^1(0,T)$, $w \geq 0$.

The system has a variational structure since its T-periodic solutions (u, v) are the critical points of the energy functional on $H_p^1(0,T) \times H_p^1(0,T)$,

$$E(u, v) = \frac{1}{2}\left(\|u\|_1^2 + \|v\|_2^2 \right) - \int_0^T F(u(s), v(s))\, ds.$$

For $i = 1, 2$, let $c_i > 0$ be the embedding constant of the inclusion $X_i \subset C[0,T]$, that is, $\|w\|_{C[0,T]} \leq c_i \|w\|_i$ for all $w \in H_p^1(0,T)$.

Theorem 8.18. *Let r_i, R_i be positive constants such that $a_i\sqrt{T} r_i \leq R_i$ $(i = 1, 2)$. Assume that for $i = 1, 2$, f_i is nondecreasing in each of the variables on $[r_1, c_1R_1] \times [r_2, c_2R_2]$ and that the following conditions hold:*

$$E(r_1, r_2) < \frac{R_i^2}{2} - TF(c_1R_1, c_2R_2),$$

and

$$f_i(r_1, r_2) \geq a_i^2 r_i, \quad f_i(c_1R_1, c_2R_2) \leq \frac{R_i}{c_i T}.$$

Then system (8.72) has a T-solution (u, v) with $r_1 \leq u(t) \leq c_1R_1$ and $r_2 \leq v(t) \leq c_2R_2$ for all $t \in [0,T]$, which minimizes E in the set $K_{rR} := (K_1)_{r_1R_1} \times (K_2)_{r_2R_2}$.

Example 8.4. The potential of the system

$$-u'' + a_1^2 u = \alpha_1\sqrt{u} + \gamma v$$

$$-v'' + a_2^2 v = \alpha_2 \sqrt{v} + \gamma u$$

is

$$F(u, v) = \frac{2}{3}\left(\alpha_1 u^{\frac{3}{2}} + \alpha_2 v^{\frac{3}{2}}\right) + \gamma uv.$$

As for Example 8.3, we have the following result: for every numbers $\alpha_i > 0$, $i = 1, 2$, $T > 0$ and $0 \leq \gamma < \min\left\{1/\left(2c_i^2 T\right) : i = 1, 2\right\}$, the system has a T-periodic solution with $u(t) \geq \alpha_1^2/a_1^4$ and $v(t) \geq \alpha_2^2/a_2^4$ and all $t \in [0, T]$. For the proof, take $r_i = \alpha_i^2/a_i^4$ $(i = 1, 2)$ and a sufficiently large $R := R_1 = R_2$.

8.5.3 Case of a Nonvariational System

We now consider the system (8.72) for two arbitrary continuous functions f_1, f_2 and use the notations from the previous section. The energy functionals associated to the equations of the system are $E_i : H_p^1(0, T) \times H_p^1(0, T) \to \mathbb{R}$,

$$E_1(u, v) = \frac{1}{2}\|u\|_1^2 - \int_0^T F_1(u(t), v(t))\, dt,$$
$$E_2(u, v) = \frac{1}{2}\|v\|_2^2 - \int_0^T F_2(u(t), v(t))\, dt,$$

where

$$F_1(\tau_1, \tau_2) = \int_0^{\tau_1} f_1(s, \tau_2)\, ds, \quad F_2(\tau_1, \tau_2) = \int_0^{\tau_2} f_2(\tau_1, s)\, ds.$$

The identification of the dual $\left(H_p^1(0, T)\right)'$ to the space $H_p^1(0, T)$ via the mapping $J_i : \left(H_p^1(0, T)\right)' \to H_p^1(0, T)$, $J_i(v) = w$, where w is the weak solution of the problem

$$-w'' + a_i^2 w = v \text{ on } (0, T),$$
$$w(0) - w(T) = w'(0) - w'(T) = 0$$

yields to the representations

$$E_{11}(u, v) = u - N_1(u, v), \quad E_{22}(u, v) = v - N_2(u, v),$$

where E_{11}, E_{22} stand for the partial derivatives of E_1, E_2 with respect to u and v, respectively, and

$$N_i(u, v) = J_i(f_i(u(\cdot), v(\cdot))).$$

Let $r = (r_1, r_2)$ and $R = (R_1, R_2)$ be such that

$$0 < a_i \sqrt{T} r_i \leq R_i, \quad i = 1, 2.$$

In what follows, we verify that the hypotheses of Theorem 8.16 are fulfilled.

Check of condition (H1): For every $(u, v) \in K_{rR} = (K_1)_{r_1 R_1} \times (K_2)_{r_2 R_2}$ and $t \in [0, T]$, we have

$$r_1 \leq u(t) \leq \|u\|_{C[0,T]} \leq c_1 \|u\|_1 \leq c_1 R_1,$$

and similarly $r_2 \leq v(t) \leq c_2 R_2$. It follows that

$$|f_i(\tau_1, \tau_2)| \leq \rho_i$$

for every $\tau_1 \in [r_1, c_1 R_1]$, $\tau_2 \in [r_2, c_2 R_2]$ and some $\rho_i \in \mathbb{R}_+$ $(i = 1, 2)$. Then

$$\begin{aligned} E_1(u, v) &\geq -\int_0^T \int_0^{u(t)} |f_1(s, v(t))| \, ds dt \geq -\int_0^T \int_0^{c_1 R_1} |f_1(s, v(t))| \, ds dt \\ &\geq -c_1 R_1 T \rho_1 > -\infty, \end{aligned}$$

and similarly $E_2(u, v) \geq -c_2 R_2 T \rho_2 > -\infty$. Hence condition (H1) holds.

Next, we assume in addition that for $i \in \{1, 2\}$,

$$\begin{aligned} &f_i(\tau_1, \tau_2) \quad \text{is nonnegative and nondecreasing} \\ &\text{in both variables} \;\; \tau_1 \text{ and } \tau_2 \text{ in } [r_1, c_1 R_1] \times [r_2, c_2 R_2], \end{aligned} \tag{8.73}$$

$$f_i(r_1, r_2) \geq a_i^2 r_i, \tag{8.74}$$

$$f_i(c_1 R_1, c_2 R_2) \leq R_i / (T c_i), \tag{8.75}$$

and

$$F_i(c_1 R_1, c_2 R_2) - F_i(r_1, r_2) < \frac{1}{2T}\left(R_i^2 - a_i^2 T r_i^2\right). \tag{8.76}$$

Check of condition (H2): Let $(u, v) \in K_{rR}$. Then from $u(t) \geq r_1$, $v(t) \geq r_2$ and the monotonicity of f_1, we have

$$f_i(u(t), v(t)) \geq f_i(r_1, r_2).$$

This together with (8.74) implies

$$l_i(N_i(u, v)) \geq l_i(J_i(f_i(r_1, r_2))) = \frac{f_i(r_1, r_2)}{a_i^2} \geq r_i.$$

Thus, the first part of (H2) is verified. For the second part, assume that there exists $(u, v) \in K_{rR}$ with $\|u\|_1 = R_1$ and $\lambda > 0$ such that

$$N_1(u, v) = (1 + \lambda) u.$$

Then

$$(1 + \lambda)\left(-u'' + a_1^2 u\right) = f_1(u, v),$$

which gives

$$R_1^2 < (1+\lambda) R_1^2 = (1+\lambda) \|u\|_1^2 = \langle f_1(u,v), u\rangle_{L^2} \\ \leq T f_1(c_1 R_1, c_2 R_2) c_1 R_1,$$

whence

$$f_1(c_1 R_1, c_2 R_2) > R_1/(T c_1),$$

which contradicts (8.75). An analogue reasoning applies if $N_2(u,v) = (1+\lambda) v$ for some $(u,v) \in K_{rR}$ with $\|v\|_2 = R_2$ and $\lambda > 0$. Therefore (H2) holds.

Check of condition (H3): The constant function r_1 belongs to $(K_1)_{r_1 R_1}$ and for any $v \in (K_2)_{r_2 R_2}$, we have

$$E_1(r_1, v) = \frac{1}{2} a_1^2 T r_1^2 - \int_0^T F_1(r_1, v(t))\, dt \\ \leq \frac{1}{2} a_1^2 T r_1^2 - T F_1(r_1, r_2).$$

Also, for any $(u,v) \in K_{rR}$ with $l_1(u) = r_1$ and $\|u\|_1 = R_1$, one has

$$E_1(u,v) = \frac{1}{2} R_1^2 - \int_0^T F_1(u(t), v(t))\, dt \geq \frac{1}{2} R_1^2 - T F_1(c_1 R_1, c_2 R_2).$$

Therefore the first part of (H3) holds with

$$\varepsilon = \frac{1}{2} R_1^2 - T F_1(c_1 R_1, c_2 R_2) - \left(\frac{1}{2} a_1^2 T r_1^2 - T F_1(r_1, r_2) \right)$$

which is positive in view of assumption (8.76). The second part of (H3) can be checked similarly.

Finally, to guarantee (H4) we need some Lipschitz conditions on f_1 and f_2. We assume the existence of nonnegative constants σ_{ij}, $i, j = 1, 2$, such that

$$|f_i(\tau_1, \tau_2) - f_i(\overline{\tau}_1, \overline{\tau}_2)| \leq \sigma_{i1} |\tau_1 - \overline{\tau}_1| + \sigma_{i2} |\tau_2 - \overline{\tau}_2|, \quad i = 1, 2, \tag{8.77}$$
$$\text{for } \tau_1, \overline{\tau}_1 \in [r_1, c_1 R_1] \text{ and } \tau_2, \overline{\tau}_2 \in [r_2, c_2 R_2],$$

and for the matrix $M = \left[\sigma_{ij} / \left(a_i a_j\right)\right]_{1 \leq i,j \leq 2}$ one has

$$M^n \to 0 \text{ as } n \to \infty. \tag{8.78}$$

Check of condition (H4): Notice that for $w \in L^2(0,T)$, from

$$\|J_i(w)\|_i^2 = \langle w, J_i(w)\rangle_{L^2} \leq \|w\|_{L^2} \|J_i(w)\|_{L^2} \leq \frac{1}{a_i} \|w\|_{L^2} \|J_i(w)\|_i,$$

one has

$$\|J_i(w)\|_i \leq \frac{1}{a_i}\|w\|_{L^2}, \quad w \in L^2(0,T). \tag{8.79}$$

Then using (8.79) and (8.77) we obtain

$$\begin{aligned}
\|N_1(u,v) - N_1(\overline{u},\overline{v})\|_1 &= \|J_1(f_1(u,v) - f_1(\overline{u},\overline{v}))\|_1 \\
&\leq \frac{1}{a_1}\|f_1(u,v) - f_1(\overline{u},\overline{v})\|_{L^2} \\
&\leq \frac{\sigma_{11}}{a_1}\|u-\overline{u}\|_{L^2} + \frac{\sigma_{12}}{a_1}\|v-\overline{v}\|_{L^2} \\
&\leq \frac{\sigma_{11}}{a_1^2}\|u-\overline{u}\|_1 + \frac{\sigma_{12}}{a_1 a_2}\|v-\overline{v}\|_2 .
\end{aligned}$$

Similarly,

$$\|N_2(u,v) - N_2(\overline{u},\overline{v})\|_2 \leq \frac{\sigma_{21}}{a_2 a_1}\|u-\overline{u}\|_1 + \frac{\sigma_{22}}{a_2^2}\|v-\overline{v}\|_2 .$$

Hence (8.63) holds with $m_{ij} = \sigma_{ij}/a_i a_j$.

Therefore we have the following result.

Theorem 8.19. *Under assumptions* (8.73)–(8.76), (8.77) *and* (8.78), *there exists a T-periodic solution $(u,v) \in K_{rR}$ of system* (8.72) *which is a Nash-type equilibrium on K_{rR} of the pair of energy functionals (E_1, E_2).*

Let us underline the fact that all the assumptions on f_1 and f_2 in the above theorem are given with respect to the bounded region $[r_1, c_1R_1] \times [r_2, c_2R_2]$. This makes possible to apply Theorem 8.19 to several disjoint such regions obtaining this way multiple solutions of Nash-type.

Example 8.5. Consider the problem of positive T-periodic solutions for the system

$$\begin{aligned}
-u'' + a_1^2 u &= \alpha_1\sqrt{u} + \gamma_1 v \\
-v'' + a_2^2 v &= \alpha_2\sqrt{v} + \gamma_2 u
\end{aligned} \tag{8.80}$$

where α_i, γ_i are nonnegative coefficients with $\gamma_i < a_i^2$ $(i = 1, 2)$.

We try to localize a positive solution (u,v) with $r \leq u(t)$ and $r \leq v(t)$ for all $t \in [0,T]$. We apply the previous result with $r_1 = r_2 =: r$ and $R_1 = R_2 =: R$.

(a) The positivity and monotonicity of f_1 and f_2 on $\mathbb{R}_+ \times \mathbb{R}_+$ required by (8.73) are obvious.

(b) Condition (8.74): We have

$$f_1(r,r) = \alpha_1\sqrt{r} + \gamma_1 r.$$

Thus we need

$$\alpha_1\sqrt{r}+\gamma_1 r \geq a_1^2 r.$$

Under the assumption $\gamma_1 < a_1^2$ this gives

$$r \leq \left(\frac{\alpha_1}{a_1^2-\gamma_1}\right)^2.$$

Similarly, for f_2,

$$r \leq \left(\frac{\alpha_2}{a_2^2-\gamma_2}\right)^2.$$

(c) Condition (8.75): We have

$$f_1\left(c_1 R, c_2 R\right) = \alpha_1\sqrt{c_1 R}+\gamma_1 c_2 R.$$

Hence we need

$$\alpha_1\sqrt{c_1 R}+\gamma_1 c_2 R \leq \frac{R}{T c_1}.$$

This implies $\gamma_1 < 1/\left(T c_1 c_2\right)$ and

$$R \geq \frac{\alpha_1^2 T^2 c_1^3}{\left(1-T\gamma_1 c_1 c_2\right)^2}.$$

Similarly, $\gamma_2 < 1/\left(T c_1 c_2\right)$ and

$$R \geq \frac{\alpha_2^2 T^2 c_2^3}{\left(1-T\gamma_2 c_1 c_2\right)^2}.$$

(d) Condition (8.76) for $i=1$ reads as

$$\frac{2}{3}\alpha_1\left(c_1 R\right)^{\frac{3}{2}}+\gamma_1 c_1 c_2 R^2 - F_1\left(r,r\right) < \frac{1}{2T}\left(R^2-a_1^2 T r^2\right)$$

and holds for a sufficiently large R provided that

$$\gamma_1 < \frac{1}{2T c_1 c_2}.$$

Similarly,

$$\gamma_2 < \frac{1}{2T c_1 c_2}.$$

(e) Condition (8.77): For $\tau_1 \in [r, c_1 R]$ and $\tau_2 \in [r, c_2 R]$, one has

$$\frac{\partial f_1(\tau_1, \tau_2)}{\partial \tau_1} = \frac{\alpha_1}{2\sqrt{\tau_1}} \le \frac{\alpha_1}{2\sqrt{r}}, \quad \frac{\partial f_2(\tau_1, \tau_2)}{\partial \tau_2} \le \frac{\alpha_2}{2\sqrt{r}}.$$

In addition,

$$\frac{\partial f_1(\tau_1, \tau_2)}{\partial \tau_2} = \gamma_1, \quad \frac{\partial f_2(\tau_1, \tau_2)}{\partial \tau_1} = \gamma_2.$$

Hence (8.77) holds with

$$\sigma_{ii} = \frac{\alpha_i}{2\sqrt{r}} \quad \text{and} \quad \sigma_{ij} = \gamma_i \quad \text{for } i \ne j \quad (i, j = 1, 2). \tag{8.81}$$

Consequently we have the following result.

Theorem 8.20. *Assume that*

$$\gamma_i < a_i^2, \quad \gamma_i < \frac{1}{2Tc_1c_2} \quad \textit{for } i = 1, 2,$$

and there exists $r > 0$ with

$$r \le \min\left\{ \left(\frac{\alpha_i}{a_i^2 - \gamma_i} \right)^2 : i = 1, 2 \right\},$$

such that the matrix $M = \left[\sigma_{ij} / \left(a_i a_j\right)\right]_{1 \le i,j \le 2}$ where σ_{ij} are given by (8.81) satisfies (8.78). Then (8.80) has a unique T-periodic solution (u, v) such that $u(t) \ge r$ and $v(t) \ge r$ for every $t \in [0, T]$, which is a Nash-type equilibrium of the pair of corresponding energy functionals.

Proof. The existence follows from Theorem 8.19 and the uniqueness is a consequence of the Perov contraction property of the operator N. □

In particular, if $a_1 = a_2 =: a$ (when $c_1 = c_2 =: c$) and $\alpha_1 = \alpha_2 =: \alpha$, the assumptions of Theorem 8.20 reduce to the following ones:

$$\gamma_i < \frac{1}{2Tc^2}, \quad r \le \frac{\alpha^2}{\left(a^2 - \min\{\gamma_1, \gamma_2\}\right)^2}$$

and

$$4\left(a^4 - \gamma_1\gamma_2\right) r - 4\alpha a^2 \sqrt{r} + \alpha^2 > 0$$

(the condition for M to satisfy (8.78)). We may choose

$$r = \frac{\alpha^2}{\left(a^2 - \min\{\gamma_1, \gamma_2\}\right)^2}$$

if

$$\min\{\gamma_1, \gamma_2\} > 2\sqrt{\gamma_1\gamma_2} - a^2.$$