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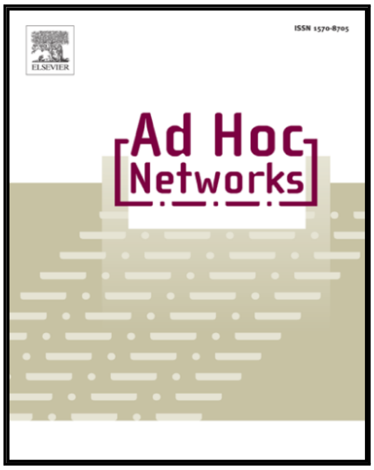

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Minimum Connected Dominating Sets in Heterogeneous 3D Wireless Ad Hoc Networks

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***Abstract*—The Minimum Connected Dominating Set (MCDS) problem is a fundamental problem in wireless ad hoc networks. The majority of approximation algorithms for this NP-hard problem follow a two-phased approach: The first phase is to construct a Maximal Independent Set (MIS), and the second phase is to connect the nodes in it. The upper bounds of the MISs play a key role in the design of constant approximation MCDS algorithms. This paper considers this problem for 3D heterogeneous ad hoc networks, where the transmission ranges of nodes are allowed to be different. We prove upper bounds of MISs with two classical mathematical problems, the Spherical Code Problem and the Sphere Packing Problem. When the transmission range ratio (the ratio of the maximum transmission range over the minimum transmission range) is $(1, 1.023]$, $(1.023, 1.055]$, $(1.055, 1.082]$, ... , we reduce the MIS upper bounds from the best-known results $22|OPT|+1$, $23|OPT|+1$, $24|OPT|+1$, ... , to $12|OPT|+1$, $13|OPT|+1$, $14|OPT|+1$, ... , where OPT is an optimal CDS and $|OPT|$ is the size of OPT. With the bounds of MISs, the approximation ratio of MCDS algorithms can be reduced from 25.02 to 16.02 in heterogeneous 3D wireless ad hoc networks.**

***Index Terms*—maximal independent set, minimum connected dominating set, wireless ad hoc network, heterogeneous**

I. Introduction

Connected dominating sets (CDSs) are used to serve as virtual backbones in wireless ad hoc networks [1]–[4]. For a wireless ad hoc network abstracted as a graph $G = (V, E)$, a connected subset $C \subset V$ is a CDS of G, if (1) the subgraph induced by C is connected, and (2) for any node v in $V \backslash C$, there exists a node u in C such that $uv \in E$. A node in the CDS is called a dominator and a non-CDS node is called dominated. The dominators serve as relay nodes in the network and form a virtual backbone. Naturally, a small virtual backbone brings up less signal interference and less energy consumption. Therefore, many researches have focused on the *minimum connected dominating set* (MCDS) problem. Since the MCDS problem has been proven NP-hard [5], approximation algorithms are used to solve this problem.

Fig. 1. The MIS-based MCDS algorithms have constant approximation ratios, which can be proved through three steps: First, proving the independence number is bounded by a constant; Second, proving the upper bounds of MISs; Third, proving the upper bounds of the constructed CDSs.

The majority of existing MCDS approximation algorithms follow a two-phased approach [6]–[8]: In the first phase, a *maximal independent set* (MIS) is constructed; In the second phase, nodes in the MIS are connected to form a CDS. In graph theory, an *independent set* (IS) of $G = (V, E)$ is a set of nodes in V such that no two of which are adjacent, an MIS is an IS that is not a subset of any other IS. Existing researches have shown that the MIS-based, two-phased MCDS algorithms can achieve constant approximation ratios in *unit disk graphs* (UDGs) [8], [9], general *disk graphs* (DGs) [7], [10], [11], and *unit ball graphs* (UBGs) [12]–[15]. But few work has been done on the MCDS problem in general *ball graphs* (BGs), where nodes in a BG are considered to cover balls with different radii, and two nodes are adjacent if they are in the ball of each other. BGs can represent heterogeneous wireless ad hoc networks with different transmission ranges of nodes.

This paper shows that the MIS-based MCDS algorithms can also achieve constant approximation ratios in BGs. As shown in Fig. 1, existing works for UDGs, DGs, and UBGs prove that the MIS-based MCDS algorithms have constant approximation ratios through three steps:

(1) First, proving that the *independence number* is bounded by a constant, where the independence number is defined as the maximum number of independent neighbors of a node. Two neighbors of a node are independent if they are not adjacent. In UDGs, it can be proved the independence

TABLE I
UPPER BOUNDS OF MISS IN EXISTING WORKS

Network model	Upper bounds of MISs	References
UDG	$3.399 \cdot \vert OPT\vert + 4.874$	[9]
DG	$(4 + 8\lceil\log_{\frac{1+\sqrt{5}}{2}} R\rceil) \cdot \vert OPT\vert + 1$	[7]
UBG	$10.917 \cdot \vert OPT\vert + 1.083$	[12]

number is 5 [16]. In UBGs, Kim et al. in [12] show that the independence number is 12 using a famous mathematical problem, the *kissing number problem*, where a kissing number is defined as the number of non-overlapping unit spheres that can be arranged such that each touch another given unit sphere. In DGs, the best-known upper bound of the independence number is $5 + 8\lceil log_{\frac{1+\sqrt{5}}{2}} R\rceil$ proposed by Wang et al. in [7], where R is the *transmission range ratio* that is defined as the ratio of the maximum transmission range over the minimum transmission range in the network.

(2) Second, proving the upper bounds of MISs with the upper bounds of the independence number. When the independence number is bounded by constants, it can be proved that any MIS is constant times bounded by $|OPT|$, where $|OPT|$ denotes the size of an optimal CDS. Table I shows the best-known upper bounds of MISs in existing works.

(3) Third, proving the upper bounds of the constructed CDSs. With the upper bounds of MISs, we can prove that the size of a CDS constructed by an MIS-based algorithm is also constant times bounded by $|OPT|$, where an MIS-based algorithm first constructs an MIS as a dominating set, and then adds additional nodes to connect the MIS to form a CDS. That is, an MIS-based MCDS algorithm has constant approximation ratio.

Consequently, the upper bounds of the independence number and the upper bounds of MISs play the key roles in the design of constant-approximation MCDS algorithms. In this paper, we first compute the upper bounds of the independence number for BGs with two classical mathematical problems, the *spherical code problem* and the *sphere packing problem*. Specially, when the transmission range ratio is close to 1, we prove that 13 is the tightest upper bound of the independence number, which means the independence number is exactly 13. Then, with the upper bounds of the independence number, we propose the upper bounds of MISs for BGs as shown in Table II. Furthermore, we show that the MIS-based MCDS algorithms can achieve constant approximation ratios in BGs. We reduce the approximation ratio from the best-known result 25.02 [11] to 16.02 when the transmission range ratio is close to 1.

The rest of this paper is organized as follows: Section II briefly reviews existing works about the MCDS problem. Some preliminaries about the spherical code problem and the sphere packing problem are introduced in Section III. In Section IV, we propose the upper bounds of the independence number for BGs. In Section V, we show how to compute the MIS bounds, and prove the constant approximation ratios of MIS-based MCDS algorithms for BGs. At last, we conclude this paper in Section VI.

TABLE II
THE PROPOSED UPPER BOUNDS OF MISS FOR BGS

R	Upper bounds of MISs in BGs
$1 \sim 1.023$	$12 \cdot \vert OPT\vert + 1$
$1.023 \sim 1.055$	$13 \cdot \vert OPT\vert + 1$
$1.055 \sim 1.082$	$14 \cdot \vert OPT\vert + 1$
$1.082 \sim 1.115$	$15 \cdot \vert OPT\vert + 1$
$1.115 \sim 1.146$	$16 \cdot \vert OPT\vert + 1$
$1.146 \sim 1.540$	$(\lfloor 0.780 \cdot (2R+1)^3\rfloor - 1) \cdot \vert OPT\vert + 1$
$1.540 \sim +\infty$	$(16 + 36 \cdot \lceil log_{\frac{1+\sqrt{5}}{2}}(0.872 \cdot R)\rceil) \cdot \vert OPT\vert + 1$

II. RELATED WORK

The MCDS problem is a fundamental problem in graph theory and wireless ad hoc networks. Besides the virtual backbone construction, CDSs have many applications, including energy harvest [17], data aggregation [18], [19], etc. In this section, we briefly review existing MCDS construction algorithms. Interested readers can refer to the surveys [20], [21] for more details about the MCDS problem.

Guha and Khuller in [22] firstly studied the MCDS approximation algorithms in general graphs. They proposed two centralized, polynomial-time algorithms with approximation ratio $O(H(\Delta))$, where Δ is the maximum degree of the graph and H is a harmonic function. Afterwards, a lot of effort was taken on the MCDS problem in UDGs. Most of these researches follow the MIS-based, two-phased approaches to devise constant-approximation MCDS algorithms. Wan et al. in [23] proposed the first distributed MCDS algorithm with constant approximation ratio of 8. Their algorithm first generates a two-hops MIS, and then connects the MIS to form a CDS. Some later researches [4], [6], [8], [9], [16] follow this work to obtain better approximation ratios. To our knowledge, the best-known approximation ratio of MCDS algorithms in UDGs is $(4.8 + \ln 5)$ [4]. For 3D wireless ad hoc networks, Kim et al. in [12] proposed an MCDS algorithm with approximation ratio 14.937 in UBGs. Gao et al. in [14] presented more discussions to refine the proof in [12].

Another line of researches focused on MCDS algorithms optimizing other parameters of the network, such as the routing path length, load balancing, fault tolerance, etc. Kim et al. in [24] proposed two algorithms to construct CDSs with bound diameters, where the diameter of a graph is the longest shortest path in it. [25]–[27] studied CDS algorithms to optimize the routing paths between any pair of nodes. Xin et al. in [28] proposed CDS algorithms to optimize latency of networks with acoustic communications. He et al. in [29] proposed a CDS algorithm to balance the load of backbone nodes. The construction of k-connected m-dominating sets to generate fault-tolerant virtual backbones has been studied in [2], [3], [30] . Besides, some researches focused on the MCDS problem under other network models, such as the cooperative communication model [31], the beeping model [32], and in battery-free networks [1].

For heterogeneous wireless ad hoc networks, Thai et al. in [10] proved the first upper bound of MISs in DGs and proposed an MIS-based, constant-approximation MCDS algorithm. Wang et al. in [7] improved Thai's results through some geometric methods and obtained a better approximation ratio. Bai et al. in [11] further improved the approximation ratio of MIS-based MCDS algorithms in DGs referencing the classical circle packing problem. Further, for heterogeneous 3D wireless ad hoc networks, Bai et al. in [11] discussed about the upper bound of MISs for BGs using the sphere packing problem. However, we show in this paper that their bounds in BGs are rather loose, and proposed better results that are close to the tight bounds. With the proposed bounds of MISs, we significantly improve the approximation ratio of MCDS algorithms in BGs.

III. Preliminaries

To compute the independence number N_{id} for BGs, we first introduce two mathematical problems, the spherical code problem and the sphere packing problem.

A. The Spherical Code Problem

The spherical code problem studies how can n points be distributed on a unit sphere such that the minimum distance between any pair of points is maximized. The spherical code problem has not been completely solved. However, an upper bound was given in [33] as shown in Theorem 1.

Theorem 1. *For n points on a unit sphere, there always exist two points whose distance d is*

$$d \leq \sqrt{4 - \csc^2\left(\frac{\pi \cdot n}{6(n-2)}\right)}.$$

Proof. See [33]. □

According to Theorem 1, we have Corollary 1.

Corollary 1. *For n points on a unit sphere, if the distance between any pair of points is bigger than d, then*

$$n \leq \frac{\pi}{3(arccsc\sqrt{4-d^2} - \frac{\pi}{6})} + 2$$

Corollary 2 generalizes Corollary 1 to general spheres.

Corollary 2. *For n points on a sphere, if the angle between any pair of points and the center is bigger than θ, then*

$$n \leq \frac{\pi}{3(arccsc\sqrt{4-(2\sin\frac{\theta}{2})^2} - \frac{\pi}{6})} + 2$$

The upper bound shown in Theorem 1 is not tight. Bachoc et al. in [34] proposed some better bounds as shown in Theorem 2.

Theorem 2. *For n points deployed on a unit sphere such that the minimum distance between any pair of points is maximized, let θ denote the angle between the center of the sphere and the closest pair of nodes on the sphere, then θ is upper bounded by Table III.*

Proof. See [34]. □

TABLE III
Upper Bounds for the Spherical Code Problem

n	Upper bounds on θ (degree)
13	58.50
14	56.58
15	55.03
16	53.27
17	51.69

TABLE IV
Lower Bounds for the Spherical Code Problem

n	Lower bounds on θ (degree)
12	63.4349488
13	57.1367031
14	55.6705700
15	53.6578501
16	52.2443957
17	51.0903285

In addition to upper bounds, some previous works try to find lower bounds for the spherical code problem. In other words, these works focus on optimizing the deployment for n nodes on a unit sphere. Through numerical approaches, these works obtained lower bounds for the spherical code problem as shown in Theorem 3.

Theorem 3. *For n points on a unit sphere, there exists a deployment such that the minimum angle θ between any two points on the sphere and the center of the sphere is as Table IV.*

Proof. See [35]. □

B. The Sphere Packing Problem

The sphere packing problem studies how to arrange non-overlapping spheres within a given containing space, such that the spheres fill as large a proportion of the space as possible. In this paper, we focus on a special instance of the general sphere packing problem, the *sphere packing in a sphere problem*, which studies how many unit spheres can be packed in a given sphere.

As well as the spherical code problem, the sphere packing in a sphere problem has not been completely solved either. However, an upper bound for the sphere packing in a sphere problem as shown in Theorem 4 has been proved.

Theorem 4. *If $n > 1$, there are n non-overlapping unit spheres packing in another given sphere, then the density of this packing is always less than $0.77963\cdots$.*

Proof. See [36], page 875. □

IV. THE INDEPENDENCE NUMBER IN BALL GRAPHS

We propose the upper bounds of the independence number for BGs in this section. Three methods are introduced to compute the independence number. The spherical code problem is used in the first and the third methods, and the sphere packing problem is used in the second method. By calculations, we found that the three methods respectively perform better when $R \in (1, 1.146]$, $R \in (1.146, 1.540]$, and $R \in (1.540, +\infty]$. Therefore, we compute the independence number respectively for these three ranges. Our results are compared with the best-known results at the end of this section.

Without loss of generality, we suppose that the minimum transmission range of nodes is 1 and the maximum transmission range of nodes is R. Thus, the upper bound of the independence number equals to the upper bound on number of independent neighbors of a node with transmission range R. This is the main idea to compute the upper bounds of the independence number in this section.

A. Upper Bounds of the independence number when $R \in (1, 1.146]$

In the rest of this paper, we use N_{id} to denote the independence number. This subsection proves upper bounds for N_{id} when $R \in (1, 1.146]$ with the spherical code problem. The main results are presented in Theorem 5. The main idea to prove Theorem 5 is to divide the ball with radius R into two sub-volumes by the sphere with radius 0.5, and then consider the upper bounds of independent nodes in each sub-volume.

Another major result in this section, as shown in Theorem 6, is that we prove 13 is a tight upper bound of the independence number when $R \in (1, 1.023]$, which means the independence number is exactly 13 when $R \in (1, 1.023]$. Accordingly, the MIS bound proposed later is tight when the transmission range ratio $R \to 1^+$.

Theorem 5. *When $R \in (1, 1.146]$, N_{id} has upper bounds*

$$\begin{cases} 13, & R \in (1, 1.023]; \\ 14, & R \in (1.023, 1.055]; \\ 15, & R \in (1.055, 1.082]; \\ 16, & R \in (1.082, 1.115]; \\ 17, & R \in (1.115, 1.146]. \end{cases}$$

Proof. Hereafter in this paper, for two spheres τ_i and τ_j with the same center, we use $D(\tau_i, \tau_j)$ to denote the volume outside the smaller sphere τ_i and inside the bigger sphere τ_j. Specifically, $D(\tau_i, c)$ denotes the ball inside sphere τ_i centered at c.

Consider a node c with transmission range R as Fig. 2. We divide τ_2 into two parts: $D(\tau_1, c)$ and $D(\tau_1, \tau_2)$. We discuss upper bounds on number of independent neighbors of c in $D(\tau_1, c)$ and $D(\tau_1, \tau_2)$ respectively.

Observe that $1.146 < \frac{1+\sqrt{17}}{4}$, according to Lemma 1 (proved later), for any two nodes x and y which are independent neighbors of c, if $x, y \in D(\tau_1, \tau_2)$, then we have

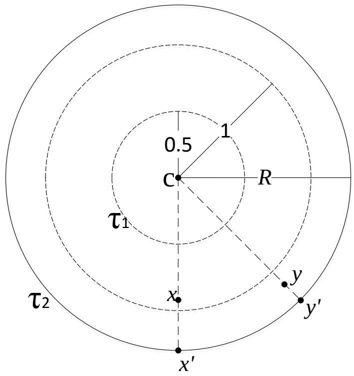

Fig. 2. On the proof of Theorem 5. τ_1 denotes the sphere centered at c with radius 0.5. τ_2 denotes the sphere centered at c with radius R. The ball with radius $R \in (1, 1.146]$ is divided into two parts by τ_1.

$$\angle xcy > 2arcsin\frac{1}{2R},$$

when $R \in (1, 1.146]$.

Let x' denote the point intersected by line cx and sphere τ_2. Let y' denote the point intersected by line cy and sphere τ_2. Hence,

$$\angle x'cy' > 2arcsin\frac{1}{2R}.$$

That is to say, if there are n independent neighbors of c in $D(\tau_1, \tau_2)$, then there exist n points on sphere τ_2 such that the angle between any pair of them and c is bigger than $2arcsin\frac{1}{2R}$.

Therefore, according to Theorem 2, when $R \le \frac{1}{2sin\frac{\theta}{2}}$, a group of upper bounds for number of independent neighbors of c in $D(\tau_1, \tau_2)$ are obtained as:

$$\begin{cases} 12, & R \in (1, 1.023]; \\ 13, & R \in (1.023, 1.055]; \\ 14, & R \in (1.055, 1.082]; \\ 15, & R \in (1.082, 1.115]; \\ 16, & R \in (1.115, 1.146]. \end{cases}$$

Moreover, it can be seen that in $D(\tau_1, c)$, there is at most 1 independent neighbor of c since the distance between each pair of independent nodes has to be more than 1.

This completes the proof. □

Lemma 1. *If x and y are two independent neighbors of c, $R \in (1, \frac{1+\sqrt{17}}{4}]$, $0.5 < |cx| \le R$ and $0.5 < |cy| \le R$, then*

$$\angle xcy > 2arcsin\frac{1}{2R}.$$

Proof. Since x and y are independent, we have

$$|xy| > 1.$$

As shown in Fig. 3, let x'' denote the point intersected by line cx and sphere τ_1, let y'' denote the point intersected by line cy and sphere τ_1.

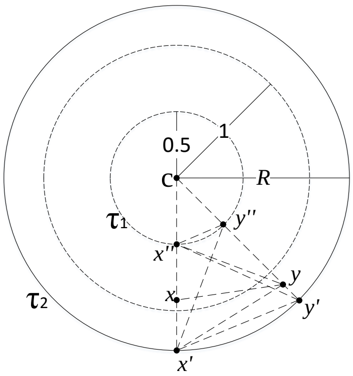

Fig. 3. On the proof of Lemma 1.

In $\triangle x'x''y$, according to Lemma 2 (proved later), either

$$|x''y| \geq |xy| > 1$$

or

$$|x'y| \geq |xy| > 1.$$

(i). If $|x''y| > 1$, in $\triangle y''x''y'$, according to Lemma 2, we have that either

$$|x''y''| \geq |x''y| > 1$$

or

$$|x''y'| \geq |x''y| > 1.$$

However, we know that $|x''y''| < 1$, hence,

$$|x''y'| > 1.$$

That is,

$$|x'y''| > 1.$$

(ii). If $|x'y| > 1$, in $\triangle y''x'y'$, according to Lemma 2, we have that either

$$|x'y''| \geq |x'y| > 1$$

or

$$|x'y'| \geq |x'y| > 1.$$

Overall, according to (i) and (ii), we have that either

$$|x'y''| > 1$$

or

$$|x'y'| > 1.$$

Now we prove by contradiction that if $|x'y''| > 1$, then we also have $|x'y'| > 1$.

Assume that $|x'y''| > 1$ and $|x'y'| \leq 1$. According to the cosine formula, in $\triangle cx'y''$, we have

$$\begin{aligned}\cos\angle x'cy' &= \frac{|cx'|^2 + |cy''|^2 - |x'y''|^2}{2|cx'||cy''|} \\ &= \frac{R^2 + 0.5^2 - |x'y''|^2}{2 \cdot R \cdot 0.5} \\ &= \frac{R^2 + \frac{1}{4} - |x'y''|^2}{R}\end{aligned}$$

and in $\triangle cx'y'$, we have

$$\begin{aligned}\cos\angle x'cy' &= \frac{|cx'|^2 + |cy'|^2 - |x'y'|^2}{2|cx'||cy'|} \\ &= \frac{R^2 + R^2 - |x'y'|^2}{2 \cdot R \cdot R} \\ &= \frac{2R^2 - |x'y'|^2}{2R^2}.\end{aligned}$$

Hence,

$$\frac{R^2 + \frac{1}{4} - |x'y''|^2}{R} = \frac{2R^2 - |x'y'|^2}{2R^2}$$

That is,

$$|x'y'|^2 = 2R^2 - 2R^3 + (2|x'y''|^2 - \frac{1}{2})R.$$

Since we have assumed that $|x'y'| \leq 1$, we have

$$2R^2 - 2R^3 + (2|x'y''|^2 - \frac{1}{2})R \leq 1.$$

Moreover, we have assumed that $|x'y''| > 1$, hence,

$$2R^2 - 2R^3 + \frac{3}{2}R < 2R^2 - 2R^3 + (2|x'y''|^2 - \frac{1}{2})R \leq 1.$$

Let

$$f(R) = 2R^2 - 2R^3 + \frac{3}{2}R - 1,$$

then $f(R) < 0$.

We have

$$f'(R) = 4R - 6R^2 + \frac{3}{2}.$$

With simple calculations, we have

$$f'(R) \geq 0, \ when \ R \in [\frac{1}{2}, \frac{2 + \sqrt{13}}{6}],$$

and

$$f'(R) \leq 0, \ when \ R \in [\frac{2 + \sqrt{13}}{6}, \frac{1 + \sqrt{17}}{4}].$$

That is, $f(R)$ is monotonically increasing when $R \in [\frac{1}{2}, \frac{2+\sqrt{13}}{6}]$ and $f(R)$ is monotonically decreasing when $R \in [\frac{2+\sqrt{13}}{6}, \frac{1+\sqrt{17}}{4}]$.

Besides, we have $f(\frac{1}{2}) = 0$ and $f(\frac{1+\sqrt{17}}{4}) = 0$, hence,

$$f(R) \geq 0, \ when \ R \in (1, \frac{1 + \sqrt{17}}{4}].$$

This completes the proof that if $|x'y''| > 1$, then we also have $|x'y'| > 1$.

Therefore, we have $|x'y'| > 1$.

In $\triangle cx'y'$, now it can be proved that

$$\angle x'cy' > 2arcsin\frac{1}{2R}.$$

This completes the proof. □

Lemma 2. *Given a $\triangle abc$ (as shown in Fig. 4), d is a point on edge bc, then either $ab > ad$ or $ac > ad$.*

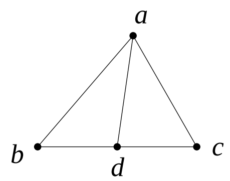

Fig. 4. On the proof of Lemma 2.

Proof. Since

$$\angle adb + \angle adc = \pi,$$

we have that either

$$\angle adb \geq \pi/2$$

or

$$\angle adc \geq \pi/2.$$

Hence, either

$$\angle adb \geq \angle abc$$

or

$$\angle adc \geq \angle acb.$$

That is, $ab > ad$ or $ac > ad$. □

In Theorem 6, we prove that the proposed upper bounds of N_{id} is tight when the transmission range ratio is close to 1. In real-world wireless ad hoc networks, the transmission range is usually not very big, where the proposed upper bounds of N_{id} are close to the tight bounds.

Theorem 6. *When $R \in (1, 1.023]$, N_{id} is 13.*

Proof. To prove the theorem, we only need to prove that for any $\delta > 0$, when $R = 1 + \delta$, 13 independent neighbors can be deployed for a given node c.

According to Theorem 3, 12 points $x_i(1 \leq i \leq 12)$ can be deployed in the unit sphere centered at c, such that the distance between any pair of x_i is bigger than 1. We use $(1, \varphi_i, \theta_i)$ to denote the polar coordinates of x_i.

Let $n_i(1 \leq i \leq 13)$ be 13 nodes that (1) for $1 \leq i \leq 12$, n_i is located at $(1 + i \cdot \varepsilon, \varphi_i, \theta_i)$, where $\varepsilon \rightarrow \frac{1}{+\infty}$, and (2) for $i = 13$, n_{13} is at center c.

Let the transmission range of $n_i(1 \leq i \leq 13)$ be $|cn_i|$. Then it can be verified that $n_i(1 \leq i \leq 13)$ are independent, and they are all neighbors of node c. □

B. Upper Bounds of the independence number when $R \in (1.146, 1.540]$

The second method uses the sphere packing problem to compute the upper bounds for N_{id}. With simple calculations, it can be proved that the second method performs better than the other two methods when $R \in (1.146, 1.540]$. The main result of this subsection is shown in Theorem 7.

Theorem 7. *When $R \in (1.146, 1.540]$, N_{id} has upper bound*

$$\lfloor 0.780 \cdot (2R + 1)^3 \rfloor.$$

Proof. Let τ_1 denote the sphere centered at c with radius R. Consider a sphere τ_2 centered at c with radius $R + 0.5$. If there are N_{id} independent neighbors of c in τ_1, then there exist N_{id} non-overlapping spheres with radius 0.5 in τ_2. According to Theorem 4,

$$N_{id} \leq \frac{0.780 \cdot \frac{4}{3}\pi(R + 0.5)^3}{\frac{4}{3}\pi \cdot 0.5^3}$$

That is, $N_{id} \leq \lfloor 0.780 \cdot (2R + 1)^3 \rfloor$. □

C. Upper Bounds of the independence number when $R \in (1.540, +\infty]$

This subsection proposes the upper bounds of N_{id} when $R \in (1.540, +\infty]$. The main results of this subsection are shown in Theorem 8. The basic idea to prove this theorem is to divide the ball with radius R into sub-volumes with a group of inside spheres as shown in Fig. 5, and then respectively consider the upper bounds of the number of independent neighbors in each sub-volume.

The idea to compute the upper bounds of the number of independent neighbors in each sub-volume is as follows: We first prove that the angle between the center and a pair of independent neighbors in the same sub-volume is at least a constant, and then prove that there are at most constant independent neighbors in each sub-volume with the spherical code problem shown in Theorem 1.

Our division of the coverage ball relies on a constant β. Accordingly the obtained upper bound of the independence number is a function of β. We at last discuss that with what value of β we can obtain the smallest upper bound of the independence number.

Theorem 8. *When $R \in (1.540, +\infty]$, N_{id} has upper bound*

$$17 + 36 \cdot \lceil log_{\frac{1+\sqrt{5}}{2}}(0.872 \cdot R) \rceil.$$

Proof. As shown in Fig. 5, suppose c is a node with transmission range $R \in (1.540, +\infty]$. We divide the coverage ball of c by following spheres

$$\{\tau_1, \tau_2, \cdots, \tau_{k-1}, \tau_k\},$$

which are centered at c with radii $\{r_1, r_2, \cdots, r_{k-1}, r_k\}$, where

$$r_1 = 1.146,$$
$$r_{i+1} = \beta \cdot r_i, (1 \leq i \leq k - 2),$$
$$r_k = R$$

and β is a constant.

Notice that the division of the coverage ball relies on a constant β. The rest of the proof follows two phases. In the first phase, we prove the upper bound of independence number which is a function of β. In the second phase, we show that when $\beta = \frac{1+\sqrt{5}}{2}$, the best upper bound can be obtained.

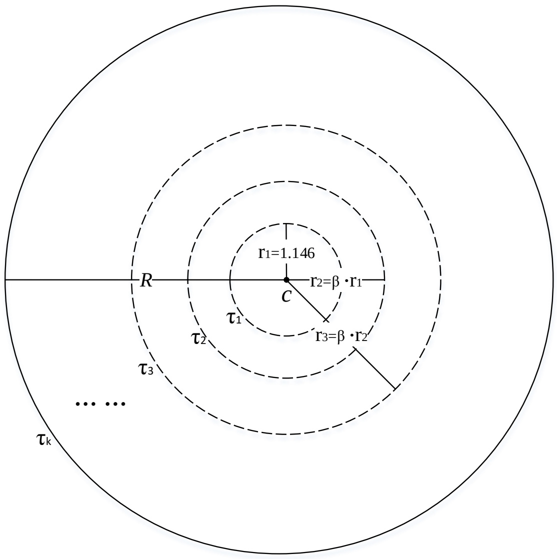

Fig. 5. On the proof of Theorem 8. A ball with radius $R \in (1.540, +\infty]$ is divided by spheres $\tau_i (1 \le i \le k)$.

According to Lemma 5 (proved later), for any two pair of independent nodes of c in $D(\tau_i, \tau_{i+1})$, $(1 \le i < k)$, the angle between them and c is bigger than

$$\theta = \begin{cases} 2arcsin\dfrac{1}{2\beta}, \ when \ \beta \in (1, \dfrac{1+\sqrt{5}}{2}]; \\ arccos\dfrac{\beta}{2}, \ when \ \beta \in [\dfrac{1+\sqrt{5}}{2}, 2). \end{cases}$$

Therefore, according to Corollary 2, in $D(\tau_i, \tau_{i+1})$, $(1 \le i < k)$, there are at most

$$\frac{\pi}{3(arccsc\sqrt{4-(2\sin\frac{\theta}{2})^2} - \frac{\pi}{6})} + 2$$

independent neighbors of c.

Overall, we have: (1) According to Theorem 5, in sphere τ_1 there are at most 17 independent neighbors of c; (2) in each $D(\tau_i, \tau_{i+1})$, $(1 \le i < k)$, there are at most $\frac{\pi}{3(arccsc\sqrt{4-(2\sin\frac{\theta}{2})^2} - \frac{\pi}{6})} + 2$ independent neighbors of c; (3) $k \le \lceil log_\beta(\frac{R}{1.146}) \rceil$. Therefore, N_{id} has upper bounds

$$17 + (\frac{\pi}{3(arccsc\sqrt{4-(2\sin\frac{\theta}{2})^2} - \frac{\pi}{6})} + 2) \cdot \lceil log_\beta(\frac{R}{1.146}) \rceil.$$

When $\beta = \frac{1+\sqrt{5}}{2}$, N_{id} has upper bounds

$$17 + 36 \cdot \lceil log_{\frac{1+\sqrt{5}}{2}}(0.872 \cdot R) \rceil.$$

□

Remark: In the proof of Theorem 8, we compute the upper bounds of N_{id} when $\beta = \frac{1+\sqrt{5}}{2}$ because let

$$\varphi(\beta) = 17 + (\frac{\pi}{3(arccsc\sqrt{4-(2sin\frac{\theta}{2})^2} - \frac{\pi}{6})} + 2) \cdot \lceil log_\beta(\frac{R}{1.146}) \rceil,$$

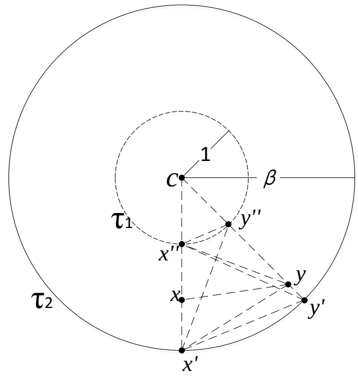

Fig. 6. On the proof of Lemma 3 and Lemma 4.

then according to Lemma 6 and Lemma 7, $\varphi(\beta)$ is monotonically decreasing when $\beta \in (1, \frac{1+\sqrt{5}}{2}]$ and $\varphi(\beta)$ is monotonically increasing when $\beta \in [\frac{1+\sqrt{5}}{2}, 2)$. Therefore, we can get the best upper bound when $\beta = \frac{1+\sqrt{5}}{2}$.

To prove Lemma 5, let us introduce Lemma 3 and Lemma 4 first.

Lemma 3. *When $\beta \in (1, \frac{1+\sqrt{5}}{2}]$, if x and y in $D(\tau_1, \tau_2)$ are two independent neighbors of node c, where τ_1 and τ_2 are two spheres centered at c with radii 1 and β, then*

$$\angle xcy > 2arcsin\frac{1}{2\beta}.$$

Proof. Since x and y are independent, we have $|xy| > 1$. As shown in Fig. 6, it can be proved similarly to Lemma 1 that either $|x'y'| > 1$ or $|x'y''| > 1$.

Now we prove by contradiction that when $\beta \in (1, \frac{1+\sqrt{5}}{2}]$, if $|x'y''| > 1$, then we also have $|x'y'| > 1$. Assume that when $|x'y''| > 1$, $|x'y'| \le 1$.

According to the cosine formula, in $\triangle cx'y''$, we have

$$\cos\angle x'cy' = \frac{|cx'|^2 + |cy''|^2 - |x'y''|^2}{2|cx'||cy''|} = \frac{\beta^2 + 1 - |x'y''|^2}{2\beta}$$

and in $\triangle cx'y'$, we have

$$\cos\angle x'cy' = \frac{|cx'|^2 + |cy'|^2 - |x'y'|^2}{2|cx'||cy'|} \quad \frac{\beta^2 + \beta^2 - |x'y'|^2}{2 \cdot \beta \cdot \beta} \quad \frac{2\beta^2 - |x'y'|^2}{2\beta^2}.$$

Hence,

$$|x'y''|^2 = \beta^2 + 1 - 2\beta + \frac{|x'y'|^2}{\beta}.$$

Since we have assumed that $|x'y''| > 1$, $|x'y'| \leq 1$, we have

$$\beta^2 + 1 - 2\beta + \frac{1}{\beta} \geq \beta^2 + 1 - 2\beta + \frac{|x'y'|^2}{\beta} = |x'y''|^2 > 1.$$

That is,

$$\beta^3 - 2\beta^2 + 1 > 0$$

Let $f(\beta) = \beta^3 - 2\beta^2 + 1$. Then $f(\beta) > 0$, and we have

$$f'(\beta) = 3\beta^2 - 4\beta.$$

With simple calculations, we have

$$f'(\beta) \leq 0, \; when \; \beta \in (1, \frac{4}{3}],$$

and

$$f'(\beta) \geq 0, \; when \; \beta \in [\frac{4}{3}, \frac{1+\sqrt{5}}{2}].$$

That is, $f(\beta)$ is monotonically decreasing when $\beta \in (1, \frac{4}{3})$ and $f(\beta)$ is monotonically increasing when $\beta \in [\frac{4}{3}, \frac{1+\sqrt{5}}{2}]$.

Besides, we have $f(1) = 0$ and $f(\frac{1+\sqrt{5}}{2}) = 0$, hence

$$f(\beta) \leq 0, \; when \; \beta \in (1, \frac{1+\sqrt{5}}{2}].$$

This completes the proof that when $\beta \in (1, \frac{1+\sqrt{5}}{2}]$, if $|x'y''| > 1$, then we also have $|x'y'| > 1$.

In $\triangle cx'y'$, with above results, it can be proved that

$$\angle x'cy' > 2arcsin\frac{1}{2\beta}.$$

□

Lemma 4. *When $\beta \in [\frac{1+\sqrt{5}}{2}, 2)$, if x and y in $D(\tau_1, \tau_2)$ are two independent neighbors of node c, where τ_1 and τ_2 are two spheres centered at c with radii 1 and β, then*

$$\angle xcy > arccos\frac{\beta}{2}.$$

Proof. Since x and y are independent, we have $|xy| > 1$. As shown in Fig. 6, it can be proved similarly to Lemma 1 that either $|x'y'| > 1$ or $|x'y''| > 1$.

Now we prove by contradiction that when $\beta \in [\frac{1+\sqrt{5}}{2}, 2)$, if $|x'y'| > 1$, then we also have $|x'y''| > 1$. Assume that when $|x'y'| > 1$, $|x'y''| \leq 1$.

As in Lemma 3, we have

$$|x'y''|^2 = \beta^2 + 1 - 2\beta + \frac{|x'y'|^2}{\beta}.$$

Therefore, let $f(\beta) = \beta^3 - 2\beta^2 + 1$, then

$$f(\beta) < 0 \; when \; \beta \in [\frac{1+\sqrt{5}}{2}, 2).$$

We have

$$f'(\beta) > 0, \; when \; \beta \in [\frac{1+\sqrt{5}}{2}, 2).$$

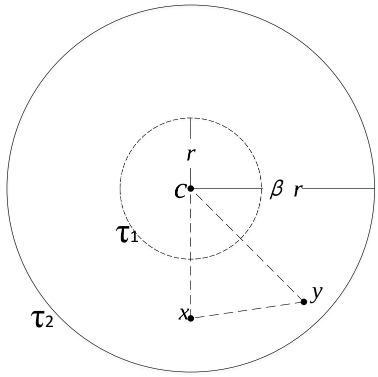

Fig. 7. On the proof of Lemma 5.

That is, $f(\beta)$ is monotonically increasing when $\beta \in [\frac{1+\sqrt{5}}{2}, 2)$.

Besides, we have $f(\frac{1+\sqrt{5}}{2}) = 0$, hence,

$$f(\beta) \geq 0, \; when \; \beta \in [\frac{1+\sqrt{5}}{2}, 2).$$

Therefore, when $\beta \in [\frac{1+\sqrt{5}}{2}, 2)$, if $|x'y'| > 1$, we also have $|x'y''| > 1$.

Therefore, we have $|x'y''| > 1$.

In $\triangle cx'y''$, with above results, we have

$$\angle xcy > arccos\frac{\beta}{2}.$$

□

Lemma 5. *If x and y in $D(\tau_1, \tau_2)$ are two independent neighbors of node c, where τ_1 and τ_2 are two spheres centered at c with radii r and $\beta \cdot r$, then*

$$\angle xcy > arccos\frac{\beta}{2}, \; when \; \beta \in [\frac{1+\sqrt{5}}{2}, 2);$$
$$\angle xcy > 2arcsin\frac{1}{2\beta}, \; when \; \beta \in (1, \frac{1+\sqrt{5}}{2}].$$

Proof. As shown in Fig. 7, since x and y are neighbors of c, we have that the transmission ranges of x and y are at least r.

Besides, x and y are independent, so we have $|xy| > r$. Thus, Lemma 5 can be proved according to Lemma 3 and Lemma 4. □

Lemma 6. *Let*

$$f(x) = \frac{\frac{\pi}{3(arccsc\sqrt{4-(2sin\frac{1}{2}(2arcsin\frac{1}{2x}))^2}-\frac{\pi}{6})} + 2}{\log x},$$

then $f(x)$ is monotonically decreasing when $x \in [1, \frac{1+\sqrt{5}}{2}]$.

Proof. See the appendix. □

Lemma 7. *Let*

$$f(x) = \frac{\frac{\pi}{3(arccsc\sqrt{4-(2sin\frac{1}{2}(2arccos\frac{x}{2}))^2}-\frac{\pi}{6})} + 2}{\log x},$$

TABLE V
COMPARISON FOR UPPER BOUNDS OF THE INDEPENDENCE NUMBER IN BGS

R	Upper bounds of N_{id} in this paper	Upper bounds of N_{id} in [11]
1.023	13	22
1.055	14	23
1.082	15	24
1.115	16	26
1.146	17	27
1.6	53	57
3	89	267
4.5	125	780
7	161	2632

then $f(x)$ is monotonically decreasing when $x \in [\frac{1+\sqrt{5}}{2}, 2]$.

Proof. See the appendix. □

D. Comparison with the Best-known Results

Table V compares the independence number N_{id} proposed in this section with the best-known results in [11]. Sen et al. in [11] proposed the upper bounds of N_{id} simply with the sphere packing problem. It can be seen that the results in this paper significantly improve existing results.

V. MIS BOUNDS AND MCDS CONSTRUCTION IN BALL GRAPHS

In this section, we first use the upper bounds of N_{id} proposed in the last section to compute the upper bounds of MISs. Second, we discuss about the approximation ratios of the MIS-based MCDS algorithms for BGs.

A. Upper Bounds of MISs

As an extension of Lemma 9 in [23], Theorem 9 extends the MIS bounds from UDGs to BGs.

Theorem 9. *The size of any IS is at most $(N_{id}-1)\cdot|OPT|+1$ for BGs.*

Proof. For a BG, let OPT be an optimal solution of the MCDS problem, and let T be a spanning tree of OPT. Suppose $v_1, v_2, \ldots, v_{|OPT|}$ is a preorder traversal of T.

Let I be any independent set of the BG. We construct a partition of I as follows.

First, for v_1,

(1) if $v_1 \in I$, let $I_1 = \{v_1\}$;

(2) if $v_1 \notin I$, let I_1 be the set of nodes in I that are adjacent to v_1.

Second, for any $2 \le i \le |OPT|$,

(1) if $v_i \in I - (I_1 \cup I_2 \cup \cdots \cup I_{i-1})$, let $I_i = \{v_i\}$;

(2) if $v_i \notin I - (I_1 \cup I_2 \cup \cdots \cup I_{i-1})$, let I_i be the set of nodes in $I - (I_1 \cup I_2 \cup \cdots \cup I_{i-1})$ that are adjacent to v_i.

Hence, $I_1, I_2, \ldots, I_{|OPT|}$ form a partition of I.

Since v_1 has at most N_{id} independent neighbors, we have $|I_1| \le N_{id}$.

For $2 \le i \le |OPT|$,

(1) if $I_i \ne \{v_i\}$, since at least one node in $\{v_1, v_2, \ldots, v_{i-1}\}$ is adjacent to v_i, we let v_* be a node in $\{v_1, v_2, \ldots, v_{i-1}\}$ that is adjacent to v_i. Then v_* is independent to nodes in I_i (otherwise some nodes in I_i will be divided into I_* in the partition step). Therefore, $\{v_*\} \cup I_i$ is a set of independent neighbors of v_i. Hence, for $2 \le i \le |OPT|$, $1 + |I_i| \le N_{id}$.

(2) if $I_i = \{v_i\}$, we also have $1 + |I_i| = 2 \le N_{id}$.

Therefore, for $2 \le i \le |OPT|$, $|I_i| \le N_{id} - 1$.

Moreover,

$$\begin{aligned} |I| &\le N_{id} + (N_{id}-1)\cdot(|OPT|-1) \\ &= (N_{id}-1)\cdot|OPT|+1. \end{aligned}$$

□

According to Theorem 9, when R is

$$(1, 1.023], (1.023, 1.055], (1.055, 1.082], \cdots,$$

we reduce the MIS upper bounds from

$$22|OPT|+1, 23|OPT|+1, 24|OPT|+1, \cdots,$$

in [11] to

$$12|OPT|+1, 13|OPT|+1, 14|OPT|+1, \cdots.$$

Details of the results are shown in Table II.

B. The MCDS Construction Algorithm

For the MCDS problem, we consider the algorithm proposed by Kim et al. in [12] in this section. In their algorithm, the construction of CDSs only relies on the topology information of the graph, and the graph is considered to be an UBG only when discussing the approximation ratio. That is, the algorithm can be trivally extended to BGs, and we only need to show that besides in UBGs, the algorithm can still guarantee constant approximation ratio in BGs.

The MCDS algorithm in [12] follows a two-phased approach: First, the algorithm constructs a two-hop MIS I, such that for any $I' \subset I$, the distance between I' and $I \backslash I'$ is exactly two hops. We have proved in this paper that $|I|$ is constantly bounded by $|OPT|$ in BGs. According to Theorem 4.2 and Lemma 4.4 in [12], given the two-hop maximal independent set I, the algorithm finds a set C that $|C| \le 4.02|OPT|$, and $I \cup C$ forms a CDS. In Theorem 10, we show that MCDS algorithm in [12] has constant approximation ratio in BGs.

Theorem 10. *The MIS-based MCDS algorithm in [12] has an approximation ratio $(N_{id}+3.02)$ in BGs.*

Proof. Let CDS be a connected dominating set constructed by the algorithm in [12], I be the MIS constructed in the algorithm, and C be the additional nodes used to connect I. According to Lemma 4.4 in [12],

$$\begin{aligned} |CDS| &\le |I| + |C| \\ &\le (N_{id}-1)\cdot|OPT| + 1 + 4.02\cdot|OPT| \\ &= (N_{id}+3.02)\cdot|OPT|+1. \end{aligned}$$

□

The approximation ratio of the algorithm in [12] in BGs is shown in Table VI. When the transmission ratio R is close to 1, this paper reduces the approximation ratio of the MCDS algorithm from 25.02 [11] to 16.02.

TABLE VI
APPROXIMATION RATIOS OF THE MIS-BASED MCDS ALGORITHM IN BGS

R	Approximation ratio in this paper	Approximation ratio in [11]
1.023	16.02	25.02
1.055	17.02	26.02
1.082	18.02	27.02
1.115	19.02	29.02
1.146	20.02	30.02
1.6	56.02	60.02
2	92.02	100.02
3	92.02	270.02
4	128.02	571.02

VI. CONCLUSION

This paper considers MISs and MCDSs in heterogeneous 3D wireless ad hoc networks abstracted as general ball graphs. With the help of two classical mathematical problems, the spherical code problem and the sphere packing problem, we prove that an MIS has upper bounds $12|OPT|+1$, $13|OPT|+1$, $14|OPT|+1$, $\cdots$, when the transmission range ratio is $(1, 1.023]$, $(1.023, 1.055]$, $(1.055, 1.082]$, $\cdots$, where $|OPT|$ is the size of an optimal CDS. Accordingly, we prove the MIS-based MCDS algorithm has approximation ratio 16.02, 17.02, $18.02, \cdots$. Our results significantly outperform the best-known results in existing works. When the transmission range ratio is in $(1, 1.023]$, we prove that the independence number is exactly 13, and $12|OPT|+1$ is a tight upper bound of the MIS.

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REFERENCES

[1] T. Shi, S. Cheng, J. Li, and Z. Cai, "Constructing connected dominating sets in battery-free networks," in *IEEE International Conference on Computer Communications*. IEEE, 2017, pp. 1–9.

[2] Z. Zhang, J. Zhou, Y. Mo, and D.-Z. Du, "Performance-guaranteed approximation algorithm for fault-tolerant connected dominating set in wireless networks," in *IEEE International Conference on Computer Communications*. IEEE, 2016, pp. 1–8.

[3] J. Zhou, Z. Zhang, S. Tang, X. Huang, Y. Mo, and D.-Z. Du, "Fault-tolerant virtual backbone in heterogeneous wireless sensor network," *IEEE/ACM Transactions on Networking*, vol. 25, no. 6, pp. 3487–3499, 2017.

[4] J. P. Mohanty, C. Mandal, and C. Reade, "Distributed construction of minimum connected dominating set in wireless sensor network using two-hop information," *Computer Networks*, vol. 123, pp. 137–152, 2017.

[5] B. N. Clark, C. J. Colbourn, and D. S. Johnson, "Unit disk graphs," *Discrete mathematics*, vol. 86, no. 1, pp. 165–177, 1990.

[6] J. P. Mohanty, C. Mandal, C. Reade, and A. Das, "Construction of minimum connected dominating set in wireless sensor networks using pseudo dominating set," *Ad Hoc Networks*, vol. 42, no. C, pp. 61–73, 2016.

[7] L. Wang, P.-J. Wan, and F. Yao, "Minimum cds in multihop wireless networks with disparate communication ranges," *IEEE Transactions on Mobile Computing*, vol. 12, no. 5, pp. 909–916, 2013.

[8] P.-J. Wan, L. Wang, and F. Yao, "Two-phased approximation algorithms for minimum cds in wireless ad hoc networks," in *International Conference on Distributed Computing Systems*. IEEE, 2008, pp. 337–344.

[9] Y. L. Du and H. W. Du, "A new bound on maximum independent set and minimum connected dominating set in unit disk graphs," *Journal of Combinatorial Optimization*, vol. 30, no. 4, pp. 1173–1179, 2015.

[10] M. T. Thai, F. Wang, D. Liu, S. Zhu, and D. Z. Du, "Connected dominating sets in wireless networks with different transmission ranges," *IEEE Transactions on Mobile Computing*, vol. 6, no. 7, pp. 721–730, 2007.

[11] S. Bai, X. Che, X. Bai, and X. Wei, "Maximal independent sets in heterogeneous wireless ad hoc networks," *IEEE Transactions on Mobile Computing*, vol. 15, no. 8, pp. 2023–2033, 2016.

[12] D. Kim, Z. Zhang, X. Li, W. Wang, W. Wu, and D. Z. Du, "A better approximation algorithm for computing connected dominating sets in unit ball graphs," *IEEE Transactions on Mobile Computing*, vol. 9, no. 8, pp. 1108–1118, 2010.

[13] S. Bai, Y. Liu, Z. Li, and X. Bai, "Detecting wormhole attacks in 3d wireless ad hoc networks via 3d forbidden substructures," *Computer Networks*, vol. 150, pp. 190–200, 2019.

[14] X. Gao, J. Li, and G. Chen, "A better approximation for constructing virtual backbone in 3d wireless ad-hoc networks," *Theoretical Computer Science*, vol. 607, no. 3, pp. 363–380, 2015.

[15] X. Bai, X. Wei, and S. Bai, "Flooding in 3-dimensional mobile ad hoc networks using 1-hop information," *International Journal of Sensor Networks*, vol. 26, no. 2, pp. 69–80, 2018.

[16] W. Wu, H. Du, X. Jia, Y. Li, and S. C.-H. Huang, "Minimum connected dominating sets and maximal independent sets in unit disk graphs," *Theoretical Computer Science*, vol. 352, no. 1-3, pp. 1–7, 2006.

[17] T. Shi, S. Cheng, Z. Cai, Y. Li, and J. Li, "Exploring connected dominating sets in energy harvest networks," *IEEE/ACM Transactions on Networking*, vol. 25, no. 3, pp. 1803–1817, 2017.

[18] L. Guo, Y. Li, and Z. Cai, "Minimum-latency aggregation scheduling in wireless sensor network," *Journal of Combinatorial Optimization*, vol. 31, no. 1, pp. 279–310, 2016.

[19] P. Wang, Y. He, and L. Huang, "Near optimal scheduling of data aggregation in wireless sensor networks," *Ad Hoc Networks*, vol. 11, no. 4, pp. 1287–1296, 2013.

[20] S. P., "A survey of connected dominating set algorithms for virtual backbone construction in ad hoc networks," *International Journal of Computer Applications*, vol. 143, no. 9, pp. 30–36, 2016.

[21] X. Cheng, F. Wang, and D.-Z. Du, "Connected dominating set," 2016.

[22] S. Guha and S. Khuller, "Approximation algorithms for connected dominating sets," *Algorithmica*, vol. 20, no. 4, pp. 374–387, 1998.

[23] P. J. Wan, K. M. Alzoubi, and O. Frieder, "Distributed construction of connected dominating set in wireless ad hoc networks," *Mobile Networks and Applications*, vol. 9, no. 2, pp. 141–149, 2004.

[24] D. Kim, Y. Wu, Y. Li, F. Zou, and D. Z. Du, "Constructing minimum connected dominating sets with bounded diameters in wireless networks," *IEEE Transactions on Parallel and Distributed Systems*, vol. 20, no. 2, pp. 147–157, 2009.

[25] L. Wu, H. Du, W. Wu, Y. Zhu, A. Wang, and W. Lee, "Ptas for routing-cost constrained minimum connected dominating set in growth bounded graphs," *Journal of Combinatorial Optimization*, vol. 30, no. 1, pp. 18–26, 2015.

[26] L. Ding, W. Wu, J. Willson, H. Du, and W. Lee, "Efficient virtual backbone construction with routing cost constraint in wireless networks using directional antennas," *IEEE Transactions on Mobile Computing*, vol. 11, no. 7, pp. 1102–1112, 2012.

[27] H. Du, W. Wu, Q. Ye, D. Li, W. Lee, and X. Xu, "Cds-based virtual backbone construction with guaranteed routing cost in wireless sensor networks," *IEEE Transactions on Parallel and Distributed Systems*, vol. 24, no. 4, pp. 652–661, 2013.

[28] X. Bai, X. Wei, and S. Bai, "Latency-optimal virtual backbone construction with acoustic communication in underwater sensor networks," *International Journal of Distributed Sensor Networks*, vol. 13, no. 11, 2017.

[29] J. He, S. Ji, Y. Pan, and Z. Cai, "Approximation algorithms for load-balanced virtual backbone construction in wireless sensor networks," *Theoretical Computer Science*, vol. 507, no. 7, pp. 2–16, 2013.

[30] A. Buchanan, J. S. Sung, S. Butenko, and E. L. Pasiliao, "An integer programming approach for fault-tolerant connected dominating sets," *Informs Journal on Computing*, vol. 27, no. 1, pp. 178–188, 2017.

[31] J. Wu, M. Cardei, F. Dai, and S. Yang, "Extended dominating set and its applications in ad hoc networks using cooperative communication," *IEEE Transactions on Parallel and Distributed Systems*, vol. 17, no. 8, pp. 851–864, 2006.

[32] J. Yu, L. Jia, D. Yu, and G. Li, "Minimum connected dominating set construction in wireless networks under the beeping model," in *Computer Communications*, 2015, pp. 972–980.

[33] L. F. Toth, "On the densest packing of spherical caps," *American Mathematical Monthly*, vol. 56, no. 5, pp. 330–331, 1949.

[34] C. Bachoc and F. Vallentin, "New upper bounds for kissing numbers from semidefinite programming," *Journal of the American Mathematical Society*, vol. 21, no. 3, pp. 909–924, 2008.

[35] N. J. A. Sloane, "Spherical codes," http://neilsloane.com/packings/index.html, 2017.

[36] K. Bezdek, "Sphere packings revisited," *European Journal of Combinatorics*, vol. 27, no. 6, pp. 864–883, 2006.

Appendix 1: Proof of Lemma 6

Proof. We have

$$f'(x) = -\frac{2+\frac{\pi}{3(-\frac{\pi}{6}+arccsc\sqrt{4-\frac{1}{x^2}})}}{x(\log x)^2} + \frac{\pi}{3\sqrt{1-\frac{1}{4-\frac{1}{x^2}}}(4-\frac{1}{x^2})^{\frac{3}{2}}x^3(-\frac{\pi}{6}+arccsc\sqrt{4-\frac{1}{x^2}})^2\log x}$$

Let

$$u(x) = 2+\frac{\pi}{3(-\frac{\pi}{6}+arccsc\sqrt{4-\frac{1}{x^2}})};$$
$$v(x) = \frac{1}{x\cdot\log x};$$
$$s(x) = \frac{\pi}{3\sqrt{1-\frac{1}{4-\frac{1}{x^2}}}(4-\frac{1}{x^2})^{\frac{3}{2}}x^3};$$
$$t(x) = \frac{1}{(-\frac{\pi}{6}+arccsc\sqrt{4-\frac{1}{x^2}})^2}.$$

Then $f'(x) = (-u(x)\cdot v(x) + s(x)\cdot t(x))\cdot\frac{1}{\log x}$.

To prove that $f(x)$ is monotonically decreasing when $x \in [1, \frac{1+\sqrt{5}}{2}]$, we only need to prove that $f'(x) < 0$, when $x \in [1, \frac{1+\sqrt{5}}{2}]$.

Since $\frac{1}{\log x} > 0$, we have $f'(x) < 0$, $x \in [1, \frac{1+\sqrt{5}}{2}]$ if and only if $-u(x)v(x) + s(x)t(x) < 0$, $x \in [1, \frac{1+\sqrt{5}}{2}]$.

In the rest of the proof, we prove that $-u(x)v(x) + s(x)t(x) < 0$, when $x \in [1, \frac{1+\sqrt{5}}{2}]$.

By simple calculations, we have that when $x \in [1, \frac{1+\sqrt{5}}{2}]$,

(i). $v(x)$ is monotonically decreasing and $v(x) > 0$;

(ii). $s(x)$ is monotonically decreasing and $s(x) > 0$.

Now we show that

(iii). $u(x)$ is monotonically increasing and $u(x) > 0$;

(iv). $t(x)$ is monotonically increasing and $t(x) > 0$.

Consider the compound function

$$u(y) = 2+\frac{\pi}{3(-\frac{\pi}{6}+arccsc\ y)}$$
$$y(x) = \sqrt{4-\frac{1}{x^2}}$$

Since $y(x) = \sqrt{4-\frac{1}{x^2}}$ is monotonically increasing when $x \in [1, \frac{1+\sqrt{5}}{2}]$, we have

$$y \in [y(1), y(\frac{1+\sqrt{5}}{2})].$$

That is ,

$$y \in [\sqrt{3}, \sqrt{4-\frac{1}{(2\cos\frac{\pi}{5})^2}}].$$

Since when $y \in [\sqrt{3}, \sqrt{4-\frac{1}{(2\cos\frac{\pi}{5})^2}}]$, $(-\frac{\pi}{6}+arccsc\ y)$ is monotonically decreasing and $(-\frac{\pi}{6}+arccsc\ y) > 0$, we have $u(y)$ is monotonically increasing when $y \in [\sqrt{3}, \sqrt{4-\frac{1}{(2\cos\frac{\pi}{5})^2}}]$. Therefore, $u(x)$ is monotonically increasing.

Moreover,

$$u(x) \geq u(1) > 0$$

Consider the compound function

$$t(y) = \frac{1}{(-\frac{\pi}{6}+arccsc\ y)^2}$$
$$y(x) = \sqrt{4-\frac{1}{x^2}}$$

As the previous proof, when $x \in [1, \frac{1+\sqrt{5}}{2}]$, $y(x) \in [\sqrt{3}, \sqrt{4-\frac{1}{(2\cos\frac{\pi}{5})^2}}]$ is monotonically increasing. In addition, we have $t(y)$ is monotonically increasing when $y \in [\sqrt{3}, \sqrt{4-\frac{1}{(2\cos\frac{\pi}{5})^2}}]$.

Hence, $t(y)$ is monotonically increasing when $x \in [1, \frac{1+\sqrt{5}}{2}]$.

Moreover,

$$t(x) \geq t(1) > 0.$$

Therefore, according to conditions (i), (ii), (iii), and (iv), for any $1 \leq x_1 \leq x_2 \leq 2\cos\frac{\pi}{5}$, when $x_1 \leq x \leq x_2$, we have

$$f'(x) \leq -u(x_1)v(x_2) + s(x_1)t(x_2).$$

We divide $[1, \frac{1+\sqrt{5}}{2}]$ into

$$\{[1, 1.001], [1.001, 1.002], \cdots, [1.609, 1.61], [1.61, \frac{1+\sqrt{5}}{2}]\}.$$

With simple calculations, we have that for any $[x_1, x_2] \in$

$$\{[1, 1.001], [1.001, 1.002], \cdots, [1.609, 1.61], [1.61, \frac{1+\sqrt{5}}{2}]\},$$

$$-u(x_1)v(x_2) + s(x_1)t(x_2) < 0.$$

That is,

$$f'(x) < 0, x \in [1, \frac{1+\sqrt{5}}{2}].$$

□

Appendix 2: Proof of Lemma 7

Proof. We have $f'(x) =$

$$\frac{6(2\sqrt{1+x}(2+x)arccsc\sqrt{2+x}(\pi - 6arccsc\sqrt{2+x}) + \pi x \log x)}{x\sqrt{1+x}(2+x)(\pi - 6arccsc\sqrt{2+x})^2(\log x)^2}$$

Let

$$u(x) = 12(arccsc\sqrt{2+x})^2 - 2\pi arccsc\sqrt{2+x};$$
$$v(x) = \sqrt{1+x}(2+x);$$
$$w(x) = \pi x \log x.$$

Then,

$$f'(x) = \frac{6(w(x) - u(x)v(x))}{x\sqrt{1+x}(2+x)(\pi - 6arccsc\sqrt{2+x})^2(\log x)^2}$$

To prove that $f(x)$ is monotonically increasing when $x \in [\frac{1+\sqrt{5}}{2}, 2]$, we only need to prove $f'(x) > 0$, $x \in [\frac{1+\sqrt{5}}{2}, 2]$.

Since

$$x\sqrt{1+x}(2+x)(\pi - 6arccsc\sqrt{2+x})^2(\log x)^2 > 0,$$

we have $f'(x) > 0$, $x \in [\frac{1+\sqrt{5}}{2}, 2]$ if and only if $w(x) - u(x)v(x) > 0$, $x \in [\frac{1+\sqrt{5}}{2}, 2]$.

In the rest of this proof, we prove that $w(x) - u(x)v(x) > 0$ when $x \in [\frac{1+\sqrt{5}}{2}, 2]$.

By simple calculations, we have that when $x \in [\frac{1+\sqrt{5}}{2}, 2]$,

(i). $w(x)$ is monotonically increasing and $w(x) > 0$;

(ii). $v(x)$ is monotonically increasing and $v(x) > 0$.

Now we prove that

(iii). $u(x)$ is monotonically decreasing and $u(x) > 0$.

Consider the compound function

$$u(y) = 6y^2 - \pi y,$$
$$y(x) = arccsc\sqrt{2+x}$$

We have

$$u(y) = 6y^2 - \pi y = 6(y - \frac{\pi}{12})^2 - \frac{\pi^2}{24}$$

Since $y(x) = arccsc\sqrt{2+x}$ is monotonically decreasing when $x \in [\frac{1+\sqrt{5}}{2}, 2]$, we have

$$y \in [y(2), y(\frac{1+\sqrt{5}}{2})].$$

That is,

$$y \in [arccsc\ 2, arccsc\sqrt{3+\sqrt{5}}].$$

In addition, $arccsc\ 2 > \frac{\pi}{12}$. Therefore, $u(y)$ is monotonically increasing when $y \in [arccsc\ 2, arccsc\sqrt{3+\sqrt{5}}]$. Hence, $u(x)$ is monotonically decreasing when $x \in [\frac{1+\sqrt{5}}{2}, 2]$.

Moreover, $u(x) > u(2) > 0$.

Let $x_1 = \frac{1+\sqrt{5}}{2}$, $x_2 = 2$, $x_3 = 1.8$.

According to conditions (i), (ii), and (iii), when $x \in [x_1, x_3]$, we have

$$w(x) - u(x)v(x) \geq w(x_1) - u(x_1)v(x_3) > 0.$$

When $x \in [x_3, x_2]$, we have

$$w(x) - u(x)v(x) \geq w(x_3) - u(x_3)v(x_2) > 0.$$

That is, $f'(x) > 0$, $x \in [\frac{1+\sqrt{5}}{2}, 2]$. □

Declaration of interests

☐ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Xiaohui Wei, Jilin University
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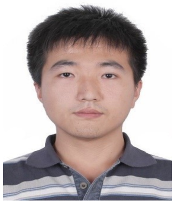

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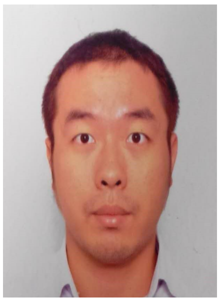

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