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Portfolio selection with mental accounts: An equilibrium model with endogenous risk aversion

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A B S T R A C T

In Das et al. (2010), an agent divides his or her wealth among mental accounts that have different goals and optimal portfolios. While the moments of the distribution of asset returns are *exogenous* in their normative model, they are *endogenous* in our corresponding positive model. We obtain the following results. First, there are multiple equilibria that we parameterize by the implied risk aversion coefficient of the agent's aggregate portfolio. Second, equilibrium asset prices and the composition of optimal portfolios within accounts depend on this coefficient. Third, altering the goal of any given account affects the composition of each portfolio.

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1. Introduction

Das et al. (2010, DMSS) combine certain aspects of behavioral and mean-variance (MV) portfolio selection models. Like Shefrin and Statman (2000), DMSS consider an agent who divides his or her wealth among mental accounts (hereafter 'accounts') with different goals such as retirement or bequests.[1] For each account, short sales are allowed and the agent maximizes its expected return subject to: (1) fully investing the wealth assigned to it; and (2) the probability of its return being less than or equal to some threshold return (e.g., −20%) not exceeding some threshold probability (e.g., 5%).[2] The threshold return and threshold probability (hereafter 'thresholds') can vary across accounts to reflect different goals. Assuming that a risk-free asset is absent and risky asset returns have a multivariate normal distribution, DMSS show that optimal portfolios within accounts and the resulting aggregate portfolio are all on the MV frontier of Markowitz (1952). In their *normative* model, the moments of this distribution are *exogenous*.

Our paper develops a corresponding *positive* model where these moments are *endogenous* in four types of economies. The first is a single-agent economy where the agent has an objective function defined over the expected value and variance of his or her future wealth as well as a single account (hereafter 'MV agent'). The second is also a single-agent economy but the agent has an objective function as in DMSS (hereafter 'DMSS agent') and a single account. The third is a single-agent economy with a DMSS agent but with multiple accounts. The fourth is a two-agent economy with an MV agent (who has a single account) and a DMSS agent who has multiple accounts.

There is ample motivation for considering such types of economies. First, a comparison of single-agent economies where the agents have a single account but differ in their objective functions (first two types of economies) allows us to identify any differences in the results that are due to these functions differing. Second, a single-agent economy with a DMSS agent and multiple accounts (third type) allow us to explore the heterogeneity of his or her preferences across accounts. Third, a two-agent economy

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[1] For an introduction to mental accounting, see Thaler (1999) and Statman (2017, pp. 200–208). Choi et al. (2009) provide empirical support for mental accounting in 401(k) plans. They find that investors allocate their own contributions to retirement plans without consideration of how they allocate their firms' matching contributions to such plans.

[2] In Telser (1955), the agent has one account but maximizes its expected return subject to a probability constraint reflecting its thresholds as in DMSS. In Roy (1952), the agent also has one account but minimizes the probability of its return being less than the threshold return. Elton et al. (2014, Ch. 11) compare the models of Roy and Telser.

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with MV and DMSS agents (fourth type) allows us to examine the heterogeneity of their preferences. Fourth, an examination of economies with a DMSS agent and multiple accounts (last two types) sheds light on the extent to which portfolio selection with accounts differs in the cases of endogenous and exogenous moments of the distribution of asset returns.

The fact that many practitioners consider accounts in portfolio decision-making also motivates our model. For example, Statman (2017, pp. 208–217) describes real-world advising programs that incorporate the idea that investors have accounts with different goals (expressed with thresholds). He notes that these programs are used in practice by financial advisers working either independently or at companies such as Fidelity and Vanguard. The business press recognizes that some individuals indeed have accounts with different goals (see, e.g., *Financial Times*, June 16, 2016, p. 11).

We obtain the following results.[3] First, in economies with a DMSS agent, there are multiple equilibria that correspond to an endogenous interval for the implied risk aversion coefficient of his or her aggregate portfolio. Since this result holds even in single-agent economies where a DMSS agent has one or more accounts, it is due to his or her objective function but not to the number of accounts. In contrast, the risk aversion coefficient of an MV agent is (by definition) exogenous.

Second, Black's (1972) zero-beta CAPM holds in all four types of economies.[4] Equilibrium expected asset returns are thus in line with influential empirical work finding that portfolios with betas less (more) than one have positive (negative) alphas instead of zero alphas as in Sharpe's (1964) CAPM; see, e.g., Black (1972). However, reflecting the heterogeneity of the preferences of MV and DMSS agents, the equilibrium expected asset returns in a single-agent economy with an MV agent generally differ from those in the other three types of economies.

Third, in economies with a DMSS agent having multiple accounts, the size of the thresholds of any given account affects the optimal portfolios within *all* accounts because the moments of the distribution of asset returns are endogenous. In DMSS, the size of the thresholds of any given account affects the optimal portfolio within only that account because such moments are exogenous. However, in both their paper and ours, optimal portfolios within accounts differ notably due to the heterogeneity in preferences across accounts.

Past work extends DMSS in various ways. In Alexander and Baptista (2011), an agent delegates the management of his or her wealth to portfolio managers. In Baptista (2012), Jiang et al. (2013), and Alexander et al. (2017), agents face, respectively, background, exchange rate, and estimation risks. Our paper differs in that: (1) agents do not delegate the management of their wealth to portfolio managers nor do they face such risks; and (2) the moments of the distribution of asset returns are endogenous.

Rodrigues and Lleo (2018) propose a practical implementation of the DMSS model when a risk-free asset is present and expected excess asset returns (over the risk-free rate) are estimated by combining equilibrium expected excess asset returns and the views of an agent as in Black and Litterman (1992). In contrast, as in DMSS we assume that a risk-free asset and estimation risk are absent (we obtain similar results when a risk-free asset is present). Also, we compare the equilibrium implications of using MV and DMSS objective functions.

An extensive literature examines models where agents have endogenous preferences.[5] In Becker and Mulligan (1997), the discount factors used to compute the utility of future consumption depend on wealth, mortality, addiction, and other variables. In Palacios-Huerta and Santos (2004), the risk aversion coefficient depends on the degree of financial market incompleteness as well as exposure to uncertainty that is both related to and unrelated to the structure of this market. Stutzer (2003) shows that minimizing the probability of the growth rate of an agent's invested wealth not exceeding a target growth rate corresponds to maximizing a power utility function with a risk aversion coefficient that depends on the investment opportunity set. Ang et al. (2013) show that a downside risk penalty on liability shortfall corresponds to increasing the risk aversion coefficient of an objective function defined over the expected value and variance of returns. In Lan et al. (2013), a risk-neutral hedge fund manager who trades off the benefits of leveraging an alpha-generating strategy against the costs of liquidation becomes risk-averse after poor performance. We contribute by showing that in economies with a DMSS agent there is an endogenous interval for the implied risk aversion coefficient of his or her aggregate portfolio.

We proceed as follows. Section 2 describes the model and characterizes optimal portfolios. Section 3 examines the equilibrium implications of the model for portfolio selection, risk aversion, and asset pricing. Section 4 illustrates such implications with an example. Section 5 concludes.[6]

2. The model

Using two dates, 0 and 1, we next describe the assets and agents in the model.

2.1. Assets

The set of assets available for trade at date 0 is $\mathbb{J} \equiv \{1, \ldots, J\}$ where $J \geq 2$. Each asset is in positive net supply and has one share outstanding. Asset payoffs at date 1 (hereafter 'payoffs') are given by a $J \times 1$ random vector $\tilde{\boldsymbol{d}}$ with a multivariate normal distribution ($\tilde{d}_j$ is asset j's random payoff).[7] The $J \times 1$ vector of expected asset payoffs is $\boldsymbol{d} \in \mathbb{R}^J_{++}$ (d_j is asset j's expected payoff). The $J \times J$ variance-covariance matrix of asset payoffs is $\boldsymbol{S}$ (S_{j_1, j_2} is the covariance between the payoffs of assets j_1 and j_2). We assume that: (a) $rank(\boldsymbol{S}) = J$; and (b) $rank([\boldsymbol{S}\mathbf{1}\ \boldsymbol{d}]) = 2$ where $\mathbf{1}$ denotes the $J \times 1$ unit vector.[8] Hence, a risk-free asset is absent (it is present in Appendix D).

[3] Here, we list only the results that either differ from or add to those in DMSS. Some of our other results (not listed here) coincide with theirs such as when the optimal portfolios within accounts and aggregate portfolio are on the MV frontier.

[4] Assuming that a risk-free asset is present and each agent has a single account as well as a lexicographic objective function incorporating the ideas in Roy (1952) and Telser (1955), Arzac and Bawa (1977) show that Sharpe's (1964) CAPM holds; see footnote 2. Following DMSS, our paper differs from that of Arzac and Bawa in four respects. First, we assume that a risk-free asset is absent. Second, we consider an agent with multiple accounts. Third, we assume that for each account the agent maximizes its expected return subject to a probability constraint involving its thresholds. Fourth, we analyze the implied risk aversion coefficients of the optimal portfolios within accounts and the aggregate portfolio.

[5] Here, we discuss only part of this literature. Becker (1996) incorporates personal capital (past personal consumption and experiences) and social capital (past actions of others) into preferences. Bowles (1998) reviews work on the impact of economic institutions on preferences.

[6] Online Appendices A and B summarize, respectively, the notation and the implications of our model. Online Appendix C (hereafter 'Appendix C') contains our proofs. Online Appendix D (hereafter 'Appendix D') adds a risk-free asset to our model.

[7] Since DMSS assume that asset *returns* have a multivariate normal distribution, we assume that asset *payoffs* have this type of distribution. However, our results hold more generally if asset payoffs are assumed to have a multivariate elliptical distribution (e.g., t-distribution) with finite first and second moments. Our results also hold, at least as an approximation, if the distribution of asset payoffs is unknown but has finite first and second moments. Das and Statman (2013) examine optimal portfolios within accounts when asset returns are assumed to have non-elliptical distributions by considering derivatives such as options. Rockenbach (2004) provides experimental evidence of mental accounting in option pricing.

[8] Note that $[\boldsymbol{S}\mathbf{1}\ \ \boldsymbol{d}]$ is $J \times 2$ matrix. Assumption (b) is related to the usual condition that at least two assets have different expected returns when the moments of

A portfolio is a $J \times 1$ vector of quantities of asset shares (hereafter 'holdings') and is denoted by $\boldsymbol{q}$ (q_j is asset j's holding). A positive (negative) q_j represents a long (short) position in asset j. Portfolio $\boldsymbol{q}$'s random payoff, expected payoff, and payoff variance are, respectively, $\widetilde{d}_{\boldsymbol{q}} \equiv \widetilde{\boldsymbol{d}}'\boldsymbol{q}$, $d_{\boldsymbol{q}} \equiv \boldsymbol{d}'\boldsymbol{q}$, and $s^2_{\boldsymbol{q}} \equiv \boldsymbol{q}'\boldsymbol{S}\boldsymbol{q}$. Since each asset has one share outstanding, the market portfolio is $\mathbf{1}$.

A vector of date-0 asset prices (hereafter 'asset prices') is denoted by $\boldsymbol{p} \in \mathbb{R}^J_{++}$ (p_j is asset j's price). For any portfolio $\boldsymbol{q}$, its price is $p_{\boldsymbol{q}} \equiv \boldsymbol{p}'\boldsymbol{q}$. If $p_{\boldsymbol{q}} > 0$, then its random return, expected return, and return variance are, respectively, $\widetilde{r}_{\boldsymbol{q}} \equiv \frac{\widetilde{d}_{\boldsymbol{q}}}{p_{\boldsymbol{q}}} - 1$, $r_{\boldsymbol{q}} \equiv \frac{d_{\boldsymbol{q}}}{p_{\boldsymbol{q}}} - 1$, and $\sigma^2_{\boldsymbol{q}} \equiv \frac{s^2_{\boldsymbol{q}}}{p^2_{\boldsymbol{q}}}$.

2.2. Agents

We now describe an 'MV agent' and a 'DMSS agent.'

2.2.1. MV agent

The MV agent has a single account and asset endowments given by $\boldsymbol{q}_0 \in \mathbb{R}^J_{++}$. He or she solves:

$$\max_{\boldsymbol{q}\in\mathbb{R}^J} \; d_{\boldsymbol{q}} - \frac{\gamma_0}{2} s^2_{\boldsymbol{q}} \tag{1}$$

$$s.t. \quad p_{\boldsymbol{q}} = p_{\boldsymbol{q}_0} \tag{2}$$

where $\gamma_0 > 0$ is his or her risk aversion coefficient and $p_{\boldsymbol{q}_0} > 0$ is his or her date-0 wealth. Eqs. (1)–(2) indicate that the MV agent: (a) has a linear objective function defined over the expected value and variance of his or her date-1 wealth; and (b) fully invests his or her date-0 wealth.[9]

2.2.2. DMSS agent

The set of the DMSS agent's accounts is $\mathbb{M} \equiv \{1, \ldots, M\}$ where $M \geq 1$.[10] In each account $m \in \mathbb{M}$, the agent's asset endowments are given by $\boldsymbol{q}_m \in \mathbb{R}^J_{++}$ and he or she solves:

$$\max_{\boldsymbol{q}\in\mathbb{R}^J} \; r_{\boldsymbol{q}} \tag{3}$$

$$s.t. \quad p_{\boldsymbol{q}} = p_{\boldsymbol{q}_m} \tag{4}$$

$$P[\widetilde{r}_{\boldsymbol{q}} \leq H_m] \leq \alpha_m \tag{5}$$

where $p_{\boldsymbol{q}_m} > 0$ is the date-0 wealth in the account, $H_m \in \mathbb{R}$ is its threshold return, and $\alpha_m \in (0, 0.5)$ is its threshold probability. Eqs. (3)–(5) indicate that the DMSS agent maximizes its expected return subject to: (i) fully investing the date-0 wealth in it, $p_{\boldsymbol{q}_m}$; and (ii) the probability of its return being less than or equal to threshold return H_m not exceeding threshold probability α_m.[11]

Let $\mathbf{0}$ denote the $J \times 1$ zero vector. Fix any portfolio $\boldsymbol{q} \neq \mathbf{0}$ and $\alpha \in (0, 0.5)$. Since $\widetilde{d}_{\boldsymbol{q}}$ has a univariate normal distribution, $\boldsymbol{q}$'s *payoff Value-at-Risk (VaR)* at confidence level $1 - \alpha$ is:

$$v_{1-\alpha,\boldsymbol{q}} \equiv z_\alpha s_{\boldsymbol{q}} - d_{\boldsymbol{q}} \tag{6}$$

where $z_\alpha \equiv -\Phi^{-1}(\alpha) > 0$ and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function. Similarly, for any portfolio $\boldsymbol{q}$ with $p_{\boldsymbol{q}} > 0$, its *return VaR* at confidence level $1 - \alpha$ is:

$$V_{1-\alpha,\boldsymbol{q}} \equiv z_\alpha \sigma_{\boldsymbol{q}} - r_{\boldsymbol{q}}. \tag{7}$$

Note that constraint (5) is equivalent to the following constraint on VaR:[12]

$$V_{1-\alpha_m,\boldsymbol{q}} \leq -H_m. \tag{8}$$

It follows from Eqs. (7) and (8) that:

$$r_{\boldsymbol{q}} \geq H_m + z_{\alpha_m}\sigma_{\boldsymbol{q}}. \tag{9}$$

Using Eq. (9), any portfolio $\boldsymbol{q}$ with $p_{\boldsymbol{q}} > 0$ meets constraint (5) if it lies on or above a line with intercept H_m and slope z_{α_m} in ($r_{\boldsymbol{q}}$, $\sigma_{\boldsymbol{q}}$) space, but does not meet constraint (5) if it lies below this line; see Fig. 1A. While the use of a larger value of H_m increases the intercept and thus tightens the constraint, the use of a higher value of α_m decreases the slope and thus loosens the constraint.

2.3. MV frontier

Let $A \equiv \boldsymbol{d}'\boldsymbol{S}^{-1}\boldsymbol{p}$, $B \equiv \boldsymbol{d}'\boldsymbol{S}^{-1}\boldsymbol{d}$, $C \equiv \boldsymbol{p}'\boldsymbol{S}^{-1}\boldsymbol{p}$, and $D \equiv BC - A^2$. Suppose that $A \neq 0$ and $rank([\boldsymbol{d}\ \boldsymbol{p}]) = 2$. Since $\text{rank}(\boldsymbol{S}) = J$ and $rank([\boldsymbol{d}\ \boldsymbol{p}]) = 2$, we have $B > 0$, $C > 0$, and $D > 0$.

A portfolio with positive price is on the *MV frontier* if there is no portfolio with the same expected payoff, a smaller payoff variance, and the same price.[13] For any given expected payoff $d \in \mathbb{R}$ and any given price $p \in \mathbb{R}_{++}$, the corresponding portfolio on this frontier is:

$$\boldsymbol{q}_{d,p} = p(\boldsymbol{q}_{A/C,1}) + \phi_{d,p}(\boldsymbol{q}_{B/A,1} - \boldsymbol{q}_{A/C,1}) \tag{10}$$

where $\boldsymbol{q}_{A/C,1} \equiv \frac{\boldsymbol{S}^{-1}\boldsymbol{p}}{C}$, $\boldsymbol{q}_{B/A,1} \equiv \frac{\boldsymbol{S}^{-1}\boldsymbol{d}}{A}$, and $\phi_{d,p} \equiv \frac{dAC - pA^2}{D}$.[14] Portfolios on it with price p are represented in ($d_{\boldsymbol{q}}$, $s_{\boldsymbol{q}}$) space by the hyperbola:

$$s_{\boldsymbol{q}} = \sqrt{p^2(1/C) + \frac{[d_{\boldsymbol{q}} - p(A/C)]^2}{D/C}}. \tag{11}$$

Since $s_{\boldsymbol{q}} = p\sigma_{\boldsymbol{q}}$ and $d_{\boldsymbol{q}} = p(1 + r_{\boldsymbol{q}})$, such portfolios are represented in ($r_{\boldsymbol{q}}$, $\sigma_{\boldsymbol{q}}$) space by the hyperbola:

$$\sigma_{\boldsymbol{q}} = \sqrt{1/C + \frac{[r_{\boldsymbol{q}} - (A/C - 1)]^2}{D/C}}. \tag{12}$$

Their location in this space depends on: (a) A/C (the expected payoff of portfolio $\boldsymbol{q}_{A/C,1}$); (b) $\sqrt{1/C}$ (its payoff standard deviation); and (c) $\sqrt{D/C}$ (the asymptotic slope of this hyperbola).

2.4. Optimal portfolios

This section characterizes the agents' optimal portfolios.

the asset return distribution are exogenous; see Huang and Litzenberger (1988, p. 62). All assets have the same equilibrium expected return in an economy with an MV agent if $rank([\boldsymbol{S}\mathbf{1}\ \boldsymbol{d}]) = 1$; see Section 3.1.

[9] Like DMSS, we assume that each agent in our model fully invests the wealth in each of his or her accounts.

[10] Like DMSS, we assume that the number of accounts is exogenous. While they focus on the multiple-account case, we consider both the single- and multiple-account cases. Section 1 provides motivation for doing so.

[11] DMSS justify the use of their model with the assumptions that agents: (1) specify account goals more precisely by using thresholds instead of risk aversion coefficients; and (2) identify thresholds more precisely by stating them for accounts instead of for the aggregate portfolio. Alexander et al. (2017) find that its use reduces estimation risk relative to the use of the MV model with plausible risk aversion coefficients.

[12] In terms of preference, a DMSS agent orders the portfolios for account m as follows. While the ones that meet constraint (8) are ordered by their expected returns in line with Eq. (3), the ones that do not meet it are "undesirable." The indifference curve for account m for any given level of expected return r is thus represented in ($r_{\boldsymbol{q}}$, $\sigma_{\boldsymbol{q}}$) space by $r_{\boldsymbol{q}} = r$ if $0 \leq \sigma_{\boldsymbol{q}} \leq (r - H_m)/z_{\alpha_m}$. Any portfolio $\boldsymbol{q}$ with $\sigma_{\boldsymbol{q}} > (r - H_m)/z_{\alpha_m}$ does not lie on this curve because it does not meet constraint (8).

[13] Our definition of the MV frontier differs from that in Huang and Litzenberger (1988, HL) in two respects. First, we define a portfolio as a vector of asset holdings, whereas HL define a portfolio as a vector of fractions of wealth that sum to one. Second, while we define a portfolio with a positive price to be on the MV frontier if there is no portfolio with the same expected payoff, a smaller payoff variance, and the same price, HL define a portfolio to be on the MV frontier if there is no portfolio with the same expected return and a smaller return variance.

[14] While $\boldsymbol{q}_{A/C,1}$ has minimum return variance among all portfolios with a price of one, $\boldsymbol{q}_{B/A,1}$ lies in ($d_{\boldsymbol{q}}$, $s^2_{\boldsymbol{q}}$) space where a ray from the origin crosses the curve representing portfolios on the MV frontier with this price after passing through $\boldsymbol{q}_{A/C,1}$.

2.4.1. MV agent

Next, we examine the MV agent's optimal portfolio.

Theorem 1. *The MV agent's optimal portfolio is:*

$$\boldsymbol{q}_0^* = (p_{\boldsymbol{q}_0})(\boldsymbol{q}_{A/C,1}) + (A/\gamma_0)(\boldsymbol{q}_{B/A,1} - \boldsymbol{q}_{A/C,1}). \tag{13}$$

Using Eqs. (10) and (13), q_0^* is on the MV frontier. Also, q_0^* depends on ($\boldsymbol{q}_0$, γ_0) and ($\boldsymbol{d}, \boldsymbol{S}, \boldsymbol{p}$).

2.4.2. DMSS agent

Let:

$$\overline{\alpha} \equiv \Phi(-\sqrt{D/C}). \tag{14}$$

Since $D/C > 0$, Eq. (14) implies that $\overline{\alpha} \in (0, 0.5)$. For any $\alpha < \overline{\alpha}$, let:

$$H_\alpha \equiv A/C - 1 - \sqrt{\frac{z_\alpha^2 - D/C}{C}}. \tag{15}$$

In Appendix C (Lemma 1), we show that the portfolio with minimum return VaR at confidence level $1 - \alpha$ among portfolios with a positive price has a return VaR at this confidence level of $-H_\alpha$.

We now examine the DMSS agent's optimal portfolios within accounts.

Theorem 2. *Fix any account $m \in \mathbb{M}$. (i) If either (a) $\alpha_m \geq \overline{\alpha}$, or (b) $\alpha_m < \overline{\alpha}$ and $H_m > H_{\alpha_m}$, then the DMSS agent's optimal portfolio within account m does not exist. (ii) If $\alpha_m < \overline{\alpha}$ and $H_m \leq H_{\alpha_m}$, then it exists and is:*

$$\boldsymbol{q}_m^* = (p_{\boldsymbol{q}_m})(\boldsymbol{q}_{A/C,1}) + (A/\gamma_m^*)(\boldsymbol{q}_{B/A,1} - \boldsymbol{q}_{A/C,1}) \tag{16}$$

where its implied risk aversion coefficient, γ_m^*, is provided in Appendix C.

It follows from Theorem 2 that the existence of $\boldsymbol{q}_m^*$ depends on (α_m, H_m) and ($\boldsymbol{d}, \boldsymbol{S}, \boldsymbol{p}$) through terms $\overline{\alpha}$ and H_{α_m}. If $\alpha_m \geq \overline{\alpha}$, then it does not exist regardless of the size of H_m and H_{α_m}. As Figs. 1A and B show, it does not exist because the expected returns of portfolios satisfying constraint (9) do not have a finite upper bound. If $\alpha_m < \overline{\alpha}$, then its existence depends on the size of H_m and H_{α_m}. When $H_m > H_{\alpha_m}$, it does not exist because no portfolio satisfies constraint (9); see Fig. 1C. When $H_m \leq H_{\alpha_m}$, it exists. Fig. 1D shows that it lies at the tangent point Q_m if $H_m = H_{\alpha_m}$. Fig. 1E shows that it lies at the point Q_m where the line crosses the top half of the curve if $H_m < H_{\alpha_m}$.

Using Eqs. (10) and (16), $\boldsymbol{q}_m^*$ is on the MV frontier if $\alpha_m < \overline{\alpha}$ and $H_m \leq H_{\alpha_m}$. Also, $\boldsymbol{q}_m^*$ depends on ($\boldsymbol{q}_m$, α_m, H_m) and ($\boldsymbol{d}, \boldsymbol{S}, \boldsymbol{p}$). As in DMSS, γ_m^* is the risk aversion coefficient that would make a hypothetical MV agent with an asset endowment of $\boldsymbol{q}_m$ optimally select $\boldsymbol{q}_m^*$.

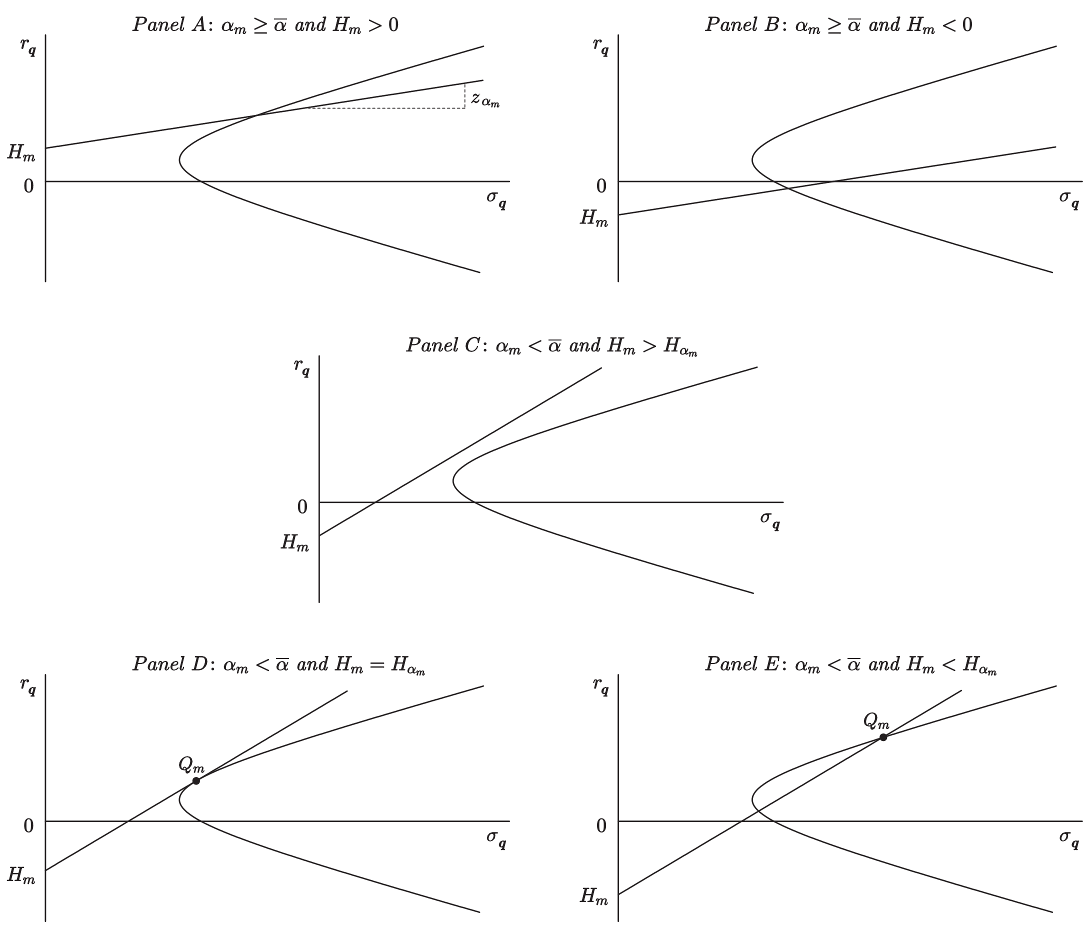

Fig. 1. Existence of an optimal portfolio within a given account. The expected return and return standard deviation of portfolio $\boldsymbol{q}$ are denoted by, respectively, $r_{\boldsymbol{q}}$ and $\sigma_{\boldsymbol{q}}$. The curve in each panel shows the portfolios on the MV frontier in ($r_{\boldsymbol{q}}$, $\sigma_{\boldsymbol{q}}$) space. Fix any account $m \in \mathbb{M}$ with threshold probability and return given by, respectively, α_m and H_m. The line in each panel has intercept H_m and slope z_{α_m}. Portfolios with a positive price that lie on or above this line satisfy constraint (9), whereas portfolios with a positive price that lie below it do not. Note that the constraint is tightened if either α_m decreases or H_m increases. Recall that $\overline{\alpha}$ is defined in Eq. (14). Also, H_{α_m} is given by Eq. (15) with $\alpha = \alpha_m$. When $\alpha_m \geq \overline{\alpha}$, the optimal portfolio within account m does not exist regardless of the threshold return (see panels A and B). When $\alpha_m < \overline{\alpha}$, the optimal portfolio within account m does not exist if $H_m > H_{\alpha_m}$ (see panel C), but it exists if either $H_m = H_{\alpha_m}$ (see panel D) or $H_m < H_{\alpha_m}$ (see panel E). In panels D and E, the optimal portfolio within account m is represented by point Q_m. In panel D, this portfolio is located where the line is tangent to the curve. In panel E, the portfolio is located where the line crosses the top half of the curve.

The DMSS agent's aggregate asset endowments are given by $\boldsymbol{q}_a \equiv \sum_{m\in\mathbb{M}} \boldsymbol{q}_m$. We next examine his of her aggregate portfolio, $\boldsymbol{q}_a^* \equiv \sum_{m\in\mathbb{M}} \boldsymbol{q}_m^*$ (assuming that $\{\boldsymbol{q}_m^*\}_{m\in\mathbb{M}}$ exist).

Theorem 3. *Suppose that $\alpha_m < \overline{\alpha}$ and $H_m \le H_{\alpha_m}$ for any account $m \in \mathbb{M}$. Then, the DMSS agent's aggregate portfolio is:*

$$\boldsymbol{q}_a^* = (p_{\boldsymbol{q}_a})(\boldsymbol{q}_{A/C,1}) + (A/\gamma_a^*)(\boldsymbol{q}_{B/A,1} - \boldsymbol{q}_{A/C,1}) \tag{17}$$

where:

$$\gamma_a^* = \left[\sum_{m\in\mathbb{M}} (1/\gamma_m^*)\right]^{-1} \tag{18}$$

is its implied risk aversion coefficient and γ_m^* is defined in Theorem 2.

Using Eqs. (10) and (17), $\boldsymbol{q}_a^*$ is on the MV frontier. Also, $\boldsymbol{q}_a^*$ depends on $\{(\boldsymbol{q}_m, \alpha_m, H_m)\}_{m\in\mathbb{M}}$ and $(\boldsymbol{d}, \boldsymbol{S}, \boldsymbol{p})$.

3. Equilibrium

We examine four types of economies: (1) a single-agent economy with an MV agent; (2) a single-agent economy with a DMSS agent and a single account; (3) a single-agent economy with a DMSS agent and multiple accounts; and (4) a two-agent economy with an MV agent and a DMSS agent with multiple accounts. In such economies, the exogenous quantities are $(\boldsymbol{d}, \boldsymbol{S})$, $(\boldsymbol{q}_0, \gamma_0)$ if there is an MV agent, and $\{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}$ if there is a DMSS agent. The endogenous quantities involve (or are based on) the asset prices and the optimal portfolio(s) of the agent(s).

3.1. Single-agent economy with an MV agent

Let $[(\boldsymbol{d}, \boldsymbol{S}), (\boldsymbol{q}_0, \gamma_0)]$ denote a single-agent economy with an MV agent. An equilibrium for it is an asset price vector $\boldsymbol{p}^*$ and a portfolio $\boldsymbol{q}_0^*$ such that: (a) $\boldsymbol{q}_0^*$ solves the agent's portfolio selection problem when $\boldsymbol{p} = \boldsymbol{p}^*$; and (b) asset markets clear so $\boldsymbol{q}_0^* = \mathbf{1}$.

For any $j \in \mathbb{J}$, let $S_{j,\mathbf{1}}$ denote the covariance between the payoffs of asset j and $\mathbf{1}$. Let $\Gamma_0 \equiv (0, \overline{\gamma}_0)$ where:

$$\overline{\gamma}_0 \equiv \min_{\{j\in\mathbb{J}:\, S_{j,\mathbf{1}}>0\}} \left\{d_j/S_{j,\mathbf{1}}\right\} \tag{19}$$

is a positive number. Suppose that $\theta_0 \in \Theta_0 \equiv \mathbb{R}_{++}$ and $\gamma_0 \in \Gamma_0$. In Appendix C, we show that $\boldsymbol{p}_{\theta_0,\gamma_0} \equiv \theta_0(\boldsymbol{d} - \gamma_0 \boldsymbol{S}\mathbf{1}) \in \mathbb{R}^J_{++}$. Let $\boldsymbol{q}^*_{0,\theta_0,\gamma_0}$ denote the MV agent's optimal portfolio when $\boldsymbol{p} = \boldsymbol{p}_{\theta_0,\gamma_0}$. In Appendix C, we show that $\boldsymbol{q}^*_{0,\theta_0,\gamma_0} = \mathbf{1}$.

The following result characterizes equilibria.

Theorem 4. *Fix any economy $[(\boldsymbol{d}, \boldsymbol{S}), (\boldsymbol{q}_0, \gamma_0)]$ where $\gamma_0 \in \Gamma_0$. For any $\theta_0 \in \Theta_0$, $(\boldsymbol{p}^*, \boldsymbol{q}_0^*) = (\boldsymbol{p}_{\theta_0,\gamma_0}, \boldsymbol{q}^*_{0,\theta_0,\gamma_0})$ is an equilibrium for it.*

For any economy considered in Theorem 4, we parameterize equilibria by the value of $\theta_0 \in \Theta_0$. In such equilibria, since asset prices are proportional to the value of θ_0, relative asset prices (e.g., the ratio of the prices of assets 1 and 2) are unique. Due to asset market clearing, the MV agent's optimal portfolio is the market portfolio regardless of this value.

Suppose that $\theta_0 \in \Theta_0$, $\gamma_0 \in \Gamma_0$, and $j \in \mathbb{J}$ are given. Then the equilibrium price of asset j, p_j^*: (a) increases in its expected payoff d_j (since $\theta_0 > 0$); and (b) decreases in the covariance between its payoffs and those of the market portfolio $S_{j,\mathbf{1}}$ (since $\theta_0 > 0$ and $\gamma_0 > 0$).

Suppose now that only $\theta_0 \in \Theta_0$ and $j \in \mathbb{J}$ are given. Then the size of γ_0 does not affect p_j^* if $S_{j,\mathbf{1}}$ is zero. However, p_j^* decreases (increases) in γ_0 if $S_{j,\mathbf{1}}$ is positive (negative).

3.2. Single-agent economy with a DMSS agent and a single account

Let $[(\boldsymbol{d}, \boldsymbol{S}), (\boldsymbol{q}_1, H_1, \alpha_1)]$ denote a single-agent economy with a DMSS agent and a single account. An equilibrium for it is an asset price vector $\boldsymbol{p}^*$ and a portfolio $\boldsymbol{q}_1^*$ such that: (a) $\boldsymbol{q}_1^*$ solves the agent's portfolio selection problem within account 1 when $\boldsymbol{p} = \boldsymbol{p}^*$; and (b) asset markets clear so $\boldsymbol{q}_1^* = \mathbf{1}$. Here, since the agent has a single account, his or her aggregate portfolio, $\boldsymbol{q}_a^*$, equals $\boldsymbol{q}_1^*$.

Let:

$$\underline{\alpha} \equiv \Phi(-d_{\mathbf{1}}/s_{\mathbf{1}}). \tag{20}$$

Since $d_{\mathbf{1}}/s_{\mathbf{1}} > 0$, Eq. (20) implies that $\underline{\alpha} \in (0, 0.5)$. Fix any $\alpha_1 > \underline{\alpha}$. Let $\Gamma_1 \equiv (0, \overline{\gamma}_1)$ where:

$$\overline{\gamma}_1 \equiv \min\{\overline{\gamma}_0, \overline{\gamma}_{\alpha_1}\}, \tag{21}$$

$\overline{\gamma}_0$ is given by Eq. (19), and:

$$\overline{\gamma}_{\alpha_1} \equiv z_{\alpha_1}/s_{\mathbf{1}} \tag{22}$$

is a positive number. Suppose that $H_1 > -1$, $\alpha_1 > \underline{\alpha}$, and $\gamma_1 \in \Gamma_1$. In Appendix C, we show that $\theta_{\gamma_1} \equiv \frac{d_{\mathbf{1}} - z_{\alpha_1} s_{\mathbf{1}}}{(H_1+1)(d_{\mathbf{1}} - \gamma_1 s_{\mathbf{1}}^2)} \in \mathbb{R}_{++}$ and $\boldsymbol{p}_{\gamma_1} \equiv \theta_{\gamma_1}(\boldsymbol{d} - \gamma_1 \boldsymbol{S}\mathbf{1}) \in \mathbb{R}^J_{++}$. Letting $\boldsymbol{q}^*_{1,\gamma_1}$ denote the DMSS agent's optimal portfolio within account 1 when $\boldsymbol{p} = \boldsymbol{p}_{\gamma_1}$, in Appendix C we show that $\boldsymbol{q}^*_{1,\gamma_1} = \mathbf{1}$.

The following result characterizes equilibria.

Theorem 5. *Fix any economy $[(\boldsymbol{d}, \boldsymbol{S}), (\boldsymbol{q}_1, H_1, \alpha_1)]$ where $H_1 > -1$ and $\alpha_1 > \underline{\alpha}$. For any $\gamma_1 \in \Gamma_1$, $(\boldsymbol{p}^*, \boldsymbol{q}_1^*) = (\boldsymbol{p}_{\gamma_1}, \boldsymbol{q}^*_{1,\gamma_1})$ is an equilibrium for it and $\gamma_1^* = \gamma_1$ is the implied risk aversion coefficient of the DMSS agent's optimal portfolio within account 1.*

For any economy considered in Theorem 5, we parameterize equilibria by the value of $\gamma_1 \in \Gamma_1$. In such equilibria, (relative) asset prices depend on this value but the DMSS agent's optimal portfolio within account 1 does not due to asset market clearing. Hence, the multiplicity of equilibria here refers to the multiplicity of equilibrium asset price vectors (but not of this portfolio).

The intuition of the multiplicity of such vectors can be seen in an example. Consider a single-agent economy with two assets ($j = 1, 2$). The expected payoff of each asset is one. While the payoff variances of assets 1 and 2 are, respectively, 0.03 and 0.06, the covariance between their payoffs is zero. The expected payoff and payoff standard deviation of market portfolio $\mathbf{1}$ are, respectively, $d_{\mathbf{1}} = 2$ $[= 1 + 1]$ and $s_{\mathbf{1}} = 0.3$ $[= \sqrt{0.03 + 0.06}]$. There is a DMSS agent with a single account ($m = 1$), a threshold probability of $\alpha_1 = 5\%$, and a threshold return of $H_1 = -15\%$. In equilibrium, the optimal portfolio within account 1 is $\boldsymbol{q}_1^* = \mathbf{1}$ and its return VaR at the 95% confidence level is $V_{0.95,\mathbf{1}} = 15\%$ (due to asset market clearing and the binding probability constraint for account 1). Since $d_{\mathbf{1}} = 2$, $s_{\mathbf{1}} = 0.3$, and $z_{0.05} = 1.645$, $V_{0.95,\mathbf{1}} = 1.645 \times (0.3/p_{\mathbf{1}}^*) - (2/p_{\mathbf{1}}^* - 1) = 1 - 1.506/p_{\mathbf{1}}^*$ where $p_{\mathbf{1}}^*$ is the equilibrium price of $\mathbf{1}$; see Eq. (7). Hence, if $p_{\mathbf{1}}^* = 1.772$, then $V_{0.95,\mathbf{1}} = 15\%$. There are multiple asset price vectors such that $\boldsymbol{p}^* = (p_1^*, p_2^*) \in \mathbb{R}^2_{++}$ and $p_{\mathbf{1}}^* = 1.772$. For example, if $\boldsymbol{p}^* = (0.900, 0.872)$, then $p_{\mathbf{1}}^* = 1.772$ $[= 0.900 + 0.872]$, $\boldsymbol{q}_1^* = \mathbf{1}$, and $\gamma_1^* = 1$. Similarly, if $\boldsymbol{p}^* = (0.915, 0.857)$, then $p_{\mathbf{1}}^* = 1.772$ $[= 0.915 + 0.857]$, $\boldsymbol{q}_1^* = \mathbf{1}$, and $\gamma_1^* = 2$.

Of interest is the mapping between thresholds (α_1 and H_1) and the implied risk aversion coefficient of the DMSS agent's optimal portfolio within account 1 (γ_1). Fixing the thresholds, Theorem 5 identifies the set to which this coefficient belongs (Γ_1). The value of γ_1 differs across equilibria because the moments of the distribution of asset returns also differ across equilibria.[15]

[15] When the moments of the distribution of asset returns are exogenous, DMSS note that the size of the implied risk aversion coefficient of the optimal portfolio within any given account depends on such moments and the thresholds. Fixing such moments, they point out that the use of different thresholds for any given account may lead to the same implied risk aversion coefficient for the optimal portfolio within that account. In comparison, fixing account 1's thresholds, Theorem 5 says that the implied risk aversion coefficient of the optimal portfolio within account 1 differs across equilibria when the moments of the distribution of asset returns are endogenous. We later extend Theorem 5 from the single-account case to the multiple-account case.

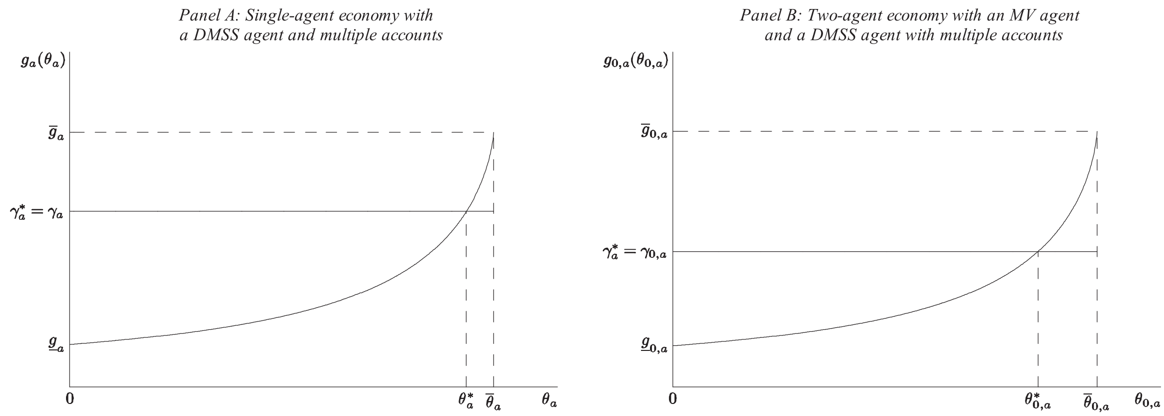

Fig. 2. Finding equilibria for a single-agent economy with a DMSS agent and multiple accounts and a two-agent economy with an MV agent and a DMSS agent with multiple accounts. Panel A illustrates how equilibria for a single-agent economy with a DMSS agent and multiple accounts can be found. While the curve plots $g_a(\theta_a)$ when θ_a ranges from 0 to $\overline{\theta}_a$, the line plots some constant $\gamma_a \in \Gamma_a$ with $\underline{g}_a < \gamma_a < \overline{g}_a$. Recall that $g_a(\theta_a)$ is the implied risk aversion coefficient of the DMSS agent's aggregate portfolio when the asset price vector is $\boldsymbol{p}_{\theta_a,\gamma_a}$. Also, $\underline{g}_a$ and $\overline{g}_a$ are the limits of $g_a(\theta_a)$ when θ_a converges to, respectively, 0 and $\overline{\theta}_a$. Since $g_a(\theta_a)$ is continuous and increases in θ_a and $\underline{g}_a < \gamma_a < \overline{g}_a$, there exists a value θ_a^* between 0 and $\overline{\theta}_a$ where $g_a(\theta_a^*) = \gamma_a$. Hence, $\boldsymbol{p}_{\theta_a^*,\gamma_a}$ is an equilibrium price vector and $\gamma_a^* = \gamma_a$ is the corresponding implied risk aversion coefficient of the DMSS agent's aggregate portfolio. Similarly, panel B illustrates how equilibria for a two-agent economy with an MV agent and a DMSS agent with multiple accounts can be found. While the curve plots $g_{0,a}(\theta_{0,a})$ when $\theta_{0,a}$ ranges from 0 and $\overline{\theta}_{0,a}$, the line plots some constant $\gamma_{0,a} \in \Gamma_{0,a}$ with $\underline{g}_{0,a} < \gamma_{0,a} < \overline{g}_{0,a}$. Recall that $g_{0,a}(\theta_{0,a})$ is the implied risk aversion coefficient of the DMSS agent's aggregate portfolio when the asset price vector is $\boldsymbol{p}_{\theta_{0,a},\varphi_{0,a}}$ where $\varphi_{0,a} = \frac{1}{1/\gamma_0 + 1/\gamma_{0,a}}$. Also, $\underline{g}_{0,a}$ and $\overline{g}_{0,a}$ are the limits of $g_{0,a}(\theta_{0,a})$ when $\theta_{0,a}$ converges to, respectively, 0 and $\overline{\theta}_{0,a}$. Since $g_{0,a}(\theta_{0,a})$ is continuous and increases in $\theta_{0,a}$ and $\underline{g}_{0,a} < \gamma_{0,a} < \overline{g}_{0,a}$, there exists a value $\theta_{0,a}^*$ between 0 and $\overline{\theta}_{0,a}$ where $g_{0,a}(\theta_{0,a}^*) = \gamma_{0,a}$. Hence, $\boldsymbol{p}_{\theta_{0,a}^*,\varphi_{0,a}}$ is an equilibrium asset price vector and $\gamma_a^* = \gamma_{0,a}$ is the corresponding implied risk aversion coefficient of the DMSS agent's aggregate portfolio.

If $\alpha_1 > \underline{\alpha}$ and $\gamma_1 \in \Gamma_1$ are given, then equilibrium asset prices decrease in H_1; see Theorem 5 and the definition of $\boldsymbol{p}_{\gamma_1}$. Similarly, if $H_1 > -1$ and $\gamma_1 \in \Gamma_1$ are given, then equilibrium asset prices increase in α_1; see again Theorem 5 and the definitions of $\boldsymbol{p}_{\gamma_1}$.

3.3. Single-agent economy with a DMSS agent and multiple accounts

Let $[(\boldsymbol{d},\boldsymbol{S}),\{(\boldsymbol{q}_m,H_m,\alpha_m)\}_{m\in\mathbb{M}}]$ denote a single-agent economy with a DMSS agent and multiple accounts. An equilibrium for it is an asset price vector $\boldsymbol{p}^*$ and portfolios $\{\boldsymbol{q}_m^*\}_{m\in\mathbb{M}}$ such that: (a) for any $m\in\mathbb{M}$, $\boldsymbol{q}_m^*$ solves the agent's portfolio selection problem within account m when $\boldsymbol{p}=\boldsymbol{p}^*$; and (b) asset markets clear so $\sum_{m\in\mathbb{M}} \boldsymbol{q}_m^* = \mathbf{1}$.

Without loss of generality, assume that $\alpha_1 \geq \alpha_m$ for any $m\in\mathbb{M}$. Also, assume that $\alpha_m > \underline{\alpha}$ for any $m\in\mathbb{M}$. Fix any $\gamma_a \in \Gamma_a \equiv \Gamma_1$. Let $\Theta_a \equiv (0,\overline{\theta}_a)$ where:

$$\overline{\theta}_a \equiv \min_{m\in\mathbb{M}} \{\overline{\theta}_m\}, \tag{23}$$

$$\overline{\theta}_m \equiv \frac{K_1 - \sqrt{K_2/K_3}}{H_m+1}, \tag{24}$$

$K_1 \equiv \frac{B-\gamma_a d_{\mathbf{1}}}{K_3}$, $K_2 \equiv z_{\alpha_m}^2 - B + \frac{(B-\gamma_a d_{\mathbf{1}})^2}{K_3}$, and $K_3 \equiv \gamma_a^2 s_{\mathbf{1}}^2 - 2\gamma_a d_{\mathbf{1}} + B$. In Appendix C, we show that $\overline{\theta}_a > 0$ and $\boldsymbol{p}_{\theta_a,\gamma_a} \equiv \theta_a(\boldsymbol{d}-\gamma_a\boldsymbol{S}\mathbf{1}) \in \mathbb{R}_{++}^J$ for any $\theta_a \in \Theta_a$. For any $m\in\mathbb{M}$, let $\boldsymbol{q}_{m,\theta_a,\gamma_a}^*$ denote the DMSS agent's optimal portfolio within account m when $\boldsymbol{p}=\boldsymbol{p}_{\theta_a,\gamma_a}$. Let $g_a(\theta_a)$ denote the implied risk aversion coefficient of his or her resulting aggregate portfolio. Also, let $\underline{g}_a \equiv \lim_{\theta_a\downarrow 0} g_a(\theta_a)$ and $\overline{g}_a \equiv \lim_{\theta_a\uparrow\overline{\theta}_a} g_a(\theta_a)$.

The following result characterizes equilibria.

Theorem 6. *Fix any economy $[(\boldsymbol{d},\boldsymbol{S}),\{(\boldsymbol{q}_m,H_m,\alpha_m)\}_{m\in\mathbb{M}}]$ where $H_m > -1$ and $\alpha_m > \underline{\alpha}$ for any account $m\in\mathbb{M}$. For any $\gamma_a \in \Gamma_a$ with $\underline{g}_a < \gamma_a < \overline{g}_a$, there exists $\theta_a^* \in \Theta_a$ such that $(\boldsymbol{p}^*,\{\boldsymbol{q}_m^*\}_{m\in\mathbb{M}}) = (\boldsymbol{p}_{\theta_a^*,\gamma_a},\{\boldsymbol{q}_{m,\theta_a^*,\gamma_a}^*\}_{m\in\mathbb{M}})$ is an equilibrium for it and $\gamma_a^* = \gamma_a$ is the implied risk aversion coefficient of the DMSS agent's aggregate portfolio.*

For any economy considered in Theorem 6, we parameterize equilibria by the value of $\gamma_a \in \Gamma_a$. In such equilibria, (relative) asset prices and the DMSS agent's optimal portfolios within accounts depend on this value but his or her aggregate portfolio does not due to asset market clearing.

Fig. 2A illustrates how equilibria can be found. While the curve plots $g_a(\theta_a)$ when θ_a ranges from 0 to $\overline{\theta}_a$, the line plots some constant $\gamma_a \in \Gamma_a$ with $\underline{g}_a < \gamma_a < \overline{g}_a$. Recall that $g_a(\theta_a)$ is the implied risk aversion coefficient of the DMSS agent's aggregate portfolio when the asset price vector is $\boldsymbol{p}_{\theta_a,\gamma_a}$. Also, $\underline{g}_a$ and $\overline{g}_a$ are the limits of $g_a(\theta_a)$ when θ_a converges to, respectively, 0 and $\overline{\theta}_a$. Since $g_a(\theta_a)$ is continuous and increases in θ_a and $\underline{g}_a < \gamma_a < \overline{g}_a$, there exists a value θ_a^* between 0 and $\overline{\theta}_a$ where $g_a(\theta_a^*) = \gamma_a$. Hence, $\boldsymbol{p}_{\theta_a^*,\gamma_a}$ is an equilibrium price vector and $\gamma_a^* = \gamma_a$ is the corresponding implied risk aversion coefficient of the DMSS agent's aggregate portfolio.

3.4. Two-agent economy with an MV agent and a DMSS agent with multiple accounts

Let $[(\boldsymbol{d},\boldsymbol{S}),(\boldsymbol{q}_0,\gamma_0),\{(\boldsymbol{q}_m,H_m,\alpha_m)\}_{m\in\mathbb{M}}]$ denote a two-agent economy with an MV agent and a DMSS agent with multiple accounts. An equilibrium for it is an asset price vector $\boldsymbol{p}^*$ and a portfolio allocation $(\boldsymbol{q}_0^*,\{\boldsymbol{q}_m^*\}_{m\in\mathbb{M}})$ such that: (a) $\boldsymbol{q}_0^*$ solves the MV agent's portfolio selection problem when $\boldsymbol{p}=\boldsymbol{p}^*$; (b) for any $m\in\mathbb{M}$, $\boldsymbol{q}_m^*$ solves the DMSS agent's portfolio selection problem within account m when $\boldsymbol{p}=\boldsymbol{p}^*$; and (c) asset markets clear so $\boldsymbol{q}_0^* + \sum_{m\in\mathbb{M}} \boldsymbol{q}_m^* = \mathbf{1}$.

Again, assume that $\alpha_1 \geq \alpha_m > \underline{\alpha}$ for any $m\in\mathbb{M}$. Fix any $\gamma_0 > 0$. Let $\Gamma_{0,a} \equiv (0,\overline{\gamma}_{0,a})$ where:

$$\overline{\gamma}_{0,a} \equiv \min\left\{\overline{\overline{\gamma}}_0, \overline{\overline{\gamma}}_{\alpha_1}\right\}, \tag{25}$$

$$\overline{\overline{\gamma}}_0 \equiv \begin{cases} \infty & \text{if} \quad \gamma_0 \leq \overline{\gamma}_0 \\ \frac{1}{1/\overline{\gamma}_0 - 1/\gamma_0} & \text{if} \quad \gamma_0 > \overline{\gamma}_0 \end{cases}, \tag{26}$$

$\overline{\gamma}_0$ is given by Eq. (19),

$$\overline{\overline{\gamma}}_{\alpha_1} \equiv \begin{cases} \infty & \text{if} \quad \gamma_0 \leq \overline{\gamma}_{\alpha_1} \\ \frac{1}{1/\overline{\gamma}_{\alpha_1} - 1/\gamma_0} & \text{if} \quad \gamma_0 > \overline{\gamma}_{\alpha_1} \end{cases}, \tag{27}$$

and $\overline{\gamma}_{\alpha_1}$ is given by Eq. (22). Fix any $\gamma_{0,a} \in \Gamma_{0,a}$. Let $\Theta_{0,a} \equiv (0,\overline{\theta}_{0,a})$ where:

$$\overline{\theta}_{0,a} \equiv \min_{m\in\mathbb{M}} \left\{\overline{\theta}_{0,m}\right\}, \tag{28}$$

$$\overline{\theta}_{0,m} \equiv \frac{K_4 - \sqrt{K_5/K_6}}{H_m + 1}, \tag{29}$$

$K_4 \equiv \frac{B-\varphi_{0,a}d_{\mathbf{1}}}{K_6}$, $K_5 \equiv z_{\alpha_m}^2 - B + \frac{(B-\varphi_{0,a}d_{\mathbf{1}})^2}{K_6}$, $K_6 \equiv \varphi_{0,a}^2 s_{\mathbf{1}}^2 - 2\varphi_{0,a}d_{\mathbf{1}} + B$, and $\varphi_{0,a} \equiv \frac{1}{1/\gamma_0 + 1/\gamma_{0,a}}$. In Appendix C, we show that $\overline{\theta}_{0,a} > 0$ and $\boldsymbol{p}_{\theta_{0,a},\varphi_{0,a}} \equiv \theta_{0,a}(\boldsymbol{d} - \varphi_{0,a}\boldsymbol{S1}) \in \mathbb{R}_{++}^J$ for any $\theta_{0,a} \in \Theta_{0,a}$. Let $\boldsymbol{q}^*_{0,\theta_{0,a},\varphi_{0,a}}$ denote the MV agent's optimal portfolio when $\boldsymbol{p} = \boldsymbol{p}_{\theta_{0,a},\varphi_{0,a}}$. Similarly, for any $m \in \mathbb{M}$, let $\boldsymbol{q}^*_{m,\theta_{0,a},\varphi_{0,a}}$ denote the DMSS agent's optimal portfolio within account m when $\boldsymbol{p} = \boldsymbol{p}_{\theta_{0,a},\varphi_{0,a}}$. Let $g_{0,a}(\theta_{0,a})$ denote the implied risk aversion coefficient of his or her resulting aggregate portfolio. Also, let $\underline{g}_{0,a} \equiv \lim_{\theta_{0,a}\downarrow 0} g_{0,a}(\theta_{0,a})$ and $\overline{g}_{0,a} \equiv \lim_{\theta_{0,a}\uparrow\overline{\theta}_{0,a}} g_{0,a}(\theta_{0,a})$.

The following result characterizes equilibria.

Theorem 7. *Fix any economy* $[(\boldsymbol{d}, \boldsymbol{S}), (\boldsymbol{q}_0, \gamma_0), \{(\boldsymbol{q}_m, H_m, \alpha_m)\}_{m\in\mathbb{M}}]$ *where* $\gamma_0 > 0$ *as well as* $H_m > -1$ *and* $\alpha_m > \underline{\alpha}$ *for any account* $m \in \mathbb{M}$. *For any* $\gamma_{0,a} \in \Gamma_{0,a}$ *with* $\underline{g}_{0,a} < \gamma_{0,a} < \overline{g}_{0,a}$, *there exists* $\theta^*_{0,a} \in \Theta_{0,a}$ *such that* $[\boldsymbol{p}^*, (\boldsymbol{q}^*_0, \{\boldsymbol{q}^*_m\}_{m\in\mathbb{M}})] = [\boldsymbol{p}_{\theta^*_{0,a},\varphi_{0,a}}, (\boldsymbol{q}^*_{0,\theta^*_{0,a},\varphi_{0,a}}, \{\boldsymbol{q}^*_{m,\theta^*_{0,a},\varphi_{0,a}}\}_{m\in\mathbb{M}})]$ *is an equilibrium for it and* $\gamma^*_a = \gamma_{0,a}$ *is the implied risk aversion coefficient of the DMSS agent's aggregate portfolio.*

For any economy considered in Theorem 7, we parameterize equilibria by the value of $\gamma_{0,a} \in \Gamma_{0,a}$. In such equilibria, (relative) asset prices and the agents' optimal portfolios depend on this value.

Fig. 2B illustrates how equilibria can be found. While the notation in Fig. 2B slightly differs from that in Fig. 2A to accommodate the presence of an MV agent, the ideas in the former figure are similar to those in the latter (discussed earlier).

3.5. Equilibrium expected returns

Fix any economy considered in Theorems 4–7 and an equilibrium for it. Let $\boldsymbol{p}^* \in \mathbb{R}_{++}^J$ be the equilibrium asset price vector. The equilibrium expected return and return standard deviation of market portfolio $\mathbf{1}$ are, respectively, $r^*_{\mathbf{1}} \equiv \frac{d_{\mathbf{1}}}{p^*_{\mathbf{1}}} - 1$ and $\sigma^*_{\mathbf{1}} \equiv \frac{s_{\mathbf{1}}}{p^*_{\mathbf{1}}}$ ($p^*_{\mathbf{1}}$ is its equilibrium price). For any asset $j \in \mathbb{J}$, its equilibrium expected return is $r^*_j \equiv \frac{d_j}{p^*_j} - 1$ and its equilibrium beta is $\beta^*_j \equiv \sigma^*_{j,\mathbf{1}}/(\sigma^*_{\mathbf{1}})^2$ where $\sigma^*_{j,\mathbf{1}} \equiv \frac{S_{j,\mathbf{1}}}{p^*_j p^*_{\mathbf{1}}}$ is the covariance between its equilibrium returns and those of $\mathbf{1}$. Let $\boldsymbol{q}_{zc}$ be the portfolio on the MV frontier with an equilibrium price of $p^*_{\mathbf{1}}$ for which the covariance between the equilibrium returns of $\boldsymbol{q}_{zc}$ and $\mathbf{1}$ is zero. The equilibrium expected return of $\boldsymbol{q}_{zc}$ is $r^*_{\boldsymbol{q}_{zc}}$.

The following result characterizes expected asset returns in equilibrium.

Theorem 8. *For any asset* $j \in \mathbb{J}$, *its equilibrium expected return is:*

$$r^*_j = r^*_{\boldsymbol{q}_{zc}} + \beta^*_j (r^*_{\mathbf{1}} - r^*_{\boldsymbol{q}_{zc}}). \tag{30}$$

Using Theorem 8, Black's (1972) zero-beta CAPM holds in all four types of economies. However, reflecting the heterogeneity of the preferences of MV and DMSS agents, the equilibrium expected asset returns in a single-agent economy with an MV agent generally differ from those in the other three types of economies.

4. Example

We now illustrate our model with an example. As before, we consider four types of economies. As in the example of DMSS, there are three assets; see the first column of Table 1A. All assets have an expected payoff of one; see the next column.

Table 1
Expected values, variances, and covariances of asset payoffs as well as asset endowments. In the economies examined in Section 4, there are three assets ($j = 1, 2, 3$) with each having one share outstanding. Panel A shows their expected payoff vector (d_j, $j = 1, 2, 3$) and the variance-covariance matrix for asset payoffs (S_{j_1,j_2}, $j_1 = 1, 2, 3$, $j_2 = 1, 2, 3$). Panel B provides the asset endowments in four cases. Case B1 shows the endowments in a single-agent economy with an MV agent ($q_{0,j}$, $j = 1, 2, 3$). Case B2 shows the endowments a single-agent economy with a DMSS agent and one account ($q_{1,j}$, $j = 1, 2, 3$). Case B3 shows the endowments in a single-agent economy with a DMSS agent and three accounts ($q_{m,j}$, $m = 1, 2, 3$, $j = 1, 2, 3$). Case B4 shows the endowments in a two-agent economy with an MV agent and a DMSS agent with three accounts.

Panel A. Expected values, variances, and covariances of asset payoffs

			j_2		
			1	2	3
j	d_j	j_1	S_{j_1,j_2}		
1	1	1	0.0025	0.0000	0.0000
2	1	2		0.0400	0.0280
3	1	3			0.1225

Panel B. Asset endowments

B1. Single-agent economy with an MV agent

j	$q_{0,j}$
1	1.0
2	1.0
3	1.0

B2. Single-agent economy with a DMSS agent and one account

j	$q_{1,j}$
1	1.0
2	1.0
3	1.0

B3. Single-agent economy with a DMSS agent and three accounts

j	$q_{1,j}$	$q_{2,j}$	$q_{3,j}$
1	0.2	0.2	0.6
2	0.2	0.2	0.6
3	0.2	0.2	0.6

B4. Two-agent economy with an MV agent and a DMSS agent with three accounts

j	$q_{0,j}$	$q_{1,j}$	$q_{2,j}$	$q_{3,j}$
1	0.5	0.1	0.1	0.3
2	0.5	0.1	0.1	0.3
3	0.5	0.1	0.1	0.3

Asset 1 has a relatively small payoff variance (0.0025) and the covariance between its payoff and that of either assets 2 or 3 is zero; see the last three columns. Assets 2 and 3 have increasingly larger payoff variances (0.0400 and 0.1225, respectively) and the covariance between their payoffs is positive (0.0280);[16] see the last two columns.

[16] In the example of DMSS, one of the assets (interpreted as a bond) has a relatively small return variance and the covariance between its return and that of either of the other two assets is zero. These two assets (interpreted as low- and high-risk stocks) have increasingly larger return variances and the covariance between their returns is positive. Hence, our assumptions on the distribution of asset payoffs are similar to DMSS's assumptions on the distribution of asset returns.

Table 2
Equilibrium in a single-agent economy with an MV agent. Consider an economy with an MV agent and three assets ($j = 1, 2, 3$) as shown in panel A of Table 1. This agent has the asset endowments in panel B1 of Table 1 and a risk aversion coefficient of $\gamma_0 = 1$. Panel A provides equilibrium asset prices (p_j^*, $j = 1, 2, 3$), expected returns (r_j^*, $j = 1, 2, 3$), return standard deviations (σ_j^*, $j = 1, 2, 3$), return VaRs at the 90%, 95%, and 99% confidence levels (respectively, $V_{0.90,j}^*$, $V_{0.95,j}^*$, and $V_{0.99,j}^*$, $j = 1, 2, 3$), and betas (β_j^*, $j = 1, 2, 3$). Panel B shows the composition of the MV agent's optimal portfolio ($\boldsymbol{q}_0^*$), zero-covariance portfolio ($\boldsymbol{q}_{zc}$), and market portfolio ($\mathbf{1}$) along with their expected returns ($r_{\boldsymbol{q}}^*$), return standard deviations ($\sigma_{\boldsymbol{q}}^*$), return VaRs at the 90%, 95%, and 99% confidence levels (respectively, $V_{0.90,\boldsymbol{q}}^*$, $V_{0.95,\boldsymbol{q}}^*$, and $V_{0.99,\boldsymbol{q}}^*$), and betas ($\beta_{\boldsymbol{q}}^*$). The equilibrium is determined by setting $\theta_0 = 0.95$ (see Theorem 4). Expected returns, return standard deviations, and return VaRs are reported in percentage points.

Panel A. Assets: prices, expected returns, and risk statistics

j	p_j^*	r_j^*	σ_j^*	$V_{0.90,j}^*$	$V_{0.95,j}^*$	$V_{0.99,j}^*$	β_j^*
1	0.948	5.527	5.276	1.235	3.152	6.748	0.032
2	0.885	12.943	22.589	16.005	24.212	39.606	0.917
3	0.807	23.912	43.369	31.668	47.424	76.980	2.228

Panel B. The MV agent's optimal portfolio, zero-covariance portfolio, and market portfolio: composition, expected returns, and risk statistics

	q_1	q_2	q_3	$r_{\boldsymbol{q}}^*$	$\sigma_{\boldsymbol{q}}^*$	$V_{0.90,\boldsymbol{q}}^*$	$V_{0.95,\boldsymbol{q}}^*$	$V_{0.99,\boldsymbol{q}}^*$	$\beta_{\boldsymbol{q}}^*$
$\boldsymbol{q}_0^*$	1.000	1.000	1.000	13.634	17.807	9.186	15.655	27.790	1.000
$\boldsymbol{q}_{zc}$	2.788	0.068	−0.077	5.263	5.364	1.611	3.559	7.215	0.000
$\mathbf{1}$	1.000	1.000	1.000	13.634	17.807	9.186	15.655	27.790	1.000

4.1. Single-agent economy with an MV agent

Consider a single-agent economy with an MV agent where his or her endowment of each asset is one; see Panel B1 of Table 1. Table 2 shows equilibrium values of various quantities when $\gamma_0 = 1$ and $\theta_0 = 0.95$; see Theorem 4. Panel A examines the assets. In the second column, asset 1's price exceeds asset 2's, which in turn exceeds asset 3's. In the last six columns, asset 1's expected return and five risk statistics (standard deviation, VaRs at three confidence levels, and beta) are smaller than asset 2's, which in turn are smaller than asset 3's.

Panel B examines the MV agent's optimal portfolio ($\boldsymbol{q}_0^*$), the zero-covariance portfolio ($\boldsymbol{q}_{zc}$), and the market portfolio ($\mathbf{1}$). Due to asset market clearing, $\boldsymbol{q}_0^* = \mathbf{1}$; see the second, third, and fourth columns. The expected return and risk statistics of $\mathbf{1}$ exceed those of $\boldsymbol{q}_{zc}$; see the last six columns.

Fig. 3A plots a curve showing the equilibrium MV frontier where point Q_0 represents $\boldsymbol{q}_0^*$ (as before, $\theta_0 = 0.95$). Since $\theta_0 = 1$

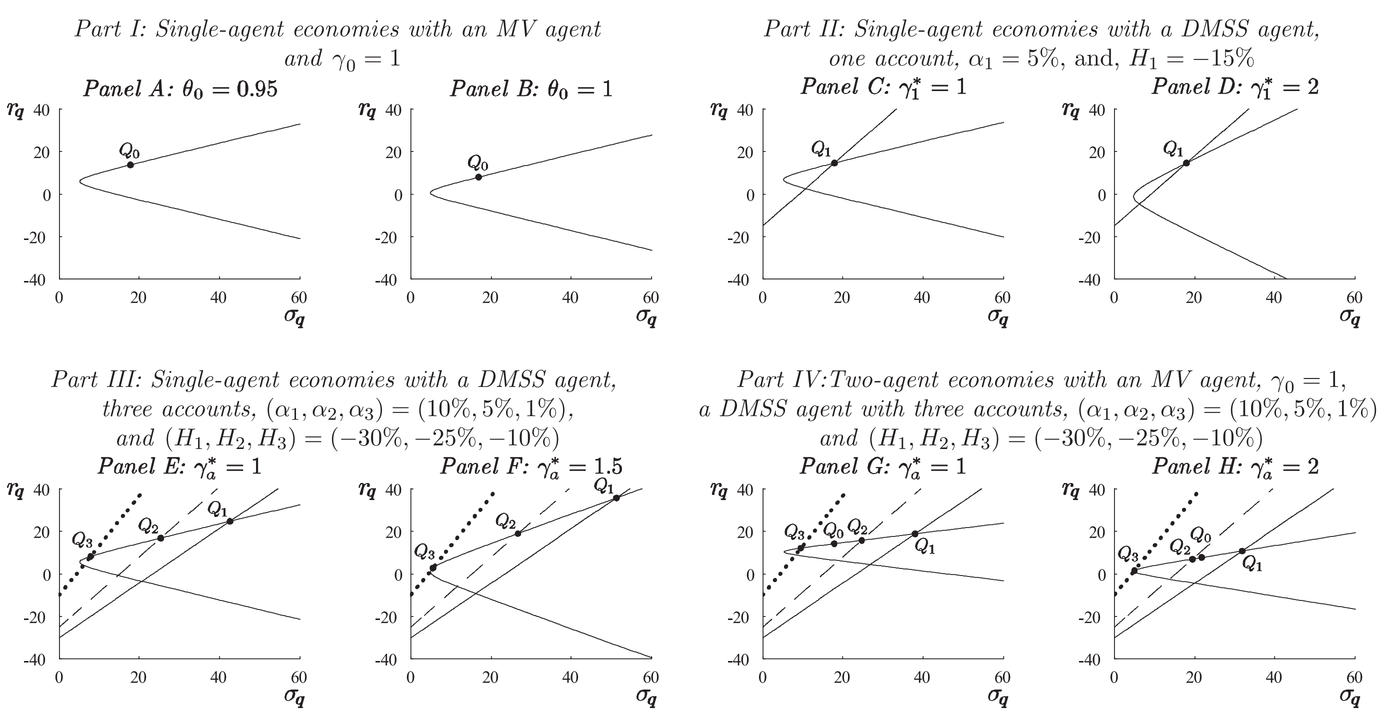

Fig. 3. Equilibrium MV frontiers and optimal portfolios. In each panel, the curve represents portfolios on the equilibrium MV frontier in ($r_{\boldsymbol{q}}$, $\sigma_{\boldsymbol{q}}$) space where $r_{\boldsymbol{q}}$ and $\sigma_{\boldsymbol{q}}$ (reported in percentage points) denote, respectively, the expected return and return standard deviation of portfolio $\boldsymbol{q}$. Panels A and B consider single-agent economies with an MV agent. Panels C and D consider single-agent economies with a DMSS agent and one account. Panels E and F consider single-agent economies with a DMSS agent and three accounts. Panels G and H consider two-agent economies with an MV agent and a DMSS agent with three accounts. In panels A, B, G, and H, point Q_0 shows the MV agent's optimal portfolio. In panels C–H, the solid line corresponds to the probability constraint for account 1, whereas point Q_1 shows the DMSS agent's optimal portfolio within account 1. In panels E–H, the dashed and dotted lines correspond to the probability constraints for, respectively, accounts 2 and 3, whereas points Q_2 and Q_3 show the DMSS agent's optimal portfolios within accounts 2 and 3, respectively. The titles of the parts and panels of the figure note the MV agent's risk aversion coefficient (γ_0), the DMSS agent's threshold probabilities (α_1, α_2, and α_3) and threshold returns (H_1, H_2, and H_3), and the parameters used in Theorems 4–7 to find equilibria (θ_0, γ_1^*, and γ_a^*). Other parameters are from Table 1.

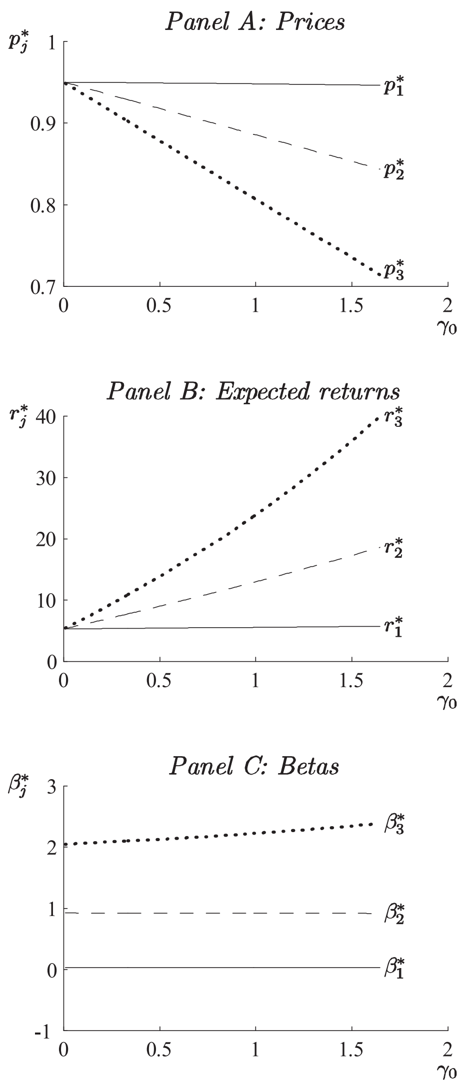

Fig. 4. Asset prices, expected returns, and betas in a single-agent economy with an MV agent. Panel A plots equilibrium asset prices (p_j^*, $j = 1, 2, 3$) as a function of the MV agent's risk aversion coefficient (γ_0) while setting θ_0 to 0.95 (see Theorem 4). Panels B and C plot, respectively, the corresponding expected asset returns (r_j^*, $j = 1, 2, 3$) and asset betas (β_j^*, $j = 1, 2, 3$). In all panels, the solid, dashed, and dotted lines refer to, respectively, assets 1, 2, and 3. In each panel, economy parameters other than γ_0 take the values in panels A and B1 of Table 1. Expected returns are reported in percentage points.

in Fig. 3B, the corresponding equilibrium asset prices are 1.0526 [$= \frac{1}{0.95}$] times those for Fig. 3A; see Theorem 4. Hence, the values of A/C and $\sqrt{1/C}$ in Fig. 3B are 0.95 times those in Fig. 3A, but the value of $\sqrt{D/C}$ is equal in the two figures.

The solid, dashed, and dotted lines in Fig. 4A display the respective equilibrium prices of assets 1, 2, and 3 for various values of γ_0 when $\theta_0 = 0.95$.[17] Since the covariance between the payoffs of any given asset and the market portfolio is positive, asset prices decrease in γ_0. In Fig. 4B, expected asset returns thus increase in γ_0. In Fig. 4C, the betas of assets 1 and 2 are unaffected by the value of γ_0, whereas the beta of asset 3 increases slightly in γ_0.

[17] We consider values of γ_0 such that the corresponding equilibrium expected asset returns do not exceed 40%. Figures that consider the entire range of values of γ_0 are available upon request. Similar remarks apply to the subsequent economies.

4.2. Single-agent economy with a DMSS agent and one account

Consider a single-agent economy with a DMSS agent and one account where the endowment of each asset is one; see Panel B2 of Table 1. Table 3 shows equilibrium values of various quantities when $\alpha_1 = 5\%$, $H_1 = -15\%$, and $\gamma_1^* = 1$;[18] see Theorem 5. Panel A examines the assets. Since Tables 2 and 3 use the same value for, respectively, γ_0 and γ_1^*, the prices in Table 3A equal those in Table 2A times a positive number, which is 0.9923 [$= \frac{\theta_{\gamma_1^*}}{\theta_0} = \frac{0.9427}{0.95}$]; see the second column as well as Theorems 4 and 5. The expected returns and return standard deviations are thus larger in Table 3A than in Table 2A, return VaRs are smaller, and betas are equal; see the last six columns.

Table 3B examines the DMSS agent's optimal portfolio within account 1 ($\boldsymbol{q}_1^*$), the zero-covariance portfolio ($\boldsymbol{q}_{zc}$), and the market portfolio ($\mathbf{1}$). Due to asset market clearing, $\boldsymbol{q}_1^* = \mathbf{1}$; see the second, third, and fourth columns. Compared to Table 2B, expected returns and return standard deviations are larger, whereas return VaRs are smaller; see the next five columns.

Fig. 3C plots a curve showing the equilibrium MV frontier, a line corresponding to the probability constraint for account 1, and point Q_1 (where the line crosses the top half of the curve) representing $\boldsymbol{q}_1^*$. Since in Fig. 3D γ_1^* is 2 instead of 1, the corresponding equilibrium asset prices are *not* proportional to those for Fig. 3C; see Theorem 5. While the values of A/C and $\sqrt{1/C}$ in Fig. 3D are smaller than those in Fig. 3C, the value of $\sqrt{D/C}$ is larger.

In Fig. 5A, we display equilibrium asset prices for various values of γ_1^*. While the prices of assets 1 and 2 (respectively, solid and dashed lines) increase in γ_1^*, that of asset 3 (dotted line) decreases in γ_1^*. Hence, as Fig. 5B shows, the expected returns of assets 1 and 2 decrease in γ_1^* but that of asset 3 increases in γ_1^*. In Fig. 5C, the betas of assets 1 and 2 are unaffected by the value of γ_1^* but that of asset 3 increases slightly in γ_1^*.

Recall that α_1 and H_1 are, respectively, 5% and -15% in panels A–C. In comparison, α_1 and H_1 are, respectively: (i) 5% and -10% in panels D–F; (ii) 10% and -15% in panels G–I; and (iii) 10% and -10% in panels J–L. Fixing the values of α_1 and γ_1^* (for which asset prices are reported), prices are lower when $H_1 = -10\%$; see, e.g., panels A and D. Also, fixing the values of H_1 and γ_1^* (for which asset prices are reported), prices are higher when $\alpha_1 = 10\%$; see, e.g., panels A and G. While expected asset returns notably depend on the values of α_1 and H_1 in panels B, E, H, and K, asset betas do not in panels C, F, I, and L.

4.3. Single-agent economy with a DMSS agent and three accounts

Consider a single-agent economy with a DMSS agent and three accounts. The agent's endowments of each asset in accounts 1, 2, and 3 are, respectively, 0.2, 0.2, and 0.6;[19] see Panel B3 of Table 1. Table 4 shows equilibrium values of various quantities when $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$, $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$, and $\gamma_a^* = 1$; see Theorem 6. Panel A examines the assets. Since Tables 4 and 3 use the same value for,

[18] Table 3 pinpoints an equilibrium from the set of equilibria by using a plausible value for γ_1^*. Alternatively, if the price of an asset increases in γ_1^*, then the use of a plausible value for its price identifies an equilibrium. More generally, if there is a monotonic relation between an equilibrium quantity (e.g., based on the return distribution of $\boldsymbol{q}_1^*$) and γ_1^*, then the use of a plausible value for this quantity identifies an equilibrium. Similar remarks apply to the subsequent economies.

[19] In the three-account example of DMSS, the agent has 20%, 20%, and 60% of his or her total wealth in, respectively, accounts 1, 2, and 3. Our assumptions on the number of accounts and endowments in this section are thus similar to theirs.

Table 3
Equilibrium in a single-agent economy with a DMSS agent and one account. Consider an economy with a DMSS agent and three assets ($j = 1, 2, 3$) as shown in panel A of Table 1. This agent has one account, the asset endowments in panel B2 of Table 1, a threshold probability of $\alpha_1 = 5\%$, and a threshold return of $H_1 = -15\%$. Panel A provides equilibrium asset prices ($p_j^*, j = 1, 2, 3$), expected returns ($r_j^*, j = 1, 2, 3$), return standard deviations ($\sigma_j^*, j = 1, 2, 3$), return VaRs at the 90%, 95%, and 99% confidence levels (respectively, $V_{0.90,j}^*$, $V_{0.95,j}^*$, and $V_{0.99,j}^*$, $j = 1, 2, 3$), and betas ($\beta_j^*, j = 1, 2, 3$). Panel B shows the composition of the DMSS agent's optimal portfolio within account 1 ($\boldsymbol{q}_1^*$), zero-covariance portfolio ($\boldsymbol{q}_{zc}$), and market portfolio ($\mathbf{1}$) along with their expected returns ($r_{\boldsymbol{q}}^*$), return standard deviations ($\sigma_{\boldsymbol{q}}^*$), return VaRs at the 90%, 95%, and 99% confidence levels (respectively, $V_{0.90,\boldsymbol{q}}^*$, $V_{0.95,\boldsymbol{q}}^*$, and $V_{0.99,\boldsymbol{q}}^*$), and betas ($\beta_{\boldsymbol{q}}^*$). The equilibrium is determined by using an implied risk aversion coefficient of the DMSS agent's optimal portfolio within account 1, γ_1^*, of one (see Theorem 5). Expected returns, return standard deviations, and return VaRs are reported in percentage points.

Panel A. Assets: prices, expected returns, and risk statistics

j	p_j^*	r_j^*	σ_j^*	$V_{0.90,j}^*$	$V_{0.95,j}^*$	$V_{0.99,j}^*$	β_j^*
1	0.940	6.347	5.317	0.468	2.399	6.023	0.032
2	0.879	13.821	22.764	15.353	23.623	39.137	0.917
3	0.801	24.875	43.706	31.137	47.016	76.801	2.228

Panel B. The DMSS agent's optimal portfolio within account 1, zero-covariance portfolio, and market portfolio: composition, expected returns, risk statistics, and implied risk aversion coefficient

	q_1	q_2	q_3	$r_{\boldsymbol{q}}^*$	$\sigma_{\boldsymbol{q}}^*$	$V_{0.90,\boldsymbol{q}}^*$	$V_{0.95,\boldsymbol{q}}^*$	$V_{0.99,\boldsymbol{q}}^*$	$\beta_{\boldsymbol{q}}^*$	γ_1^*
$\boldsymbol{q}_1^*$	1.000	1.000	1.000	14.517	17.945	8.481	15.000	27.229	1.000	1.000
$\boldsymbol{q}_{zc}$	2.788	0.068	−0.077	6.081	5.405	0.846	2.810	6.494	0.000	–
$\mathbf{1}$	1.000	1.000	1.000	14.517	17.945	8.481	15.000	27.229	1.000	–

Table 4
Equilibrium in a single-agent economy with a DMSS agent and three accounts. Consider an economy with a DMSS agent and three assets ($j = 1, 2, 3$) as shown in panel A of Table 1. This agent has three accounts, the asset endowments in panel B3 of Table 1, threshold probabilities given by $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$, and threshold returns given by $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$. Panel A provides equilibrium asset prices ($p_j^*, j = 1, 2, 3$), expected returns ($r_j^*, j = 1, 2, 3$), return standard deviations ($\sigma_j^*, j = 1, 2, 3$), return VaRs at the 90%, 95%, and 99% confidence levels (respectively, $V_{0.90,j}^*$, $V_{0.95,j}^*$, and $V_{0.99,j}^*$, $j = 1, 2, 3$), and betas ($\beta_j^*, j = 1, 2, 3$). Panel B shows the composition of the DMSS agent's optimal portfolio within accounts 1, 2, and 3 (respectively, $\boldsymbol{q}_1^*$, $\boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$), his or her aggregate portfolio ($\boldsymbol{q}_a^*$), zero-covariance portfolio ($\boldsymbol{q}_{zc}$), and market portfolio ($\mathbf{1}$) along with their expected returns ($r_{\boldsymbol{q}}^*$), return standard deviations ($\sigma_{\boldsymbol{q}}^*$), return VaRs at the 90%, 95%, and 99% confidence levels (respectively, $V_{0.90,\boldsymbol{q}}^*$, $V_{0.95,\boldsymbol{q}}^*$, and $V_{0.99,\boldsymbol{q}}^*$), and betas ($\beta_{\boldsymbol{q}}^*$). It also shows the implied risk aversion coefficients of the optimal portfolios within accounts ($\gamma_m^*, m = 1, 2, 3$). The equilibrium is determined by using an implied risk aversion coefficient of the DMSS agent's aggregate portfolio, γ_a^*, of one (see Theorem 6). Expected returns, return standard deviations, and return VaRs are reported in percentage points.

Panel A. Assets: prices, expected returns, and risk statistics

j	p_j^*	r_j^*	σ_j^*	$V_{0.90,j}^*$	$V_{0.95,j}^*$	$V_{0.99,j}^*$	β_j^*
1	0.951	5.118	5.256	1.618	3.527	7.109	0.032
2	0.889	12.505	22.501	16.331	24.506	39.840	0.917
3	0.810	23.432	43.201	31.933	47.628	77.069	2.228

Panel B. The DMSS agent's optimal portfolios within accounts and aggregate portfolio, zero-covariance portfolio, and market portfolio: composition, expected returns, risk statistics, and implied risk aversion coefficients

	q_1	q_2	q_3	$r_{\boldsymbol{q}}^*$	$\sigma_{\boldsymbol{q}}^*$	$V_{0.90,\boldsymbol{q}}^*$	$V_{0.95,\boldsymbol{q}}^*$	$V_{0.99,\boldsymbol{q}}^*$	$\beta_{\boldsymbol{q}}^*$	γ_1^*	γ_2^*	γ_3^*	γ_a^*
$\boldsymbol{q}_1^*$	−0.288	0.455	0.494	24.581	42.590	30.000	45.473	74.497	2.366	2.008	–	–	–
$\boldsymbol{q}_2^*$	0.048	0.279	0.291	16.735	25.373	15.782	25.000	42.292	1.425	–	3.417	–	–
$\boldsymbol{q}_3^*$	1.240	0.266	0.214	8.217	7.831	1.818	4.663	10.000	0.403	–	–	4.774	–
$\boldsymbol{q}_a^*$	1.000	1.000	1.000	13.194	17.738	9.538	15.982	28.070	1.000	–	–	–	1.000
$\boldsymbol{q}_{zc}$	2.788	0.068	−0.077	4.855	5.343	1.992	3.933	7.574	0.000	–	–	–	–
$\mathbf{1}$	1.000	1.000	1.000	13.194	17.738	9.538	15.982	28.070	1.000	–	–	–	–

respectively, γ_a^* and γ_1^*, the prices in Table 4A equal those in Table 3A times a positive number which exceeds one; see the second column. Compared to Table 3A, expected returns and return standard deviations are thus smaller in Table 4A, return VaRs are larger, and betas are equal; see the last six columns.

Table 4B examines the DMSS agent's optimal portfolios within accounts 1, 2, and 3 ($\boldsymbol{q}_1^*$, $\boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$, respectively), his or her aggregate portfolio ($\boldsymbol{q}_a^*$), the zero-covariance portfolio ($\boldsymbol{q}_{zc}$), and the market portfolio ($\mathbf{1}$). Asset holdings vary notably across accounts; see the second, third, and fourth columns. The expected return and risk statistics of $\boldsymbol{q}_1^*$ exceed those of $\boldsymbol{q}_2^*$, which in turn exceed those of $\boldsymbol{q}_3^*$; see the next six columns. The implied risk aversion coefficient of $\boldsymbol{q}_1^*$ is smaller than that of $\boldsymbol{q}_2^*$, which in turn is smaller than that of $\boldsymbol{q}_3^*$; see the next three columns. Due to asset market clearing, $\boldsymbol{q}_a^* = \mathbf{1}$;[20] see the second, third, and fourth columns.

Fig. 3E plots: (i) a curve showing the equilibrium MV frontier; (ii) solid, dashed, and dotted lines corresponding to the probability constraints for, respectively, accounts 1, 2, and 3; and (iii) points Q_1, Q_2, and Q_3 (located where such lines cross the top half of the curve) representing, respectively, $\boldsymbol{q}_1^*, \boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$. Since in Fig. 3F γ_a^* is 1.5 instead of 1, the corresponding equilibrium asset prices are

[20] Note that the implied risk tolerance coefficient of $\boldsymbol{q}_a^*$ $[1/\gamma_a^* = 1/1 = 1]$ is the sum of the implied risk tolerance coefficients of $\boldsymbol{q}_1^*, \boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$ $[1/\gamma_1^* + 1/\gamma_2^* + 1/\gamma_3^* = 1/2.008 + 1/3.417 + 1/4.774 = 1]$; see the last four columns. A similar result holds in the two-agent economy with MV and DMSS agents (examined later).

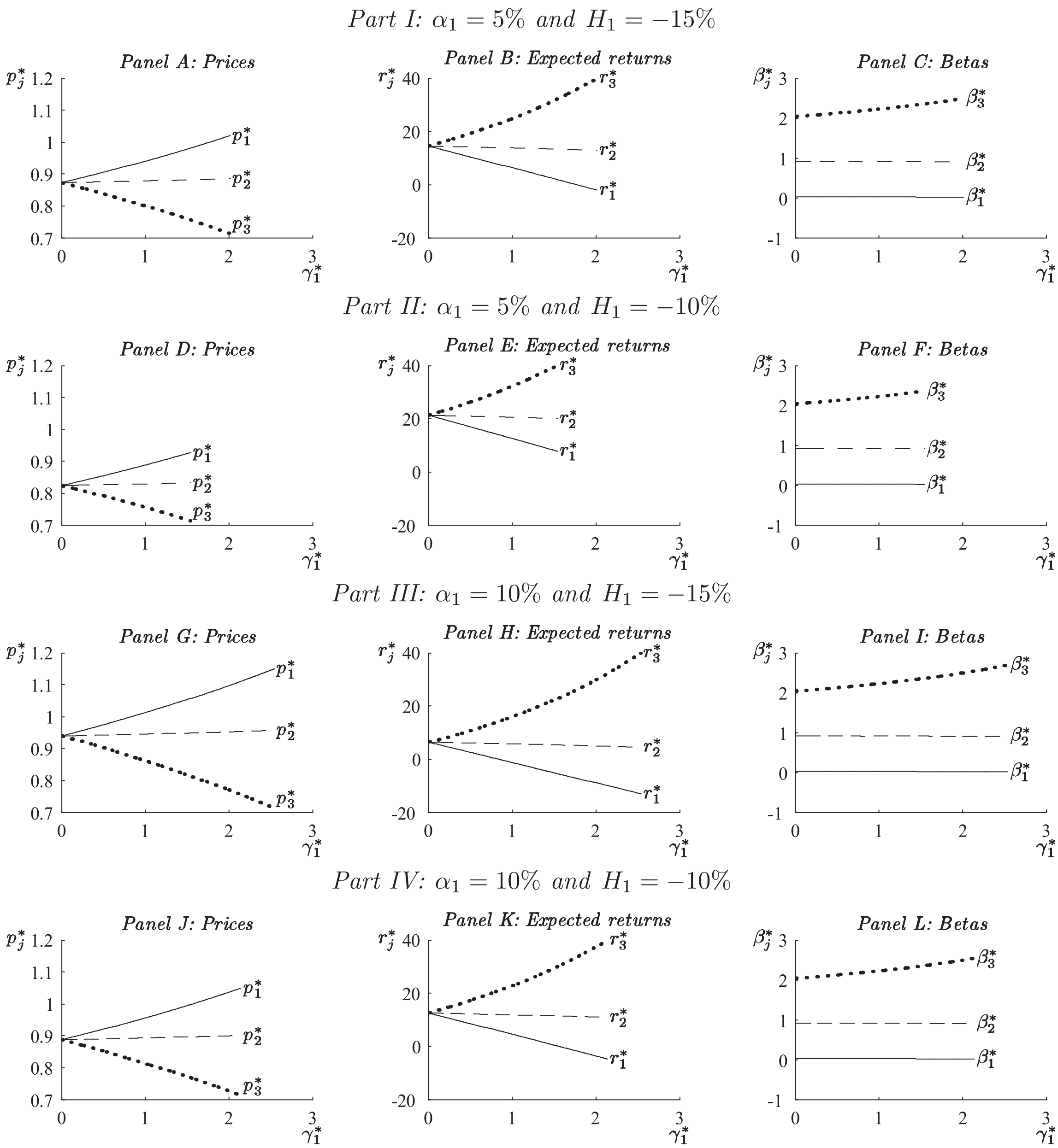

Fig. 5. Asset prices, expected returns, and betas in a single-agent economy with a DMSS agent and one account. Given a threshold probability (α_1) and a threshold return (H_1) for account 1, panels A, D, G, and J plot equilibrium asset prices (p_j^*, $j = 1, 2, 3$) as a function of the implied risk aversion coefficient of the DMSS agent's optimal portfolio within account 1 (γ_1^*). Panels B, E, H, and K plot the corresponding expected asset returns (r_j^*, $j = 1, 2, 3$). Panels C, F, I, and L plot the corresponding asset betas (β_j^*, $j = 1, 2, 3$). In all panels, the solid, dashed, and dotted lines refer to, respectively, assets 1, 2, and 3. In each panel, parameters other than α_1 and H_1 (shown in the title of the corresponding part of the figure) take the values in panels A and B2 of Table 1. Expected returns are reported in percentage points.

not proportional to those for Fig. 3E;[21] see Theorem 6. The values of A/C and $\sqrt{1/C}$ in Fig. 3F are smaller than those in Fig. 3E, but the value of $\sqrt{D/C}$ is larger. While $\boldsymbol{q}_1^*$ and $\boldsymbol{q}_2^*$ lie further up on the frontier in Fig. 3F than in Fig. 3E, $\boldsymbol{q}_3^*$ lies further down on it.

In Fig. 6A, we display equilibrium asset prices for various values of γ_a^*. While the price of asset 1 (solid line) increases in γ_a^*, the price of asset 3 (dotted line) decreases in γ_a^*. The price of asset 2 (dashed line) increases in γ_a^* except for at larger reported values of γ_a^* where the price decreases. In Fig. 6B, expected asset returns depend on the value of γ_a^*. In Fig. 6C, the betas of assets 1 and 2 are unaffected by the value of γ_a^* but that of asset 3 increases slightly in γ_a^*.

Panels D–F use the same thresholds as panel A except that α_1 is 15% instead of 10%.[22] Given any value of γ_a^* for which both figures show prices, the prices in Fig. 6D exceed those in Fig. 6A. Similarly, panel G–I use the same thresholds as panel A except that H_1 is −25% instead of −30%. Given any value of γ_a^* for which both figures show prices, the prices in Fig. 6G are smaller than those in Fig. 6A. While expected returns notably depend on the values of α_1 and H_1 in panels B, E, and H, betas do not in panels C, F, and I.

[21] Since the upper bound on γ_a^* is less than two, we use $\gamma_a^* = 1.5$ in Fig. 3F instead of $\gamma_a^* = 2$ as in Fig. 3D.

[22] Here, we focus on the threshold probability of account 1 because similar results hold for those of accounts 2 and 3. A similar remark applies to the threshold return of account 1 and to the two-agent economy examined later.

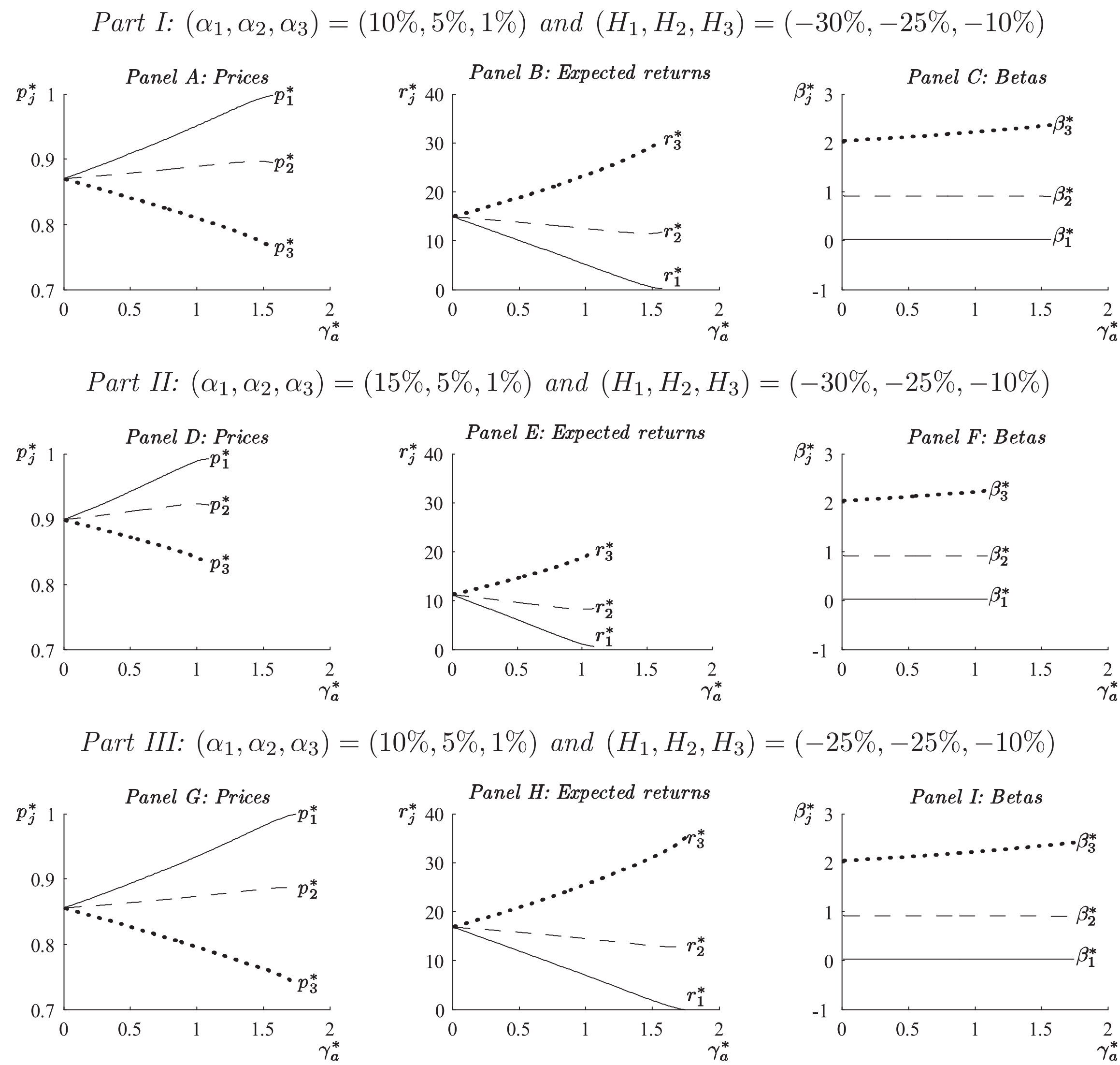

Fig. 6. Asset prices, expected returns, and betas in a single-agent economy with a DMSS agent and three accounts. Given the threshold probabilities ($\alpha_m, m = 1, 2, 3$) and threshold returns ($H_m, m = 1, 2, 3$) for three accounts, panels A, D, and G plot equilibrium asset prices ($p_j^*,\ j = 1, 2, 3$) as a function of the implied risk aversion coefficient of the DMSS agent's aggregate portfolio (γ_a^*). Panels B, E, and H plot the corresponding expected asset returns ($r_j^*,\ j = 1, 2, 3$). Panels C, F, and I plot the corresponding asset betas ($\beta_j^*,\ j = 1, 2, 3$). In all panels, the solid, dashed, and dotted lines refer to, respectively, assets 1, 2, and 3. In each panel, parameters other than thresholds (shown in the title of the corresponding part of the figure) take the values in panels A and B3 of Table 1. Expected returns are reported in percentage points.

In panels A, B, and C of Fig. 7, the value of γ_a^* notably affects $\boldsymbol{q}_1^*$, $\boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$, respectively (the thresholds are as in Table 4). For example, $\boldsymbol{q}_1^*$'s holding of asset 1 (solid line) decreases in γ_a^*, whereas those of assets 2 and 3 (respectively, dashed and dotted lines) increase in γ_a^*; see panel A.

The thresholds in panels D–F equal those in panels A–C except that α_1 is 15% instead of 10%. Given any value of γ_a^* such that panels A–F show asset holdings, the size of α_1 notably affects the holdings in *all* accounts. For example, consider the holding of asset 1. In account 1, that in panel D is more negative than that in panel A. In account 2, that in panel E exceeds that in panel B. In account 3, that in panel F exceeds that in panel C. Similarly, the thresholds in panels G–I are as in panels A–C except that H_1 is -25% instead of -30%. Given any value of γ_a^* such that panels A–C and G–I show asset holdings, the size of H_1 notably affects the holdings in all accounts. Again, consider the holding of asset 1. In account 1, this holding in panel G is less negative than that in panel A. In account 2, the holding in panel H is smaller than that in panel B. In account 3, the holding in panel I is smaller than that in panel C.

4.4. Two-agent economy with an MV agent and a DMSS agent with three accounts

Consider a two-agent economy with an MV agent and a DMSS agent with three accounts. For each asset, the MV agent's endowment of 0.5 equals the DMSS agent's aggregate endowment of 0.5 [$= 0.1 + 0.1 + 0.3$]; see Panel B4 of Table 1. Table 5 shows equilibrium values of various quantities when $\gamma_0 = 1,\ (\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%),\ (H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$, and $\gamma_a^* = 1$; see Theorem 7. Panel A examines the assets. For assets 1 and 2, the price and return VaRs in Table 5A are smaller than those in Table 4A, but the expected return, return standard deviation, and beta are larger. For asset 3, the price and return VaRs in Table 5A are larger than those in Table 4A, but the expected return, return standard deviation, and beta are smaller.

Table 5B examines the MV agent's optimal portfolio ($\boldsymbol{q}_0^*$), the DMSS agent's optimal portfolios within accounts 1, 2, and 3 ($\boldsymbol{q}_1^*$, $\boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$, respectively), his or her aggregate portfolio ($\boldsymbol{q}_a^*$), the zero-covariance portfolio ($\boldsymbol{q}_{zc}$), and the market portfolio ($\mathbf{1}$). Since $\gamma_0 = \gamma_a^*$ and the MV agent's asset endowments equal the DMSS

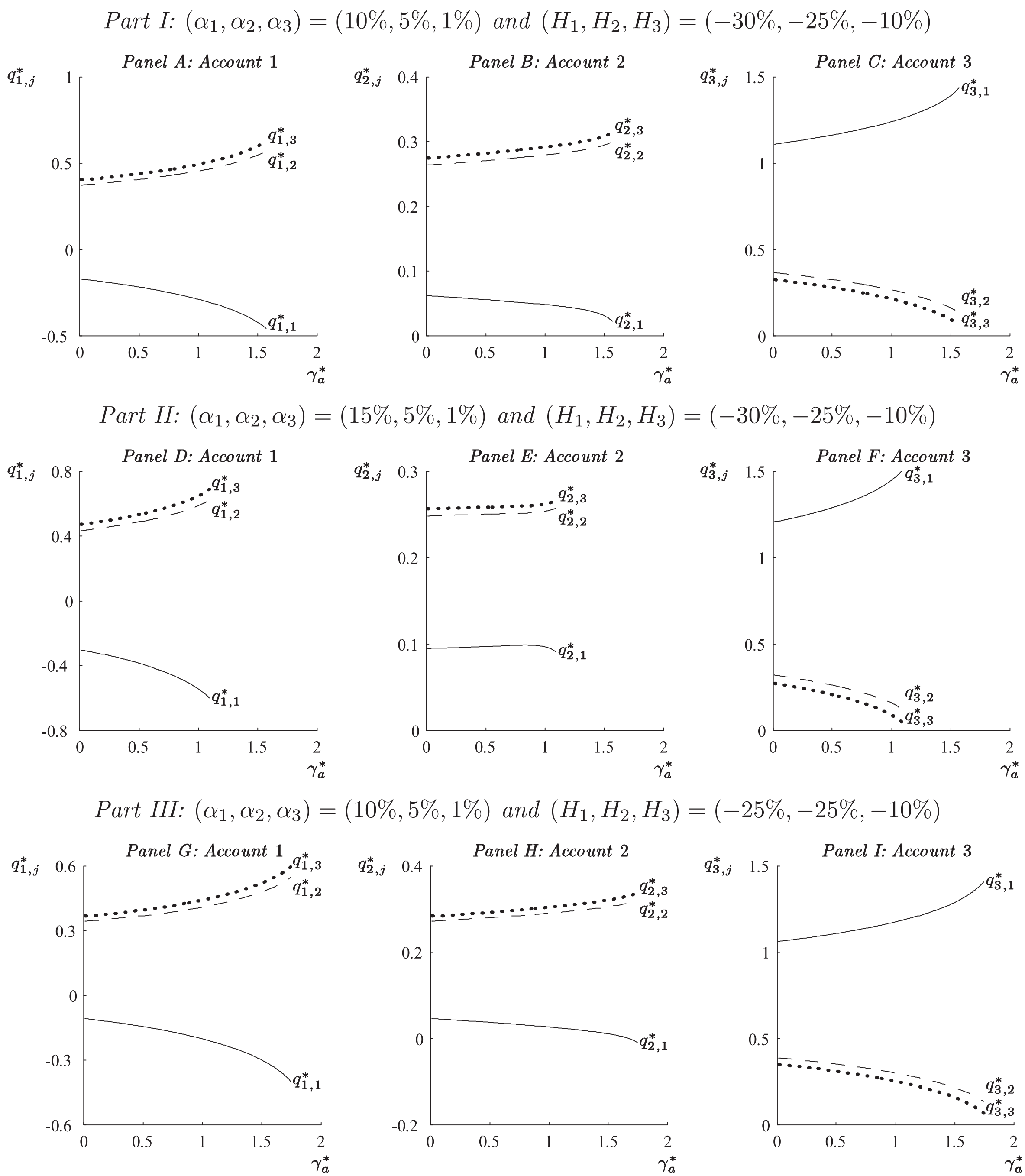

Fig. 7. Optimal portfolios in a single-agent economy with a DMSS agent and three accounts. Given the threshold probabilities ($\alpha_m, m = 1, 2, 3$) and threshold returns ($H_m, m = 1, 2, 3$) for three accounts, each panel plots the DMSS agent's optimal portfolio within a given account ($q^*_{m,j}$, $m = 1, 2, 3,\ j = 1, 2, 3$) as a function of the implied risk aversion coefficient of his or her aggregate portfolio (γ_a^*). Panels A, D, and G consider account 1. While panels B, E, and H consider account 2, panels C, F, and I consider account 3. In all panels, the solid, dashed, and dotted lines report the optimal holdings of, respectively, assets 1, 2, and 3. In each panel, parameters other than thresholds (shown in the title of the corresponding part of the figure) take the values in panels A and B3 of Table 1.

agent's aggregate asset endowments, $\boldsymbol{q}_0^* = \boldsymbol{q}_a^* = 0.500 \times \mathbf{1}$; see the second, third, and fourth columns. Hence, $\boldsymbol{q}_0^*, \boldsymbol{q}_a^*$, and $\mathbf{1}$ have the same expected return and risk statistics; see the next six columns. The expected return and risk statistics of $\boldsymbol{q}_1^*$ exceed those of $\boldsymbol{q}_2^*$, which in turn exceed those of $\boldsymbol{q}_3^*$; see also such columns. The implied risk aversion coefficient of $\boldsymbol{q}_1^*$ is smaller than that of $\boldsymbol{q}_2^*$, which in turn is smaller than that of $\boldsymbol{q}_3^*$; see the next three columns.

Fig. 3G plots: (i) a curve showing the equilibrium MV frontier; (ii) point Q_0 on it representing $\boldsymbol{q}_0^*$; (iii) solid, dashed, and dotted lines corresponding to the probability constraints for, respectively, accounts 1, 2, and 3; and (iv) points Q_1, Q_2, and Q_3 (located where such lines cross the top half of the curve) representing, respectively, $\boldsymbol{q}_1^*, \boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$. Since in Fig. 3H γ_a^* is 2 instead of 1, the corresponding equilibrium asset prices are *not* proportional to those for Fig. 3G; see Theorem 7. The values of A/C and $\sqrt{1/C}$ in Fig. 3H are smaller than those in Fig. 3G, but the value of $\sqrt{D/C}$ is larger. While $\boldsymbol{q}_0^*$ is further up on the frontier in Fig. 3H than in Fig. 3G, $\boldsymbol{q}_1^*$, $\boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$ are further down on it.

Using the same primitives of the economy as in Table 5, Fig. 8A shows that equilibrium asset prices increase in γ_a^*. In Fig. 8B, expected asset returns thus decrease in γ_a^*. In Fig. 8C, the betas of

Table 5
Equilibrium in a two-agent economy with an MV agent and a DMSS agent with three accounts. Consider an economy with an MV agent, a DMSS agent, and three assets ($j = 1, 2, 3$) as shown in panel A of Table 1. The MV agent has the asset endowments in the second column of panel B4 of Table 1 and a risk aversion coefficient of $\gamma_0 = 1$. The DMSS agent has three accounts, the asset endowments in the last three columns of this panel, threshold probabilities given by $(\alpha_1, \alpha_2, \alpha_3) = (10\%, 5\%, 1\%)$, and threshold returns given by $(H_1, H_2, H_3) = (-30\%, -25\%, -10\%)$. Panel A provides equilibrium asset prices ($p_j^*, j = 1, 2, 3$), expected returns ($r_j^*, j = 1, 2, 3$), return standard deviations ($\sigma_j^*, j = 1, 2, 3$), return VaRs at the 90%, 95%, and 99% confidence levels (respectively, $V_{0.90,j}^*$, $V_{0.95,j}^*$, and $V_{0.99,j}^*$, $j = 1, 2, 3$), and betas ($\beta_j^*, j = 1, 2, 3$). Panel B shows the composition of the MV agent's optimal portfolio ($\boldsymbol{q}_0^*$), the DMSS agent's optimal portfolios within accounts 1, 2, and 3 (respectively, $\boldsymbol{q}_1^*$, $\boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$), his or her aggregate portfolio ($\boldsymbol{q}_a^*$), zero-covariance portfolio ($\boldsymbol{q}_{zc}$), and market portfolio ($\mathbf{1}$) along with their expected returns ($r_{\boldsymbol{q}}^*$), return standard deviations ($\sigma_{\boldsymbol{q}}^*$), return VaRs at the 90%, 95%, and 99% confidence levels (respectively, $V_{0.90,\boldsymbol{q}}^*$, $V_{0.95,\boldsymbol{q}}^*$, and $V_{0.99,\boldsymbol{q}}^*$), and betas ($\beta_{\boldsymbol{q}}^*$). It also shows the implied risk aversion coefficients of the optimal portfolios within accounts ($\gamma_m^*, m = 1, 2, 3$). The equilibrium is determined by using an implied risk aversion coefficient of the DMSS agent's aggregate portfolio, γ_a^*, of one (see Theorem 7). Expected returns, return standard deviations, and return VaRs are reported in percentage points.

Panel A. Assets: prices, expected returns, and risk statistics

j	p_j^*	r_j^*	σ_j^*	$V_{0.90,j}^*$	$V_{0.95,j}^*$	$V_{0.99,j}^*$	β_j^*
1	0.909	10.065	5.503	−3.012	−1.013	2.737	0.033
2	0.879	13.797	22.759	15.371	23.639	39.150	0.920
3	0.841	18.873	41.605	34.447	49.562	77.916	2.128

Panel B. The MV agent's optimal portfolio, the DMSS agent's optimal portfolios within accounts and aggregate portfolio, zero-covariance portfolio, and market portfolio: composition, expected returns, risk statistics, and implied risk aversion coefficients

	q_1	q_2	q_3	$r_{\boldsymbol{q}}^*$	$\sigma_{\boldsymbol{q}}^*$	$V_{0.90,\boldsymbol{q}}^*$	$V_{0.95,\boldsymbol{q}}^*$	$V_{0.99,\boldsymbol{q}}^*$	$\beta_{\boldsymbol{q}}^*$	γ_1^*	γ_2^*	γ_3^*	γ_a^*
$\boldsymbol{q}_0^*$	0.500	0.500	0.500	14.131	17.885	8.789	15.286	27.475	1.000	–	–	–	–
$\boldsymbol{q}_1^*$	−0.109	0.202	0.219	18.751	38.041	30.000	43.820	69.745	2.099	2.266	–	–	–
$\boldsymbol{q}_2^*$	0.028	0.135	0.141	15.726	24.760	16.005	25.000	41.873	1.379	–	3.530	–	–
$\boldsymbol{q}_3^*$	0.581	0.163	0.140	12.060	9.483	0.093	3.538	10.000	0.507	–	–	3.632	–
$\boldsymbol{q}_a^*$	0.500	0.500	0.500	14.131	17.885	8.789	15.286	27.475	1.000	–	–	–	1.000
$\boldsymbol{q}_{zc}$	2.899	0.070	−0.080	9.927	5.601	−2.749	−0.714	3.103	0.000	–	–	–	–
$\mathbf{1}$	1.000	1.000	1.000	14.131	17.885	8.789	15.286	27.475	1.000	–	–	–	–

assets 1 and 2 are unaffected by the value of γ_a^* but that of asset 3 increases slightly in γ_a^*.

Panels D–F use the same primitives of the economy as panel A except that γ_0 is now 2 instead of 1. For any value of γ_a^* such that both figures display asset prices, these prices are smaller when γ_0 is 2. Panels G–I use the same primitives of the economy as panel A except that except that α_1 is 15% instead of 10%. For any value of γ_a^* such that both figures display asset prices, these prices are larger when α_1 is 15%. Panels J–L use the same primitives of the economy as panel A except that except that H_1 is −25% instead of −30%. For any value of γ_a^* such that both figures display asset prices, these prices are smaller when H_1 is −25%. While expected asset returns notably depend on the values of γ_0, α_1, and H_1 in panels B, E, H, and K, asset betas do not in panels C, F, I, and L.

Using the same primitives of the economy as in Table 5, panel A of Fig. 9 shows that $\boldsymbol{q}_0^*$'s holding of asset 1 (solid line) decreases in γ_a^* but those of assets 2 and 3 (respectively, dashed and dotted lines) increase in γ_a^*. In contrast, panels B, C, and D show that the holding of asset 1 of, respectively, $\boldsymbol{q}_1^*, \boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$ increases in γ_a^* but those of assets 2 and 3 decrease in γ_a^*.

The primitives of the economy in panels E–H are as in panel A except that γ_0 is 2 instead of 1. Note that the value of γ_0 affects $\boldsymbol{q}_0^*, \boldsymbol{q}_1^*, \boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$. For example, if $\gamma_a^* = 1$, then $\boldsymbol{q}_0^*$'s holding of asset 1 in panel E is larger than that in panel A, whereas those of assets 2 and 3 are smaller.

The primitives of the economy in panels I–L are also as in panel A except that α_1 is 15% instead of 10%. Fix any value of γ_a^* for which panels A–D and I–L report asset holdings. While the size of α_1 does not affect $\boldsymbol{q}_0^*$ (panels A and I), it affects $\boldsymbol{q}_1^*, \boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$ (panels B–D and J–L). Similarly, the primitives of the economy in panels M–P are as in panel A except that H_1 is −25% instead of −30%. Fix any value of γ_a^* for which panels A–D and M–P show asset holdings. While the size of H_1 does not affect $\boldsymbol{q}_0^*$ (panels A and M), it affects $\boldsymbol{q}_1^*, \boldsymbol{q}_2^*$, and $\boldsymbol{q}_3^*$ (panels B–D and N–P).

4.5. Practical plausibility

While the economies in our example simplify reality, they allow us to explore the implications of the heterogeneity of preferences across agents and accounts. Since the MV and DMSS models are both used in practice, the two-agent economies are arguably more plausible than the single-agent economies. The exogenous parameters of all of the economies in our example are nevertheless plausible as noted earlier. The size of the endogenous implied risk aversion coefficient of the DMSS agent's aggregate portfolio is also plausible. For example, when this coefficient is properly set, the difference between the expected returns of high- and low-risk assets is relatively large (or small).

The results in our example are in line with empirical work noting that portfolios with betas less (more) than one have positive (negative) alphas instead of zero alphas as in Sharpe's (1964) CAPM; see Black et al. (1972). When Black's (1972) zero-beta CAPM holds, the alpha of any given portfolio $\boldsymbol{q}$ equals $(r_{\boldsymbol{q}_{zc}} - r_f)(1 - \beta_{\boldsymbol{q}})$ where $r_{\boldsymbol{q}_{zc}}$ and r_f are, respectively, the zero-covariance portfolio's expected return and the risk-free return; see Elton et al. (2014, p. 345). In an equilibrium with risk-free lending but without risk-free borrowing, we have $r_{\boldsymbol{q}_{zc}} > r_f$; see Elton et al. (2014, pp. 317 and 318). Hence, if $\beta_{\boldsymbol{q}}$ is less (more) than one, then $\boldsymbol{q}$'s alpha is positive (negative). For example, assume that $r_{\boldsymbol{q}_{zc}} = 4.855\%$ (as in Table 4B) and $r_f = 2.855\%$ so that $r_{\boldsymbol{q}_{zc}} - r_f = 2\%$. If $\beta_{\boldsymbol{q}} = 0.5$, then $\boldsymbol{q}$'s alpha is 1% $[= 2\% \times (1 - 0.5)]$. Similarly, if $\beta_{\boldsymbol{q}} = 1.5$, then $\boldsymbol{q}$'s alpha is −1% $[= 2\% \times (1 - 1.5)]$.

4.6. Summary of implications from the comparison of economies

Our example illustrates that the preferences of MV and DMSS agents are quite different. First, the MV agent's optimal

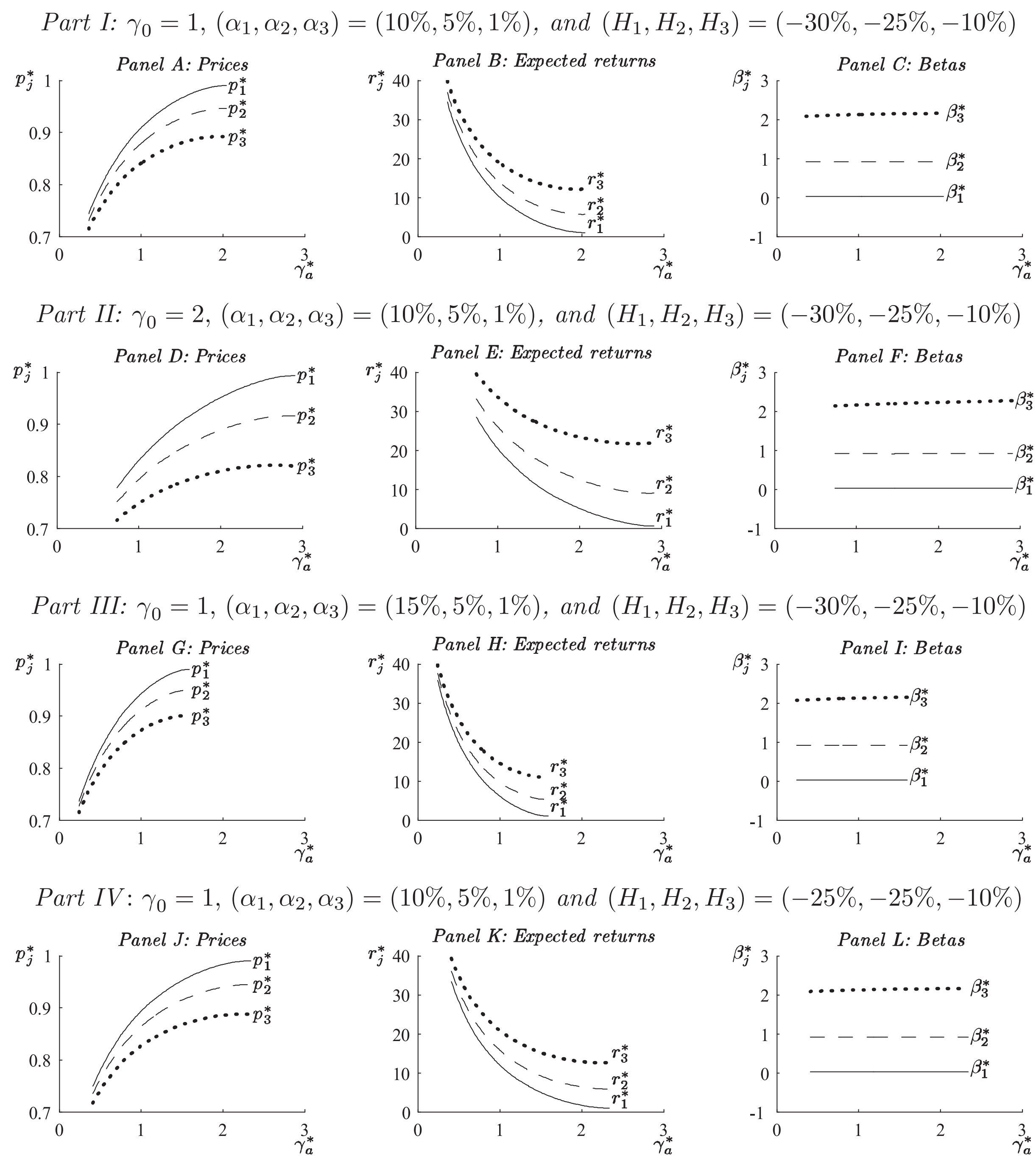

Fig. 8. Asset prices, expected returns, and betas in a two-agent economy with an MV agent and a DMSS agent with three accounts. Given the MV agent's risk aversion coefficient (γ_0) as well as the threshold probabilities (α_m, $m = 1, 2, 3$) and threshold returns (H_m, $m = 1, 2, 3$) for the DMSS agent's accounts, panels A, D, G, and J plot equilibrium asset prices (p_j^*, $j = 1, 2, 3$) as a function of the implied risk aversion coefficient of the DMSS agent's aggregate portfolio (γ_a^*). Panels B, E, H, and K plot the corresponding expected asset returns (r_j^*, $j = 1, 2, 3$). Panels C, F, I, and L plot the corresponding asset betas (β_j^*, $j = 1, 2, 3$). In all panels, the solid, dashed, and dotted lines refer to, respectively, assets 1, 2, and 3. In each panel, parameters other than the MV agent's risk aversion coefficient and thresholds (shown in the title of the corresponding part of the figure) take the values in panels A and B4 of Table 1. Expected returns are reported in percentage points.

portfolio generally differs from the optimal portfolios within accounts and the aggregate portfolio of the DMSS agent in two-agent economies with both agents. Second, while the MV agent's risk aversion coefficient is (by definition) fixed, the implied risk aversion coefficient of the DMSS agent's aggregate portfolio varies across equilibria. Third, fixing the primitives, relative asset prices are unique in equilibria for a single-agent economy with an MV agent but differ across equilibria for the other three types of economies.

4.7. Comparing the cases of endogenous and exogenous moments of the distribution of asset returns

While the moments of the distribution of asset returns are endogenous in our example, we next highlight differences relative to DMSS where such moments are exogenous. First, fixing the thresholds, the optimal portfolio within any given account depends on the implied risk aversion coefficient of the DMSS agent's aggregate portfolio with endogenous moments but is unique with

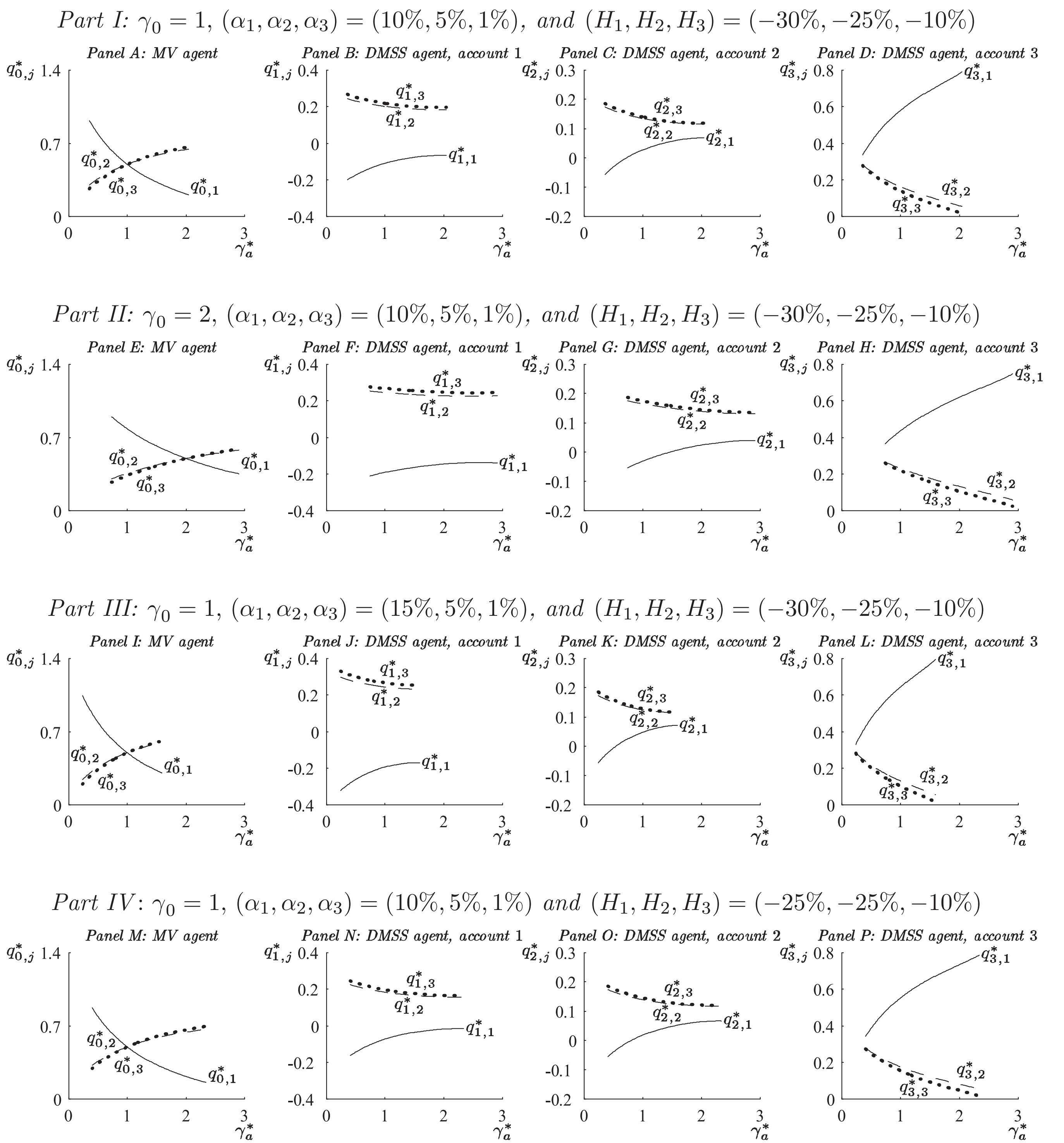

Fig. 9. Optimal portfolios in a two-agent economy with an MV agent and a DMSS agent with three accounts. Given the MV agent's risk aversion coefficient (γ_0) as well as the threshold probabilities (α_m, $m = 1, 2, 3$) and threshold returns (H_m, $m = 1, 2, 3$) for the DMSS agent's accounts, panels A, E, I, and M show the MV agent's optimal portfolio ($q^*_{0,j}$, $j = 1, 2, 3$) as a function of the implied risk aversion coefficient of the DMSS agent's aggregate portfolio (γ^*_a). Similarly, panels B–D, F–H, J–L, and N–P show the DMSS agent's optimal portfolios within accounts ($q^*_{m,j}$, $m = 1, 2, 3$, $j = 1, 2, 3$) as a function of γ^*_a. In all panels, the solid, dashed, and dotted lines report the optimal holdings of, respectively, assets 1, 2, and 3. In each panel, parameters other than the MV agent's risk aversion coefficient and thresholds (shown in the title of the corresponding part of the figure) take the values in panels A and B4 of Table 1.

exogenous moments. Second, again fixing the thresholds, this coefficient varies across equilibria with endogenous moments but is unique with exogenous moments. Third, the size of the thresholds of any given account affects the optimal portfolios within all accounts with endogenous moments whereas it affects the optimal portfolio within only that account with exogenous moments.

4.8. Economies with two or more DMSS agents

Our theoretical results for economies with a DMSS agent who has multiple accounts are applicable also to economies with two or more DMSS agents who have either a single or multiple accounts. For example, consider a two-agent economy with two

DMSS agents, 1 and 2, having, respectively, one and two accounts. In account 1 of agent 1, (a) the endowment of each asset is 0.2, (b) the threshold probability is 10%, and (c) the threshold return is -30%. In accounts 1 and 2 of agent 2: (a) the endowments of each asset are, respectively, 0.2 and 0.6; (b) the threshold probabilities are, respectively, 5% and 1%; and (c) the threshold returns are, respectively, -25% and -10%. This two-agent economy corresponds to the single-agent economy with a DMSS agent and three accounts examined in Section 4.3 where in accounts 1, 2, and 3: (a) the endowments of each asset are, respectively, 0.2, 0.2, and 0.6; (b) the threshold probabilities are, respectively, 10%, 5%, and 1%; and (c) the threshold returns are, respectively, -30%, -25%, and -10%. Our results for the latter economy are thus applicable to the former. For example, the reading of Table 4A is identical for the two economies. The reading of Table 4B for the two-agent economy differs slightly from that noted earlier for the single-agent economy. In the two-agent economy, row $\boldsymbol{q}_1^*$ refers to agent1's optimal portfolio within account 1, whereas rows $\boldsymbol{q}_2^*$ and $\boldsymbol{q}_3^*$ refer to agent 2's optimal portfolios within accounts 1 and 2, respectively. Row $\boldsymbol{q}_a^*$ refers to the combination of the aggregate portfolios of agents 1 and 2. While the implied risk aversion coefficient of agent 1's optimal portfolio within account 1 (and of his or her aggregate portfolio due to having a single account) is 2.008 (column γ_1^*), those of agent 2's optimal portfolios within accounts 1 and 2 are, respectively, 3.417 and 4.774 (columns γ_2^* and γ_3^*). The implied risk aversion coefficient of agent 2's aggregate portfolio equals $(1/3.417 + 1/4.774)^{-1} = 1.992$ (not shown in the table). Also, the implied risk aversion coefficient of the combination of the aggregate portfolios of agents 1 and 2 is 1 (column γ_a^*).

5. Conclusion

In Das et al. (2010, DMSS), an agent divides his or her wealth among mental accounts (hereafter 'accounts') with different goals. For each account, the agent maximizes its expected return subject to the probability of its return being less than or equal to some threshold return not exceeding some threshold probability. The threshold return and threshold probability (hereafter 'thresholds') can vary across accounts to reflect different goals. Assuming that a risk-free asset is absent and risky asset returns have a multivariate normal distribution, DMSS show that optimal portfolios within accounts and the resulting aggregate portfolio are all on the mean-variance (MV) frontier. In DMSS, the moments of this distribution are *exogenous.*

Our paper develops a corresponding equilibrium model where such moments are *endogenous* in four types of economies. The first three types are single-agent economies where the agent has either: (1) an MV objective function (hereafter 'MV agent'); (2) an objective function as in DMSS (hereafter 'DMSS agent') and a single account; or (3) an objective function as in DMSS but multiple accounts. The fourth is a two-agent economy with an MV agent (who has a single account) and a DMSS agent who has multiple accounts.

We obtain the following results. First, in economies with a DMSS agent, there are multiple equilibria that correspond to an endogenous interval for the implied risk aversion coefficient of his or her aggregate portfolio. Since this result holds even in single-agent economies where a DMSS agent has one or more accounts, it is due to his or her objective function but not to the number of accounts. In contrast, the risk aversion coefficient of an MV agent is (by definition) exogenous.

Second, Black's (1972) zero-beta CAPM holds in all four types of economies. Equilibrium expected asset returns are thus in line with influential empirical work finding that portfolios with betas less (more) than one have positive (negative) alphas instead of zero alphas as in Sharpe's (1964) CAPM. However, the equilibrium expected asset returns in a single-agent economy with an MV agent generally differ from those in the other three types of economies.

Third, in economies with a DMSS agent having multiple accounts, the size of the thresholds of any given account affects the optimal portfolios within *all* accounts because the moments of the distribution of asset returns are endogenous. In DMSS, the size of the thresholds of any given account affects the optimal portfolio within only that account because such moments are exogenous.

Three aspects of our contribution are worth emphasizing. First, in developing an equilibrium model with accounts, we complement the economic foundations of the DMSS model. Second, in noting certain differences in the equilibrium implications of using MV and DMSS objective functions for portfolio selection, we enrich the relation between the MV and DMSS models. Third, in showing that the size of the implied risk aversion coefficient of the DMSS agent's aggregate portfolio is found in equilibrium, we add to models where individuals have endogenous preferences.

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Supplementary materials

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