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Nash equilibrium in tariffs in a multi-country trade model[☆]

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ABSTRACT

We study a general equilibrium model of trade with two goods and many countries where each country sets its distortionary tariff non-cooperatively to maximize the payoff of the representative household. We prove the existence of pure strategy Nash equilibria by showing that there are consistent bounds on tariff rates that are common across countries and that payoff functions in the induced game are quasiconcave. Separately, we show that best responses are strictly increasing functions, and provide robust examples that show that the game need not be supermodular. The fact that a country's payoff does not respond monotonically to increases in a competitor's tariff rate, shows that the standard condition in the literature for payoff comparisons across Nash equilibria fails in our model. We then show that the participation of at most two countries in negotiated tariff changes suffices to induce a Pareto improving allocation relative to a Nash equilibrium. Further results provided concern the location of the best response in relation to the free trade point, the monotonicity of payoffs, and the bounds on equilibrium strategies. The final result is that there is no trade if and only if the equilibrium allocation is Pareto optimal.

1. Introduction

The ability to impose a tariff is arguably the tool most commonly used by a government to influence foreign trade. This is done to benefit the country as it moves the equilibrium allocation in an appropriate direction away from free trade. Since retaliation is only to be expected, the resulting strategic equilibrium becomes the object of analysis; importantly, the allocation induced is, quite generally, inefficient.[1] This sets the stage for a role for institutions that regulate and promote international trade to attempt to mitigate the inefficiency by facilitating the negotiation of multilateral agreements.[2] The heterogeneity across countries, particularly in terms of their relative size and tastes, is likely to play a key role in the determination of the rules of multilateral engagement used by these institutions to achieve the desired mitigation. An essential element of any analysis that provides the foundation for such rules would be to clarify the manner in which heterogeneity interacts with the number of countries in consideration, and our aim is to contribute to that analysis.

We study a model with many countries in which the prices that domestic agents face are the world prices distorted by a tariff, and where the revenue from the tariff is distributed by the government to the agents as a lump-sum transfer. Trade in competitive markets results in the determination of world prices for goods in general equilibrium, and each government acts non-cooperatively to set tariff rates to maximize the utility of the agents. The equilibrium concept used is pure strategy Nash equilibrium.

The literature on optimal tariffs in the presence of retaliation has drawn attention to the importance of solving for the Nash equilibrium of a non-cooperative tariff game.[3] However, the quote from Costinot et al. (2016), who present a parametric "new trade model" and study the problem faced by a single country, that "future research should strive to characterize the Nash equilibrium in which all countries attempt to manipulate

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1 See Johnson (1953) for the two country and two good exchange model. He presents a geometric analysis and provides an analytical treatment of the special case of constant elasticity offer curves wherein reaction functions are horizontal or vertical straight lines. He concludes that a tariff may be welfare improving even with retaliation. Gorman (1958) provides a more detailed consideration of the constant elasticity case for the same model.

2 The recent survey by Bagwell et al. (2016) provides a comprehensive appraisal of that argument and the rapidly developing empirical analysis that provides support or reason for scepticism. Costinot et al. (2016) confirm the role of the terms of trade externalities in motivating international trade agreements.

3 This is noted in the survey by Costinot and Rodríguez-clare (2014).

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their terms of trade'', confirms the paucity of results on existence and characterization of pure strategy Nash equilibrium.

Our paper contributes to the literature by studying a tractable multi-country framework; it addresses the issue of the existence of a Nash equilibrium and also provides qualitative results, and it does so analytically rather than numerically/computationally.[4]

Our model is sufficiently general in that we do not impose any restrictions on trade patterns and we are able to characterize the equilibrium in terms of conditions on countries' endowments. This is in contrast to the literature where results can often be traced to simplifying assumptions, like symmetry, on the structure of the model.[5]

Evidently, the introduction of tariffs enriches the model whilst creating additional technical difficulties. Sontheimer (1971) draws on Foster and Sonnenschein (1970) to note that if some good is not normal then, in the presence of a tariff, the demand set could fail to be convex. Hence, existence of a general equilibrium requires that in each country every good is normal. It is also clear that continuity of behaviour and payoffs requires that, given tariff rates, the Walrasian equilibrium is unique. These two requirements immediately restrict the set of economies that one can work with. In addition one must face the main difficulty in proving existence of Nash equilibrium in such a framework: one has to establish that each payoff function is quasiconcave in the player's own choice.

In view of the technical problems that the richness of the model forces us to confront, there is a trade-off in how general a model one can work with. We would also like to adopt a framework that allows the theory to be taken to the data. The linear expenditure system, which has been fruitfully applied in many areas, and preferences that generate it, provides a compromise that is an attractive specification for a model of tariffs. The fact that these generalized Cobb–Douglas preferences induce a demand function that is the same as the aggregate demand function induced by heterogeneous agents with sufficiently heterogeneously distributed characteristics (see Grandmont, 1992) is a bonus.

We consider an exchange economy with an arbitrary number of countries and with two goods. Preferences are restricted to be in the Cobb–Douglas class but are otherwise arbitrary—no assumptions of symmetry are made and countries are allowed to be different. Endowments are also arbitrary other than being nonnegative.

We study non-cooperative tariff equilibria in the induced strategic form game.

We first show that if for each good at least two countries have a positive endowment of the good, a mild assumption, then there are common upper and lower bounds on tariff rates such that no country's best response is on the boundary of the strategy sets that are induced. These bounds are consistent in that all potential equilibrium points are interior.

We then show that if at least two countries have a positive endowment of the good on which the tariff is imposed then each country's payoff function has the property that its second derivative is negative at any point at which the first derivative of the function is zero. When combined with the behaviour of the payoff function at the boundaries of the strategy sets, the local second order property ensures that the payoff function is quasiconcave. That suffices to prove the existence of pure strategy Nash equilibria.

Next we show that the tariff game is a game of strategic complementarity as the best response functions are strictly increasing. We also provide robust examples to show that the game can fail to be supermodular; it follows that the existence proof cannot be simplified by appealing directly to lattice theory, a simplification that would make redundant the difficult step in which we verify quasiconcavity.[6]

We then use the properties developed to provide results on payoff comparisons that culminate in a policy implication. We show that, in our model, a country's payoff increases when a competitor raises its tariff if and only if the country has set a positive net tariff rate. Therefore, the standard condition in the literature under which payoff comparisons across Nash equilibria become possible must be violated in our model since that condition (which is not implied by supermodularity) requires that the country's payoff always responds in the same direction whenever a competitor raises its tariff. We then show that a country's payoff increases as we move away from the free trade point in either direction along the best response. We are also able to show that the following surprising result holds very generally: the participation of at most two countries in negotiated tariff changes suffices to induce a Pareto improving allocation relative to a Nash equilibrium with the direction of tariff rate change easily determined; this is despite the fact that changing any tariff rate affects the payoff of every country. In view of the fact that our model is parametric and can therefore be easily taken to the data, this result on welfare improvements has the potential to play an important role in policy deliberations in the current global environment where large economies are adopting protectionist measures.

We also provide results that relate the position of a country's best response function to the trade pattern in the absence of tariffs, and that specify lower bounds on the arithmetic and harmonic means of equilibrium tariff rates; this last result implies that there is at most one symmetric equilibrium and it must be free trade. Our final result is that there is no trade if and only if the equilibrium allocation is Pareto optimal. We then comment on the extent to which comparative statics exercises can be carried out, and on the possibility of obtaining a result on uniqueness.

We reiterate that our analysis is free of any restrictions on trade patterns and our minimal assumptions are transparent since they are on the fundamentals of the economy and not on elasticities. We observe that, although a key goal of this literature is to determine the welfare effects of the introduction of non-cooperative tariffs, we do not pursue that goal analytically. This is because we know from Kennan and Riezman (1988) that, even in the two country world with identical symmetric Cobb–Douglas preferences, the pattern of endowments determines one of the three outcomes—there is a cigar shaped region around the line of slope -1 that delineates a region in which, in Nash equilibrium, both lose relative to free trade, and outside the region the bigger country wins.[7]

One imagines that in our model with many countries welfare effects of tariffs will depend intricately on the combination of preference parameters and endowments. This suggests the

[4] Hamilton and Whalley (1983) is an early contribution that recognized the computational difficulties that arise as soon as one steps beyond the two country and two good framework. That the issue remains unresolved is recognized in Abrego et al. (2005, 2006).

[5] For example, Ossa (2011) considers a two and a three country model with identical preferences, identical technology, tariffs that do not generate revenue, and a positive transportation cost. It is then shown that there is a unique Nash equilibrium with identical tariffs set at the highest possible value which is exogenously chosen. Also see footnotes 9 and 10.

[6] On the other hand, in a Ricardian production model without consumers or tariff considerations, Costinot (2009) provides strong predictions on comparative advantage by assuming logsupermodularity of functions specifying the linear technology and endowments.

[7] In a two country Ricardian model with identical Cobb–Douglas preferences, Opp (2010) shows that the relative size of the country determines the outcome of a tariff war.

use of numerical methods which are placed on firmer ground since in our tariff games solutions to the first order conditions characterize all possible interior pure strategy Nash equilibria.

One must ask whether our framework is too restrictive. In the literature one sometimes finds expression of the belief that the issue of the existence of pure strategy Nash equilibria in the general model described has been settled; yet, as we now argue, very little is known about it.[8] Wong (2004) considers the 2×2 (two good and two country) pure exchange model and provides an example to show that existence can fail if for some country, and a tariff rate set by the other country, the country's offer curve fails to enclose a convex set; this confirms that the result in Otani (1980) on the existence of compensated equilibrium in a general model with production is driven by his Assumption 11 (b), one that he refers to as "most uneasy", which convexifies the problem. Wong (2004) then proves existence with the normal goods assumption and the assumption that the area enclosed by each offer curve is a convex set for each level of the tariff chosen by the opponent, and makes the argument that his proof cannot be extended to the case with more than two countries.[9]

The first order conditions of the 2×2 pure exchange economy with Cobb–Douglas preferences have been studied by Otani (1980) as an example, and by Kennan and Riezman (1988) who revisit Johnson's original question and provide the solution described earlier.[10]

To summarize, the existence results that are known require strong restrictions and are not known to extend to the case with more than two countries.[11]

We make one final comment. A trivial modification, which amounts to no more than relabelling the variables, allows one to view the model as one of multiple tax jurisdictions. Once local tax rates are set and treated as parameters, a standard Walrasian equilibrium is played. Our model provides an "off-the-shelf" well-received parametric framework in which existence of a pure strategy Nash equilibrium is guaranteed and in which qualitative as well as quantitative analysis can be easily carried out.

We present the model in Section 2, and discuss, in order, existence in Section 3, strategic complementarity in Section 4, and properties of the solution in Section 5. Concluding comments are in Section 6, and all proofs are collected in Section 7.

2. The model

2.1. The economy

Consider a world with two goods and a set $\mathcal{I} = \{1, 2, \ldots, I\}$ of countries. The goods are traded in international markets at prices $p_1 > 0$ and $p_2 > 0$ (later we set $p_1 = 1$). The government in each country sets a nondiscriminatory tariff on each good; the gross tariff rates are denoted τ_{i1} and τ_{i2} for each $i \in \mathcal{I}$. We impose the restriction that the gross tariff rates are always positive, $\tau_{i1} > 0$ and $\tau_{i2} > 0$ for each $i \in \mathcal{I}$. The tariffs τ_{i1} and τ_{i2} induce a vector of domestic prices $(\tau_{i1}p_1, \tau_{i2}p_2)$ in country i and the revenue generated by the tariffs depends linearly on the net trade vector; the proceeds from the tariffs are redistributed to consumers in country i in the form of a lump-sum.

The representative consumer in each country has an endowment, denoted $\omega_i \in \mathbb{R}^2_+/\{0\}$ for $i \in \mathcal{I}$, and behaves competitively when faced with the vector of domestic prices $(\tau_{i1}p_1, \tau_{i2}p_2)$ in country i. The quantities of each good demanded by the consumer in country i are denoted x_{i1} and x_{i2}. Let w_i denote the income available to the consumer in country i. It follows that the budget constraint faced by consumer i is

$$\tau_{i1}p_1x_{i1} + \tau_{i2}p_2x_{i2} \le w_i.$$

Also, the revenue generated by the tariffs is given by

$$(\tau_{i1} - 1)p_1(x_{i1} - \omega_{i1}) + (\tau_{i2} - 1)p_2(x_{i2} - \omega_{i2}),$$

and since tariff revenues are redistributed in the form of a lump-sum, we have

$$\begin{aligned} w_i &= \tau_{i1}p_1\omega_{i1} + \tau_{i2}p_2\omega_{i2} + (\tau_{i1} - 1)p_1(x_{i1} - \omega_{i1}) \\ &\quad + (\tau_{i2} - 1)p_2(x_{i2} - \omega_{i2}) \\ &= (\tau_{i1} - 1)p_1x_{i1} + (\tau_{i2} - 1)p_2x_{i2} + p_1\omega_{i1} + p_2\omega_{i2}. \end{aligned}$$

We shall assume that the consumer in country i has a utility function u_i of the Cobb–Douglas form, so $u_i(x_{i1}, x_{i2}) = {x_{i1}}^{\alpha_i}{x_{i2}}^{1-\alpha_i}$, with parameter $\alpha_i \in (0, 1)$.

We also make the nondegeneracy assumptions: $\sum_i \omega_{i1} > 0$ and $\sum_i \omega_{i2} > 0$.

For ease of reference we collect the assumptions made so far; these will be treated as maintained assumptions.

Assumption 1. (i) For all $i \in \mathcal{I}$, $\omega_i \in \mathbb{R}^2_+/\{0\}$ and $u_i(x_{i1}, x_{i2}) = {x_{i1}}^{\alpha_i}{x_{i2}}^{1-\alpha_i}$, with parameter $\alpha_i \in (0, 1)$;
(ii) $\sum_i \omega_{i1} > 0$ and $\sum_i \omega_{i2} > 0$.

The optimization problem faced by i taking p_1, p_2, τ_{i1}, and τ_{i2} as given is

$$\max {x_{i1}}^{\alpha_i}{x_{i2}}^{1-\alpha_i} \qquad \text{subject to} \qquad \tau_{i1}p_1x_{i1} + \tau_{i2}p_2x_{i2} \le w_i.$$

Evidently, the first order necessary and sufficient conditions for x_i to solve the problem are

$$x_{i2} = \frac{(1-\alpha_i)\tau_{i1}p_1}{\alpha_i\tau_{i2}p_2}x_{i1} \qquad \tau_{i1}p_1x_{i1} + \tau_{i2}p_2x_{i2} = w_i.$$

Demand $x_{i1}(p_1, p_2, \tau_{i1}, \tau_{i2})$ can now be calculated by observing that the budget constraint simplifies to

$$p_1x_{i1} + p_2x_{i2} = p_1\omega_{i1} + p_2\omega_{i2}.$$

We have demand $x_{i1}(p_1, p_2, \tau_{i1}, \tau_{i2})$ is the value that satisfies

$$\begin{aligned} & p_1x_{i1} + p_2\frac{(1-\alpha_i)\,p_1}{\alpha_i\,(\tau_{i2}/\tau_{i1})\,p_2}x_{i1} = p_1\omega_{i1} + p_2\omega_{i2} \\ \iff\quad & p_1\left\{\frac{\alpha_i\,(\tau_{i2}/\tau_{i1}) + (1-\alpha_i)}{\alpha_i(\tau_{i2}/\tau_{i1})}\right\}x_{i1} = p_1\omega_{i1} + p_2\omega_{i2} \\ \iff\quad & x_{i1} = \left\{\frac{\alpha_i(\tau_{i2}/\tau_{i1})}{\alpha_i\,(\tau_{i2}/\tau_{i1}) + (1-\alpha_i)}\right\}\frac{[p_1\omega_{i1} + p_2\omega_{i2}]}{p_1}. \end{aligned}$$

World markets will clear at prices (p_1^*, p_2^*) if and only if

$$\begin{aligned} & \sum_i x_{i1}(p_1^*, p_2^*) = \sum_i \omega_{i1} \\ \iff\quad & \sum_i\left[\left\{\frac{\alpha_i(\tau_{i2}/\tau_{i1})}{\alpha_i\,(\tau_{i2}/\tau_{i1}) + (1-\alpha_i)}\right\}\frac{[p_1^*\omega_{i1} + p_2^*\omega_{i2}]}{p_1^*}\right] \\ & = \sum_i \omega_{i1}. \end{aligned}$$

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8 Kuga (1973) studies equilibrium in mixed strategies with finite sets of choices in a production economy.

9 Thursby and Jensen (1983), Syropoulos (2002) and Wong (2004), all provide sufficient conditions, in terms of elasticities, for the existence of pure strategy Nash equilibria in two country models, typically with strong assumptions on preferences. Zissimos (2009) argues that quasiconcavity of the payoff function can be established in the case of symmetric Cobb–Douglas preferences in a model with two groups of countries where groups are internally homogeneous.

10 Kennan and Riezman (1988) consider the case with identical and symmetric preferences and aggregate endowment normalized to one while Kennan and Riezman (1984) consider nonidentical and asymmetric preferences. It is not clear that the analytics for reaction functions can be meaningfully extended to more than 2 countries. Kennan and Riezman (1990) provide numerical solutions for the 3×3 case with identical symmetric preferences and different endowment specifications.

11 The model in Bagwell and Staiger (1999) has reduced form payoff functions that depend on domestic and world relative prices. They directly assume that the required second order conditions hold (see their footnote 9). The same model is used for expository purposes in Bagwell et al. (2016).

Clearly, we can normalize prices and set $p_1 = 1$. Also, the tariff rates set by each country affect prices only through the ratios of the tariffs on the two goods. Therefore, we may work with the variables $\tau_i = \tau_{i2}/\tau_{i1}$. From here onwards we set the gross tariff rate on the first good in each country at one, and work with a single tariff rate, that on the second good, chosen by each country; this is without loss of generality. The notation τ_i is used from here on to denote the strategic variable chosen by country i.

Let $\vec{\tau} = (\tau_1, \tau_2, \ldots, \tau_I)$. For $\alpha_i \in (0,1)$ and $\tau_i > 0$, set $A(\alpha_i, \tau_i) := \frac{\alpha_i \tau_i}{\alpha_i \tau_i + (1-\alpha_i)}$. Evidently, $A(\alpha_i, \tau_i) \in (0,1)$. We have

$$p_2^*(\vec{\tau}) \cdot \sum_i A(\alpha_i, \tau_i)\omega_{i2} = \sum_i [1 - A(\alpha_i, \tau_i)]\,\omega_{i1}$$

$$\iff \quad p_2^*(\vec{\tau}) = \frac{\sum_i [1 - A(\alpha_i, \tau_i)]\omega_{i1}}{\sum_i A(\alpha_i, \tau_i)\omega_{i2}}.$$

We have an explicit analytical expression for the market clearing international relative price as a function of the tariff rates set by each of the countries and the parameters specifying economic fundamentals $((\alpha_i, \omega_i)_{i \in \mathcal{I}})$. Assumption 1 guarantees that both the numerator and the denominator are positive and finite, i.e. $p_2^*(\vec{\tau}) \in (0, +\infty)$. Also, consumption of good 1 in country i at equilibrium prices is $x_{i1}\left(p_2^*(\vec{\tau})\right) = A(\alpha_i, \tau_i)[\omega_{i1} + p_2^*(\vec{\tau})\,\omega_{i2}]$.

Before proceeding further, we collect a few remarks on the model. Notice that the budget constraint takes the form

$$p_1 x_{i1} + \tau_i p_2 x_{i2} \le p_1 \omega_{i1} + \tau_i p_2 \omega_{i2} + (\tau_i - 1) p_2 (x_{i2} - \omega_{i2}).$$

Trade is free if $\tau_i = 1$ in every country. Also, the tariff generates revenue if $\tau_i > 1$ and country i is an importer of good 2 or if $\tau_i < 1$ and country i is an exporter of good 2.[12]

Since we provide analytical results, our restriction to Cobb–Douglas utility functions is driven by considerations of tractability. It should be noted that the analysis in Grandmont (1992) can be applied directly to our model to provide conditions on the distribution of characteristics that would generate Cobb–Douglas like demands in a pure exchange endowment economy.

2.2. The induced utility function

Utility at the Walrasian equilibrium obtained above can now be calculated and labelled the "induced utility function" for country i. More precisely, given a vector of tariff rates, $v_i : \mathbb{R}^I_{++} \to \mathbb{R}$ denotes the utility achieved by each country at the market clearing international relative price where $v_i(\vec{\tau}) = \left(x_{i1}\left(p_2^*(\vec{\tau})\right)\right)^{\alpha_i} \left(x_{i2}(p_2^*(\vec{\tau}))\right)^{1-\alpha_i}$.

We begin our analysis by studying the function v_i. Lemma 1 provides an explicit form for v_i and its first derivative with respect to τ_i. Evidently, v_i is well defined and continuously differentiable.

Lemma 1.

(i) $$v_i(\vec{\tau}) = (A(\alpha_i, \tau_i))^{\alpha_i}(1 - A(\alpha_i, \tau_i))^{1-\alpha_i} \cdot \left[\omega_{i1}\left(p_2^*(\vec{\tau})\right)^{\alpha_i - 1} + \left(p_2^*(\vec{\tau})\right)^{\alpha_i}\omega_{i2}\right];$$

(ii) $$\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = v_i(\vec{\tau}) \cdot \frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i) \cdot \left\{ \frac{\alpha_i(1-\tau_i)}{A(\alpha_i, \tau_i)} + \frac{(1-\alpha_i)\omega_{i1}}{\sum_j \left[1 - A(\alpha_j, \tau_j)\right]\omega_{j1}} - \frac{\alpha_i \cdot \omega_{i2}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}} \right\}.$$

[12] We do not impose the condition "$\tau_i > 1$ if and only if country i is a net importer of good 2". In doing so, we follow much of the literature, e.g. Otani (1980) and Kennan and Riezman (1988); Wong (2004) does impose the restriction.

The term within brackets in Lemma 1(ii) will play an important role since it is easy to show that its sign is the same as the sign of the partial derivative.

2.3. Consistent bounds

We specify mild conditions that ensure that at low enough values of the tariff rate set by country i (and no lower tariff rates set by any other country), the function $v_i(\vec{\tau})$ is increasing in τ_i, and that it is decreasing in τ_i at high enough values of the tariff rate set by country i (and no higher tariff rates set by any other country). A pair of such tariff rates, denoted $\underline{\tau}$ and $\overline{\tau}$, where $\infty > \overline{\tau} > \underline{\tau} > 0$, is used as boundary points to induce strategy sets by restricting the tariff rate set by country i to satisfy $\overline{\tau} \ge \tau_i \ge \underline{\tau}$.

Not only are the induced strategy sets compact, they are also consistent because Lemma 2 implies that if $\tilde{\tau}_i \notin (\underline{\tau}, \overline{\tau})$ for a profile of actions $\tau_{-i} := ((\tau_j)_{j \ne i})$, then, for some $j \in \mathcal{I}$, $\frac{\partial v_j}{\partial \tau_j}(\tilde{\tau}_j, \tau_{-j}) \ne 0$. In addition, Lemma 2 and continuity of $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})$ ensure that when $\tau_j \in [\underline{\tau}, \overline{\tau}]$ for all $j \ne i$, there is at least one value $\tilde{\tau}_i \in (\underline{\tau}, \overline{\tau})$ such that $\frac{\partial v_i}{\partial \tau_i}(\tilde{\tau}_i, \tau_{-i}) = 0$. The alternative of a strategy set where a maximizer exists but is on the boundary, and the derivative of the objective function at the maximizer is not zero, is not palatable in that changing the bound would change the solution and hence the putative Nash equilibrium.

Lemma 2. *(i) If $\frac{\omega_{i2}}{\sum_j \omega_{j2}} < 1$ for all $i \in \mathcal{I}$ then there exists $\underline{\tau} \in (0,1)$ such that, for all $\underline{\underline{\tau}} \in (0, \underline{\tau}]$, for all $i \in \mathcal{I}$, $\frac{\partial v_i}{\partial \tau_i}(\underline{\underline{\tau}}, \tau_{-i}) > 0$ for $\tau_{-i} \in [\underline{\underline{\tau}}, \infty)^{I-1}$; (ii) if $\frac{\omega_{i1}}{\sum_j \omega_{j1}} < 1$ for all $i \in \mathcal{I}$ then there exists $\overline{\tau} > 1$ such that, for all $\overline{\overline{\tau}} \in [\overline{\tau}, \infty)$, for all $i \in \mathcal{I}$, $\frac{\partial v_i}{\partial \tau_i}(\overline{\overline{\tau}}, \tau_{-i}) < 0$ for $\tau_{-i} \in (0, \overline{\overline{\tau}}]^{I-1}$.*

We observe that the conditions imposed in Lemma 2 are very mild: we require that for each good it is the case that at least two countries have a positive endowment of the good. As the example in Section 2.5 shows, the condition imposed in (i) is tight.

Under the hypotheses of Lemma 2, we have $0 < \min_{i \in \mathcal{I}} \frac{\sum_{j \ne i} \omega_{jl}}{\sum_j \omega_{jl}}$, for $l = 1, 2$; now it easily follows that there are values $\underline{\tau}$ and $\overline{\tau}$ that, respectively, satisfy the inequalities

$$\underline{\tau}\frac{\left[\underline{\tau} + (1-\underline{\alpha})/\underline{\alpha}\right]}{\left[\underline{\tau} + (1-\overline{\alpha})/\overline{\alpha}\right]} < \min_{i \in \mathcal{I}} \frac{\sum_{j \ne i} \omega_{j2}}{\sum_j \omega_{j2}}$$

$$\frac{1}{\overline{\tau}}\frac{\left[1/\overline{\tau} + \overline{\alpha}/(1-\overline{\alpha})\right]}{\left[1/\overline{\tau} + \underline{\alpha}/(1-\underline{\alpha})\right]} < \min_{i \in \mathcal{I}} \frac{\sum_{j \ne i} \omega_{j1}}{\sum_j \omega_{j1}},$$

where $\underline{\alpha} := \min_{i \in \mathcal{I}} \alpha_i$ and $\overline{\alpha} := \max_{i \in \mathcal{I}} \alpha_i$, where $0 < \underline{\alpha} \le \overline{\alpha} < 1$. (The corresponding inequalities are satisfied by any pair $\underline{\underline{\tau}}$ and $\overline{\overline{\tau}}$ such that $\underline{\underline{\tau}} \in (0, \underline{\tau}]$ and $\overline{\overline{\tau}} \in [\overline{\tau}, \infty)$, respectively.) The proof of Lemma 2 consists in showing that when $\underline{\tau}$ and $\overline{\tau}$ satisfy the inequalities, the properties asserted in the statement of Lemma 2 follow; since the values are independent of the characteristics of the economy, they serve as uniform bounds.

Evidently, when all countries have identical preferences then the infimum of the set of values $\overline{\tau}$ that satisfy the inequality is $1 + \frac{\omega_{\bar{i}1}}{\sum_{j \ne \bar{i}} \omega_{j1}}$ where $\bar{i}$ is the country with the largest endowment of the first good. This provides a simple upper bound for equilibrium tariff rates independent of the country; there is a corresponding lower bound. So our model has the following interesting implication: when countries have similar tastes, unless some country is quite literally very large, equilibrium tariff rates will be close to one.

Tighter, and so more informative bounds, are also available. The proof of Lemma 2 makes clear that values that satisfy the inequalities below can serve as bounds:

$$\underline{\tau} < \min_{i\in\mathcal{I}} \frac{\sum_{j\neq i} A(\alpha_j,\underline{\tau})\omega_{j2}}{\sum_j A(\alpha_j,\underline{\tau})\omega_{j2}}$$
$$\frac{1}{\overline{\tau}} < \min_{i\in\mathcal{I}} \frac{\sum_{j\neq i}\left[1-A(\alpha_j,\overline{\tau})\right]\omega_{j1}}{\sum_j \left[1-A(\alpha_j,\overline{\tau})\right]\omega_{j1}}.$$

Clearly, the bounds, including the uniform ones, are available even when tastes vary; it follows that they provide an easy test of whether Nash equilibrium behaviour is observed.[13]

2.4. The tariff game

The *tariff game* is a game in strategic form specified by the player set $\mathcal{I}$, the strategy set $[\underline{\tau},\overline{\tau}]$ for each i, and the payoff functions $(v_1(\vec{\tau}),\ldots,v_I(\vec{\tau}))$ which restrict the induced utility functions v_i to $[\underline{\tau},\overline{\tau}]^I$.

A *pure strategy Nash equilibrium* of the tariff game is an action profile $(\tau_1^*,\ldots,\tau_I^*)\in[\underline{\tau},\overline{\tau}]^I$, such that

$$\text{for each } i\in\mathcal{I} \quad \tau_i\in[\underline{\tau},\overline{\tau}] \quad\Rightarrow\quad v_i(\tau_i^*,\tau_{-i}^*)\geq v_i(\tau_i,\tau_{-i}^*).$$

A strategy profile is *interior* if $(\tau_1,\ldots,\tau_I)\in(\underline{\tau},\overline{\tau})^I$.

The discussion preceding Lemma 2 confirms that a profile of actions in which $\tau_i \notin (\underline{\tau},\overline{\tau})$ for some i cannot be a Nash equilibrium.

2.5. An example

We present an example to show that the bounds specified in Lemma 2 are tight.

Consider a two country world. Let country 2's endowment of the second good be 0, $\omega_{22}=0$, in particular $\omega_2\notin\mathbb{R}^2_{++}$. For given tariff rates the Walrasian equilibrium is always well-defined. Yet, the tariff game does not have an interior Nash equilibrium. To see this, use Lemma 1(ii) to obtain the sign of $\frac{\partial v_1}{\partial\tau_1}(\vec{\tau})$ by evaluating the expression within braces:

$$\begin{aligned}
&\frac{\alpha_1(1-\tau_1)}{A(\alpha_1,\tau_1)}+\frac{(1-\alpha_1)\omega_{11}}{\sum_j\left[1-A(\alpha_j,\tau_j)\right]\omega_{j1}}-\frac{\alpha_1\cdot\omega_{12}}{\sum_j A(\alpha_j,\tau_j)\omega_{j2}}\\
&=\frac{\alpha_1(1-\tau_1)}{A(\alpha_1,\tau_1)}+\frac{(1-\alpha_1)\omega_{11}}{\sum_j\left[1-A(\alpha_j,\tau_j)\right]\omega_{j1}}-\frac{\alpha_1}{A(\alpha_1,\tau_1)}\\
&=-\frac{\alpha_1\tau_1}{A(\alpha_1,\tau_1)}+\frac{(1-\alpha_1)\omega_{11}}{\sum_j\left[1-A(\alpha_j,\tau_j)\right]\omega_{j1}}\\
&=-\frac{1-\alpha_1}{1-A(\alpha_1,\tau_1)}+\frac{(1-\alpha_1)\omega_{11}}{\sum_j\left[1-A(\alpha_j,\tau_j)\right]\omega_{j1}},
\end{aligned}$$

since $\frac{\alpha_i\tau_i}{A(\alpha_i,\tau_i)}=\alpha_i\tau_i+(1-\alpha_i)=(1-\alpha_i)/\frac{(1-\alpha_i)}{\alpha_i\tau_i+(1-\alpha_i)}=\frac{1-\alpha_i}{1-A(\alpha_i,\tau_i)}$,

$$=(1-\alpha_1)\left\{\frac{\omega_{11}}{\sum_j\left[1-A(\alpha_j,\tau_j)\right]\omega_{j1}}-\frac{1}{1-A(\alpha_1,\tau_1)}\right\}<0$$

since $\omega_{21}>0$ necessarily as $\omega_{22}=0$ and $\omega_i\in\mathbb{R}^2_+/\{0\}$. This shows that regardless of the vector of tariffs chosen, country's 1's first order condition can never have an interior solution.

The example does not restrict ω_1 or preferences.

3. Existence

Our objective in this section is to investigate the conditions for the existence of a pure strategy Nash equilibrium in the tariff game. Lemma 1 in Section 2.2 established that the function v_i is well defined and continuously differentiable; however, v_i typically fails to be concave. By requiring the second derivative of the payoff function to be negative at every point at which the first derivative is zero, and also requiring the payoff function to be increasing at the left boundary and decreasing at the right boundary, we are able to ensure the existence of an interior Nash equilibrium.

Clearly, we must study the local behaviour of the payoff function at a point at which the first derivative is zero. Lemma 3 in Section 3.1 provides a very simple explicit algebraic form to determine the sign of the second partial derivative of the payoff function at such a point. This result is used in Lemma 4 to establish that, in our tariff games, either of $\alpha_i\leq 1/2$ or $\frac{\omega_{i2}}{\sum_j\omega_{j2}}<1$, the mild condition already encountered in Lemma 2(i), suffices to ensure that the sign of the second partial derivative of the payoff function is negative at every point at which the first derivative is zero.

The properties of the function v_i established in Lemmas 1, 2, and 4 lead to Theorem 1 in Section 3.2 which identifies mild conditions on the primitives of the model that ensure the existence of an interior pure strategy Nash equilibrium in the tariff game.

3.1. The best response

In this sub-section we study the local behaviour of the payoff function at points at which its first derivative is zero. Using the fact that at a strategy profile at which the first derivative is zero we must have $\sum_{j\neq i}A(\alpha_j,\tau_j)\omega_{j2}>0$, we are able to provide a simple algebraic expression to exactly evaluate the sign of the second derivative of the payoff function at a point at which the first derivative is zero.

Define the function *sign* : $\mathbb{R}\to\{-1,0,1\}$ by $sign(x)=-1$ if $x<0$, $sign(x)=0$ if $x=0$, and $sign(x)=1$ if $x>0$.

Lemma 3.

$$\begin{aligned}
&sign\left\{\left.\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0}\right\}\\
&=sign\left\{-2\alpha_i\tau_i\omega_{i2}-\left[1-2\alpha_i(1-\tau_i)\right]\left[\sum_{j\neq i}A(\alpha_j,\tau_j)\omega_{j2}\right]\right\}.
\end{aligned}$$

Evidently, $\alpha_i\leq 1/2$ implies that $1-2\alpha_i(1-\tau_i)>0$ and so we have a simple but strong sufficient condition under which the second derivative of the payoff function is negative. But we are able to do better: the expression in Lemma 3 has a monotonicity property because of which the sign of the expression is negative if and only if the tariff rate chosen is to the right of a threshold value.[14] So if the first derivative were to be zero at a point to the left of the threshold value then the second derivative at that point would have to be positive; but then, by using the boundary condition result in Lemma 2(i), the continuity of the first derivative, and the Intermediate Value Theorem, there would have to exist another point further to the left at which the first derivative would again be zero but where, by continuity, the second derivative would necessarily be negative. Since this

[13] Of course, high tariffs observed in data from actual economies might correspond to punitive measures with payoffs not captured by the functions $v_i(\vec{\tau})$.

[14] The threshold value, defined in the proof of Lemma 4, is $\hat{\tau}_i=\frac{\left[\sum_{j\neq i}A(\alpha_j,\tau_j)\omega_{j2}\right][2\alpha_i-1]}{2\alpha_i\left[\omega_{i2}+\sum_{j\neq i}A(\alpha_j,\tau_j)\omega_{j2}\right]}$.

happens at a point to the left of the threshold value it produces a contradiction.

Lemma 4 shows two properties that summarize the discussion: that, at a point at which the first derivative of the payoff function is zero, the second derivative of the function cannot be zero, $\frac{\partial^2 v_i}{\partial \tau_i \partial \tau_i}(\vec{\tau})\Big|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} \neq 0$, so any solution to the first order condition is robust (which is a much stronger result than claiming that robust intersection is a generic property in some appropriate space of parameters); and if either $\alpha_i \leq 1/2$ or the result in Lemma 2(i) holds, then at such a point the second derivative of the payoff function must be negative, $\frac{\partial^2 v_i}{\partial \tau_i \partial \tau_i}(\vec{\tau})\Big|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} < 0$.

This suffices to show that there is a unique best response and that it is continuous since the payoff function is quasiconcave.

Lemma 4. $\frac{\partial^2 v_i}{\partial \tau_i \partial \tau_i}(\vec{\tau})\Big|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} \neq 0$. *If* $\alpha_i \leq 1/2$ *or if* $\frac{\omega_{i2}}{\sum_j \omega_{j2}} < 1$ *then* $\frac{\partial^2 v_i}{\partial \tau_i \partial \tau_i}(\vec{\tau})\Big|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} < 0$.

3.2. Pure strategy equilibria in the tariff game

We are now in a position to show that interior pure strategy Nash equilibria exist in the tariff game played by countries whose fundamentals satisfy the mild conditions specified in Theorem 1. The conditions restrict the distribution of endowments by ruling out extreme cases in which a single country's endowment of a good is equal to the world's endowment of that good. The example in Section 2.5 illustrates that the conditions cannot be relaxed.

Theorem 1. *Assume that, for all* $i \in \mathcal{I}$, $\frac{\omega_{i1}}{\sum_j \omega_{j1}} < 1$ *and* $\frac{\omega_{i2}}{\sum_j \omega_{j2}} < 1$. *The tariff game of such an economy has an interior pure strategy Nash equilibrium.*

The proof of Theorem 1 follows from an intuitive result which says that when the strategy space is an interval, and player i's payoff function is (i) increasing in i's choice at the left boundary and decreasing in i's choice at the right boundary and (ii) has a negative second derivative at every point at which the first derivative is zero, the game has an interior pure strategy Nash equilibrium, and the fact that by Lemmas 1, 2, and 4, the tariff game satisfies conditions (i) and (ii). The intuitive existence result holds in a more general setting with one dimensional strategy sets and is stated and proved as Lemma S.8 in Section 7.

4. Strictly increasing best response functions

In this section we show that the tariff game is a game of strategic complementarity in that best response functions are strictly increasing. We also provide three robust examples of tariff games that fail to be supermodular. Since the tariff game is defined on the product of intervals, and the payoff functions are twice continuously differentiable, to ask whether it is a supermodular game is a natural line of enquiry.[15] Were this to be true then existence would follow immediately as would a number of other useful properties including the implication that best responses are increasing functions (Result 1 and 4 in Vives (2007)), i.e. a direct appeal to lattice theory would make redundant the detailed development to verify quasiconcavity that culminates in the proof of Lemma 4.[16]

One needs to check whether $\frac{\partial^2 v_i}{\partial \tau_i \partial \tau_j}(\vec{\tau}) \geq 0$ everywhere. In Lemma 5(i) we show that, at points at which the first derivative is zero, the sign of the cross partial derivative is indeed positive. In Lemma 5(ii), we provide an explicit algebraic expression for the sign of the cross partial derivative; so a computation allows us to confirm that a specific tariff game is not supermodular.

Lemma 5. *For* $i \neq j$,

$$\text{(i)}\quad sign\left\{\frac{\partial^2 v_i}{\partial \tau_i \partial \tau_j}(\vec{\tau})\Big|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0}\right\} > 0;$$

$$\begin{aligned}\text{(ii)}\quad sign\left\{\frac{\partial^2 v_i}{\partial \tau_i \partial \tau_j}(\vec{\tau})\right\} = sign\Bigg\{&\left[(1-\alpha_i)\omega_{i1} + (-\alpha_i)\big(p_2^*(\vec{\tau})\big)\omega_{i2}\right]^2 \\ &\cdot\left[\big(p_2^*(\vec{\tau})\big)^{-1}\omega_{j1} + \omega_{j2}\right] \\ &+\left[\omega_{i1}\big(p_2^*(\vec{\tau})\big)^{-1} + \omega_{i2}\right]\big(p_2^*(\vec{\tau})\big)^2\left[(1-\alpha_i)\big(p_2^*(\vec{\tau})\big)^{-2}\cdot\omega_{j1}\omega_{i1}\right. \\ &\left.+\alpha_i\cdot\omega_{i2}\cdot\omega_{j2}\right] \\ &+\left[(1-\alpha_i)\omega_{i1} + (-\alpha_i)\big(p_2^*(\vec{\tau})\big)\omega_{i2}\right]\cdot\left[\big(p_2^*(\vec{\tau})\big)^{-1}\omega_{j1} + \omega_{j2}\right] \\ &\cdot\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}\cdot\left[\sum_k[1 - A(\alpha_k,\tau_k)]\omega_{k1}\right]\Bigg\}.\end{aligned}$$

By Lemma 5(i), the positive cross partial condition holds on the restricted set of strategy profiles that are in the graph of the best response function for that player.

By Lemma 5(ii), the sign of the cross partial derivative is positive at $\tau_i = 1$, and around that value by continuity, since the term in the last two lines drops out and the other two terms are positive. This suggests that it might be possible to identify sets of endowments that together with the bounds specified in Lemma 2 induce strategy sets that are in the neighbourhood of one so that the cross partial derivative is positive.

Theorem 2. *Assume that* $\frac{\omega_{i1}}{\sum_j \omega_{j1}} < 1$ *and* $\frac{\omega_{i2}}{\sum_j \omega_{j2}} < 1$. *The best response function of player i is well-defined, differentiable, and strictly increasing in* τ_j *for all* $j \neq i$.

Theorem 2 may be proved as follows. By continuity of the payoff function and compactness of the strategy set, the best response is well defined. Under the conditions provided, one can combine Lemmas 4 and 5(i) and the implicit function theorem to conclude that each best response function is differentiable and strictly increasing in every other player's choice. We omit a full formal proof.

We turn to a brief presentation of three examples in each of which the tariff game fails to be supermodular and this failure is robust; the details can be found in the Appendix where, for each example, we also check whether the proposed tariff rates satisfy bounds discussed in Section 2.3.

In the first example there are two countries with identical symmetric preferences. Even in this rudimentary environment supermodularity fails at $(\tau_1, \tau_2) = (3, 1/2)$.

[15] Vives (2007) is a brief survey of supermodularity that suffices for the purpose at hand.

[16] Since the best response functions are strictly increasing, by Theorem 4 in Milgrom and Shannon (1994), the tariff game's payoffs must satisfy the single crossing property. However, it is not obvious that the payoff functions specified in Lemma 1(a) do satisfy the required condition so such an ex post result does not really help us.

Example 1. Let there be two countries with the following parameter specification

$$\alpha_1 = 1/2 \quad \omega_{11} = 2 \quad \omega_{12} = 8/67$$
$$\alpha_2 = 1/2 \quad \omega_{21} = 4 \quad \omega_{22} = 2.$$

In the next example we allow one out of many countries to have a different preference parameter and show that the violation can occur with $\tau_1 = 11/10$ and $\tau_i = 1$, $i \neq 1$, when we have a few more than six hundred countries.

Example 2. Let there be I countries with the following parameter specification

$$\alpha_1 = 1/3 \quad \omega_{11} = 2 \quad \omega_{12} = 80/22$$
$$\alpha_i = 1/2 \quad \omega_{i1} = 2 \quad \omega_{i2} = 2 \qquad \text{for } i = 2, \ldots, I.$$

The last example combines the features of the first two examples: it considers many countries with identical symmetric preferences and generates a violation of supermodularity close to free trade by introducing appropriate heterogeneity of endowments, specifically $\tau_1 = 11/10$ and $\tau_i = 1$, $i \neq 1$, and we have a few more than two hundred countries.

Example 3. Let there be I countries with the following parameter specification

$$\alpha_1 = 1/2 \quad \omega_{11} = 2 \quad \omega_{12} = 10/11$$
$$\alpha_i = 1/2 \quad \omega_{i1} = 4 \quad \omega_{i2} = 2 \qquad \text{for } i = 2, \ldots, I.$$

The specifications in the three examples vary quite a bit and suggest that, when there are many countries, for supermodularity to fail to obtain at tariff rates that satisfy reasonable bounds, we need some country to be quite different.

5. On Nash equilibria of the tariff game

In this section we highlight properties of Nash equilibria in our tariff game when Assumption 1 holds and endowments satisfy the condition: for all $i \in \mathcal{I}$ (i) $\frac{\omega_{i2}}{\sum_j \omega_{j2}} < 1$ and (ii) $\frac{\omega_{i1}}{\sum_j \omega_{j1}} < 1$. By Theorem 2, the tariff game is one of strategic complementarity.

Lemma 6 allows us to conclude that the sufficient condition identified in the literature that allows payoff comparisons across Nash equilibria must be violated in our model. Even so, in Proposition 2 we are able to show that country i's payoff is monotone increasing as we move away from the value one along i's best response function. More surprisingly, Proposition 3 allows us to determine the direction of negotiated tariff changes that induce a Pareto improvement relative to a Nash equilibrium and shows that it suffices that at most two countries participate in the negotiations. We also provide results on the location of the best response function, on lower bounds of the arithmetic and harmonic means of equilibrium tariff rates, with the further implication that there is at most one symmetric equilibrium, and, finally, that a Nash equilibrium allocation is Pareto optimal if and only if there is no trade. We then comment on the extent to which comparative statics exercises can be carried out, and we briefly touch upon the issue of uniqueness.

Our first result provides information on the location of the best response function.

Proposition 1. *Let $\vec{\tau}$ be such that $\tau_j = 1$ for all $j \neq i$ and $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$. Then $\tau_i > 1$ if and only if i is an exporter of good 1 when trade is free.*

The result adds to what we know about each best response function—that, by Theorem 2, they are strictly increasing and that, by Lemma 2, they take interior values at boundary points. The upper and lower bounds on strategy sets identified in Section 2.3 provide further restrictions, where we recall that the bounds are likely to be fairly tight in cases where the number of countries is large, and that the bounds become even tighter when the distribution of endowments is more symmetric.

The next lemma provides the tool that we need for the two propositions on payoff comparisons that follow. It shows that, at a point on the graph of the best response, the direction in which country i's payoff moves in response to changes in tariffs set by other countries is determined only by the value of its own tariff relative to free trade. It is immediate that in our model, $\frac{\partial v_i}{\partial \tau_j}(\vec{\tau})$ cannot be of uniform sign; it follows that the standard route in the literature to making Pareto comparisons across Nash equilibria, which requires that $\frac{\partial v_i}{\partial \tau_j}(\vec{\tau})$ is of uniform sign, a strong condition which is not implied by supermodularity, is not available to us.

Lemma 6. *Let $\vec{\tau}$ such that $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$. Then, for $i \neq j$, sign $\left\{\frac{\partial v_i}{\partial \tau_j}(\vec{\tau})\right\}$ = sign $\{\tau_i - 1\}$.*

Lemma 6 suffices to show that as we move along country i's best response, points that are further from $\tau_i = 1$ in either direction result in higher payoffs to country i.

Proposition 2. *Let $\vec{\tau}$ and $\vec{\tau}\,'$ be two strategy profiles such that $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = \frac{\partial v_i}{\partial \tau_i}(\vec{\tau}\,') = 0$. If either $\tau_i > \tau_i' \geq 1$ or $\tau_i < \tau_i' \leq 1$ then $v_i(\vec{\tau}) > v_i(\vec{\tau}\,')$.*

As we have already noted, we cannot invoke the standard route to making payoff comparisons. Yet, by Proposition 2 monotonicity results hold on either side of the value $\tau_i = 1$ allowing us to compare payoffs across some pairs of equilibrium strategy profiles for some players, and this despite the fact that our tariff game is induced by an economy in general equilibrium. This is possible because all Nash equilibrium strategy profiles of the tariff game can be ordered as best response functions are strictly increasing.

We turn to the determination of the nature and direction of negotiated tariff changes that induce Pareto improving allocations relative to a Nash equilibrium. As we shall shortly confirm in Proposition 5, in the generic case the allocation induced at a Nash equilibrium fails to be Pareto optimal and it follows that there will be strategy profiles at which payoffs are higher; however, since the payoff functions are quasiconcave only in the country's own choice, it is not obvious that more specific recommendations can be made. Somewhat surprisingly, Lemma 6 allows us to provide a much more succinct answer.

Proposition 3. *Assume that the endowment is not a Pareto optimal allocation and consider an interior Nash equilibrium. (i) If there are countries $i, j \in \mathcal{I}$ such that $\tau_i^* > 1 > \tau_j^*$ then a Pareto improvement can be induced by forming two groups such that if i and j are in the same group then either (a) $\tau_i^* > 1$ and $\tau_j^* > 1$ or (b) $\tau_i^* < 1$ and $\tau_j^* < 1$, and with the same number of countries in each group, and moving the tariffs of both sets towards the free trade value by appropriate amounts. (ii) If all countries are on the same side of free trade then a Pareto improvement can be induced by moving the tariff rate of any one country further away from free trade.*

So, when not all countries choose equilibrium tariff rates on the same side of free trade, the participation of just two countries suffices since the negative effects of local changes in tariff rates can be controlled by considering a "balanced" set of countries whose rates are adjusted, where balance simply requires that for each country with an equilibrium tariff rate that exceeds one that has its rate adjusted there is one and only one country with an equilibrium rate that is less than one whose rate is also adjusted. We do not know whether the alternative case, in which all countries choose equilibrium tariff rates on the same side of free trade, can arise, but if it does then changing the rate of a single country, i.e. a unilateral move, suffices to induce a Pareto improvement. Also, the result in Proposition 3 refers to local changes that induce higher payoffs.

Our next result sharply delimits the region in the strategy space where one might expect to find pure strategy Nash equilibria of the tariff game.

Proposition 4. *(i) At every interior Nash equilibrium of the tariff game, $\sum_j \tau_j^* > I-1$ and $\sum_j \frac{1}{\tau_j^*} > I-1$. (ii) If at an interior Nash equilibrium of the tariff game $\tau_i^* = \tau^*$ for all $i \in \mathcal{I}$, then, necessarily, $\tau_i^* = 1$ for all $i \in \mathcal{I}$.*

By Proposition 4(i), at any interior Nash equilibrium, for some i and i', $\tau_i^* > (I-1)/I$ and $\tau_{i'}^* < I/(I-1)$. By Proposition 4(ii), only free trade can be a symmetric Nash equilibrium, a result that we use in the proof of Proposition 5.

For our last formally stated result, it is useful to recall that, for a "generic" economy, the endowment vector is not a Pareto optimal allocation.

Proposition 5. *A Nash equilibrium allocation is Pareto optimal if and only if there is no trade.*

Proposition 5 shows that, for a "generic" economy, the equilibrium allocations of the tariff game fail to be Pareto optimal and there is trade in every equilibrium.[17]

We turn to the possibility of obtaining comparative statics results. Since the solutions to the I first order conditions completely characterize pure strategy Nash equilibria, such exercises can be undertaken. One easily checks that, for $j \neq i$,

$$\left.\frac{\partial^2 v_i}{\partial \omega_{i1}\,\partial \tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} > 0, \quad \left.\frac{\partial^2 v_i}{\partial \omega_{i2}\,\partial \tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} < 0,$$

$$\left.\frac{\partial^2 v_i}{\partial \omega_{j1}\,\partial \tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} < 0,$$

$$\left.\frac{\partial^2 v_i}{\partial \omega_{j2}\,\partial \tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} > 0, \quad \left.\frac{\partial^2 v_i}{\partial \alpha_j\,\partial \tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} > 0.$$

Lemma 4 and the implicit function theorem allow us to conclude that i raises τ_i in response to an increase in ω_{i1} or in ω_{j2}, and reduces τ_i in response to an increase in ω_{i2} or in ω_{j1}. So we can determine the shifts in the best response functions of all the countries in response to endowment changes. For example, as ω_{i1} increases, τ_i increases and τ_j decreases for every $j \neq i$[18]; it is that very feature of the shifts being in the "same direction" that prevents us from drawing general conclusions about changes in equilibrium behaviour in response to the parametric change. Since the sign of $\left.\frac{\partial^2 v_i}{\partial \alpha_i\,\partial \tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0}$ is difficult to determine, we cannot make any progress on changes in choices made in response to changes to preference parameters.

We make a brief remark on uniqueness of Nash equilibrium. The problem has quite a lot of geometric structure and we are able to show that the matrix of second derivatives of the payoff functions evaluated at a Nash equilibrium can be written in the form: $V \cdot \Delta M + V \cdot \Delta N$ where the matrix V is a diagonal matrix with generic element $v_i(\vec{\tau}) \cdot \frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i) > 0$, ΔM is another diagonal matrix with generic element $-\alpha_i - \frac{1-\alpha_i}{(\tau_i)^2} < 0$, and ΔN is a positive matrix with a very specific form.[19] The structure of the matrix, and the fact that the boundary behaviour of the system is nice, strongly suggest the use of degree theoretic methods to identify conditions that are sufficient to ensure uniqueness of equilibrium. Although we have not been able to find a useful way to work out the sign of the determinant, numerical methods appear to be a promising route to follow.

6. Concluding comments

We have presented and analysed a model of tariff retaliation with many countries. The principal restriction that we imposed was to assume that all preferences are in the Cobb–Douglas class. This was for analytical tractability and yet, as we showed, the induced tariff game fails to be supermodular. It is, however, a game in which all best response functions are increasing and that allowed us to develop a number of interesting results.

We hope that the model or its extensions will be taken to the data. Independent of that, one could ask whether the specification adopted helps in shedding light on models with some aspects of cooperative behaviour like customs unions or the Most Favoured Nation clause in trade agreements.

On the technical side there are two obvious candidates for further research. The first involves extending the model to three or more goods, and one imagines that many of the results in Section 5 will go through provided that the best response functions are increasing. To show the latter one will have to grapple with quasiconcavity and the sign of some cross partial derivatives. The model with two goods treated by us has the advantage that the relative price in the Walrasian equilibrium has a simple analytical form which allowed us to obtain explicit expressions for the payoff functions and, with some algebraic manipulations, we were able to identify the signs of various first and second order, own and cross, partial derivatives. With more than one relative price, such explicit forms are the solution of a large linear system of equations; as a result, the expressions for the payoff functions are not amenable to manipulations making the identification of the signs of the various derivatives an arduous task. The second candidate for further research asks the more fundamental question about how special a sub-class Cobb–Douglas preferences form when requiring normality of both goods and uniqueness of Walrasian equilibrium with tariff distortions in a two good world with an arbitrary number of countries and arbitrary nonnegative endowments.

7. Proofs

The proofs of Lemmas 1–6 use a number of supplementary results which are stated here; the proofs of the statements are either in this section or in the Supplementary Material available online at the link in Appendix B at the end of this article.

In what follows, we will find it easier to work with the reciprocal of the price $p_2^*(\vec{\tau})$. So define the function f by

$$f(\vec{\tau}) = \frac{\sum_i A(\alpha_i, \tau_i)\omega_{i2}}{\sum_i [1 - A(\alpha_i, \tau_i)]\omega_{i1}}.$$

Lemma S.1 is our first supplementary result; it provides the evaluation of three partial derivatives that will be used later.

17 It is easy to check that if free trade is a Nash equilibrium then the endowment must be a Pareto optimal allocation. It is also easy to show that only a subset of the set of Pareto optimal allocations can be induced by appropriate choice of tariff rates.

18 To fix ideas, consider the case of just two countries. Each country's best response moves to the right, where we assign τ_1 to the horizontal axis.

19 Lemmas S.4 and S.5 in Section 7 provide the details of the elements of ΔN.

Lemma S.1. *The functions A and f are differentiable on their domains and*

$$\frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i) = \frac{\alpha_i(1-\alpha_i)}{[\alpha_i \tau_i + (1-\alpha_i)]^2} > 0,$$
$$\frac{\partial A}{\partial \alpha_i}(\alpha_i, \tau_i) = \frac{\tau_i}{[\alpha_i \tau_i + (1-\alpha_i)]^2} > 0,$$
$$\text{and} \quad \frac{\partial f}{\partial \tau_i}(\vec{\tau}) = \frac{\frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i)}{\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}} \{f(\vec{\tau})\omega_{i1} + \omega_{i2}\}.$$

We can now proceed to prove Lemma 1.

Lemma 1.

(i) $v_i(\vec{\tau}) = (A(\alpha_i, \tau_i))^{\alpha_i}(1 - A(\alpha_i, \tau_i))^{1-\alpha_i}\left[\omega_{i1}\left(p_2^*(\vec{\tau})\right)^{\alpha_i - 1} + \left(p_2^*(\vec{\tau})\right)^{\alpha_i}\omega_{i2}\right];$

(ii) $$\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = v_i(\vec{\tau}) \cdot \frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i) \cdot \left\{\frac{\alpha_i(1-\tau_i)}{A(\alpha_i, \tau_i)} + \frac{(1-\alpha_i)\omega_{i1}}{\sum_j \left[1 - A(\alpha_j, \tau_j)\right]\omega_{j1}} - \frac{\alpha_i \cdot \omega_{i2}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}}\right\}.$$

Proof.

(i) Recall that $v_i(\vec{\tau}) = \left(x_{i1}(p_2^*(\vec{\tau}))\right)^{\alpha_i}\left(x_{i2}(p_2^*(\vec{\tau}))\right)^{1-\alpha_i}$

which, upon using $x_{i2} = \frac{(1-\alpha_i)\tau_{i1}p_1}{\alpha_i \tau_{i2} p_2} x_{i1}$, the first order condition for the consumer's choice, becomes

$$\begin{aligned} v_i(\vec{\tau}) &= \left(x_{i1}(p_2^*(\vec{\tau}))\right)^{\alpha_i}\left(\frac{(1-\alpha_i)}{\alpha_i \tau_i p_2^*(\vec{\tau})} x_{i1}(p_2^*(\vec{\tau}))\right)^{1-\alpha_i} \\ &= \left(x_{i1}(p_2^*(\vec{\tau}))\right)\left(\frac{(1-\alpha_i)}{\alpha_i \tau_i p_2^*(\vec{\tau})}\right)^{1-\alpha_i} \\ &= A(\alpha_i, \tau_i)\left[\omega_{i1} + p_2^*(\vec{\tau})\omega_{i2}\right]\left(\frac{(1-\alpha_i)}{\alpha_i \tau_i}\right)^{1-\alpha_i}\left(\frac{1}{p_2^*(\vec{\tau})}\right)^{1-\alpha_i}, \end{aligned}$$

where we incorporate the explicit form of the demand function $x_{i1}(p_2^*(\vec{\tau}))$. Now, observe that $\frac{(1-\alpha_i)}{\alpha_i \tau_i} = \frac{1}{A(\alpha_i, \tau_i)} - 1$ and obtain

$$\begin{aligned} v_i(\vec{\tau}) &= A(\alpha_i, \tau_i)\left[\omega_{i1}\left(\frac{1}{p_2^*(\vec{\tau})}\right)^{1-\alpha_i} + \left(\frac{1}{p_2^*(\vec{\tau})}\right)^{-\alpha_i}\omega_{i2}\right] \\ &\quad \cdot \left(\frac{1}{A(\alpha_i, \tau_i)} - 1\right)^{1-\alpha_i} \\ &= (A(\alpha_i, \tau_i))^{\alpha_i}(1 - A(\alpha_i, \tau_i))^{1-\alpha_i}\left[\omega_{i1}\left(p_2^*(\vec{\tau})\right)^{\alpha_i - 1} \right. \\ &\quad \left. + \left(p_2^*(\vec{\tau})\right)^{\alpha_i}\omega_{i2}\right]. \end{aligned}$$

(ii) Recall that $f(\vec{\tau}) = \frac{1}{p_2^*(\vec{\tau})}$ so that (i) may be rewritten as

$$v_i(\vec{\tau}) = (A(\alpha_i, \tau_i))^{\alpha_i}(1 - A(\alpha_i, \tau_i))^{1-\alpha_i}\left[\omega_{i1}(f(\vec{\tau}))^{1-\alpha_i} + (f(\vec{\tau}))^{-\alpha_i}\omega_{i2}\right].$$

We proceed to differentiate the function.

$$\begin{aligned} \frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) &= \left\{\alpha_i (A(\alpha_i, \tau_i))^{\alpha_i - 1}\frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i)(1 - A(\alpha_i, \tau_i))^{1-\alpha_i} \right. \\ &\quad \left. + (A(\alpha_i, \tau_i))^{\alpha_i}(1-\alpha_i)(1 - A(\alpha_i, \tau_i))^{-\alpha_i}(-1)\frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i)\right\} \\ &\quad \cdot \left\{(f(\vec{\tau}))^{1-\alpha_i}\omega_{i1} + (f(\vec{\tau}))^{-\alpha_i}\omega_{i2}\right\} \\ &\quad + (A(\alpha_i, \tau_i))^{\alpha_i}(1 - A(\alpha_i, \tau_i))^{1-\alpha_i}\left\{(1-\alpha_i)(f(\vec{\tau}))^{-\alpha_i}\frac{\partial f}{\partial \tau_i}(\vec{\tau})\omega_{i1} \right. \\ &\quad \left. + (-\alpha_i)(f(\vec{\tau}))^{-\alpha_i - 1}\frac{\partial f}{\partial \tau_i}(\vec{\tau})\omega_{i2}\right\} \end{aligned}$$

We group some terms from the first three lines in the expression above to obtain the first two lines below; we also substitute for $\frac{\partial f}{\partial \tau_i}(\vec{\tau})$ from Lemma S.1 and collect some common terms in the last two lines above to obtain the last three lines below. We have

$$\begin{aligned} \frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) &= \left\{(A(\alpha_i, \tau_i))^{\alpha_i}(1 - A(\alpha_i, \tau_i))^{1-\alpha_i}(f(\vec{\tau}))^{-\alpha_i}\right\} \\ &\quad \cdot \frac{[\alpha_i(1 - A(\alpha_i, \tau_i)) - (1-\alpha_i)A(\alpha_i, \tau_i)]}{A(\alpha_i, \tau_i)[1 - A(\alpha_i, \tau_i)]}\frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i) \\ &\quad \cdot \{(f(\vec{\tau}))\omega_{i1} + \omega_{i2}\} \\ &\quad + \left\{(A(\alpha_i, \tau_i))^{\alpha_i}(1 - A(\alpha_i, \tau_i))^{1-\alpha_i}(f(\vec{\tau}))^{-\alpha_i}\right\} \\ &\quad \cdot \frac{\frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i)}{\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}}\{f(\vec{\tau})\omega_{i1} + \omega_{i2}\} \\ &\quad \cdot \left\{(1-\alpha_i)\omega_{i1} + (-\alpha_i)(f(\vec{\tau}))^{-1}\omega_{i2}\right\}. \end{aligned}$$

We collect terms and use the expression for $v_i(\vec{\tau})$ obtained in (i) to simplify the expression to

$$\begin{aligned} \frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) &= v_i(\vec{\tau}) \cdot \frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i) \cdot \left\{\frac{\alpha_i - A(\alpha_i, \tau_i)}{A(\alpha_i, \tau_i)[1 - A(\alpha_i, \tau_i)]} \right. \\ &\quad \left. + \frac{(1-\alpha_i)\omega_{i1} + (-\alpha_i)(f(\vec{\tau}))^{-1}\omega_{i2}}{\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}}\right\} \\ &= v_i(\vec{\tau}) \cdot \frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i) \cdot \left\{\frac{\alpha_i(1-\tau_i)}{A(\alpha_i, \tau_i)} + \frac{(1-\alpha_i)\omega_{i1}}{\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}} \right. \\ &\quad \left. - \frac{\alpha_i \cdot \omega_{i2}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}}\right\}, \end{aligned}$$

where we use the fact that

$$\begin{aligned} \frac{\alpha_i - A(\alpha_i, \tau_i)}{1 - A(\alpha_i, \tau_i)} &= \frac{\alpha_i - \frac{\alpha_i \tau_i}{\alpha_i \tau_i + (1-\alpha_i)}}{1 - \frac{\alpha_i \tau_i}{\alpha_i \tau_i + (1-\alpha_i)}} \\ &= \frac{(\alpha_i - 1)\alpha_i \tau_i + \alpha_i(1-\alpha_i)}{(1-\alpha_i)} = \alpha_i(1-\tau_i) \end{aligned}$$

and we incorporate the explicit form of the function $f(\vec{\tau})$. ■

The next supplementary result distills the key implication obtained so far.

Lemma S.2. *sign* $\left\{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})\right\} = $ *sign* $\left\{\frac{\alpha_i(1-\tau_i)}{A(\alpha_i, \tau_i)} + \frac{(1-\alpha_i)\omega_{i1}}{\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}} - \frac{\alpha_i \cdot \omega_{i2}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}}\right\}$.

Proof. Since $A(\alpha_i, \tau_i) \in (0, 1)$ and $\omega_i \in \mathbb{R}_+^2/\{0\}$, from Lemma 1(i) we have $v_i(\vec{\tau}) > 0$. Also, by Lemma S.1, $\frac{\partial A}{\partial \tau_i}(\alpha_i, \tau_i) > 0$. The result follows directly from Lemma 1(ii). ■

For notational ease, we define

$$M(\alpha_i, \tau_i) := \frac{\alpha_i(1-\tau_i)}{A(\alpha_i, \tau_i)};$$
$$N_i(\vec{\tau}) := \frac{(1-\alpha_i)\omega_{i1}}{\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}} - \frac{\alpha_i \cdot \omega_{i2}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}}.$$

Our next supplementary result provides an evaluation of $\frac{\partial M}{\partial \tau_i}(\alpha_i, \tau_i)$.

Lemma S.3.

$$\frac{\partial M}{\partial \tau_i}(\alpha_i, \tau_i) = -\alpha_i - \frac{1-\alpha_i}{(\tau_i)^2}.$$

We turn to the proof of Lemma 2 which establishes conditions under which there is an interior solution to the first order condition $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$.

Lemma 2. *(i) If* $\frac{\omega_{i2}}{\sum_j \omega_{j2}} < 1$ *for all* $i \in \mathcal{I}$ *then there exists* $\underline{\tau} \in (0, 1)$ *such that, for all* $\underline{\underline{\tau}} \in (0, \underline{\tau}]$, *for all* $i \in \mathcal{I}$, $\frac{\partial v_i}{\partial \tau_i}(\underline{\underline{\tau}}, \tau_{-i}) > 0$ *for* $\tau_{-i} \in [\underline{\tau}, \infty)^{I-1}$; *(ii) if* $\frac{\omega_{i1}}{\sum_j \omega_{j1}} < 1$ *for all* $i \in \mathcal{I}$ *then there exists* $\overline{\tau} > 1$ *such that, for all* $\overline{\overline{\tau}} \in [\overline{\tau}, \infty)$, *for all* $i \in \mathcal{I}$, $\frac{\partial v_i}{\partial \tau_i}(\overline{\overline{\tau}}, \tau_{-i}) < 0$ *for* $\tau_{-i} \in (0, \overline{\overline{\tau}}]^{I-1}$.

Proof. By Lemma S.2, the sign of $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})$ is determined by the sign of the expression

$$\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)} + \frac{(1-\alpha_i)\cdot\omega_{i1}}{\sum_j [1 - A(\alpha_j,\tau_j)]\omega_{j1}} - \frac{\alpha_i \cdot \omega_{i2}}{\sum_j A(\alpha_j,\tau_j)\omega_{j2}}.$$

In (i) below, we ignore the positive second term and show that even so the sum of the remaining terms is positive for some $\underline{\tau}$ sufficiently small. In (ii) we ignore the last term, which is negative, and show that for some $\overline{\tau}$ sufficiently large the sum of the remaining terms is, nonetheless, negative.

Let $\underline{\alpha} := \min_{i\in\mathcal{I}} \alpha_i$ and $\overline{\alpha} := \max_{i\in\mathcal{I}} \alpha_i$; evidently, $0 < \underline{\alpha} \le \overline{\alpha} < 1$.

(i) By hypothesis $\frac{\omega_{i2}}{\sum_j \omega_{j2}} < 1$ for all $i \in \mathcal{I}$; so there must exist $\underline{\tau} \in (0, 1)$ such that

$$\forall\, i \in \mathcal{I} \quad \underline{\tau}\frac{[\underline{\tau} + (1-\underline{\alpha})/\underline{\alpha}]}{[\underline{\tau} + (1-\overline{\alpha})/\overline{\alpha}]} = \underline{\tau}\frac{A(\overline{\alpha},\underline{\tau})}{A(\underline{\alpha},\underline{\tau})} < 1 - \frac{\omega_{i2}}{\sum_j \omega_{j2}},$$

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad \underline{\tau}\cdot A(\overline{\alpha},\underline{\tau})\cdot\sum_j \omega_{j2} < A(\underline{\alpha},\underline{\tau})\cdot\sum_{j\neq i}\omega_{j2}.$$

By Lemma S.1, $A(\alpha_j, \tau_j)$ is increasing in α_j, and so

$$\Rightarrow \quad \forall\, i \in \mathcal{I} \quad \underline{\tau}\cdot\left[A(\alpha_i,\underline{\tau})\omega_{i2} + \sum_{j\neq i} A(\alpha_j,\underline{\tau})\omega_{j2}\right] < \sum_{j\neq i} A(\alpha_j,\underline{\tau})\omega_{j2},$$

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad \underline{\tau}\cdot A(\alpha_i,\underline{\tau})\omega_{i2} < \left(1-\underline{\tau}\right)\cdot\sum_{j\neq i} A(\alpha_j,\underline{\tau})\omega_{j2}.$$

By Lemma S.1 $A(\alpha_j, \tau_j)$ is increasing in τ_j, and so, for $\tau_j \ge \underline{\tau}$, we must have

$$\forall\, i \in \mathcal{I} \quad \underline{\tau}\cdot A(\alpha_i,\underline{\tau})\omega_{i2} < \left(1-\underline{\tau}\right)\cdot\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}$$

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad A(\alpha_i,\underline{\tau})\omega_{i2} < (1-\underline{\tau})\left[\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2} + A(\alpha_i,\underline{\tau})\omega_{i2}\right]$$

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad \frac{\omega_{i2}}{\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2} + A(\alpha_i,\underline{\tau})\omega_{i2}} < \frac{(1-\underline{\tau})}{A(\alpha_i,\underline{\tau})}$$

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad 0 < \frac{\alpha_i(1-\underline{\tau})}{A(\alpha_i,\underline{\tau})} - \frac{\alpha_i\cdot\omega_{i2}}{\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2} + A(\alpha_i,\underline{\tau})\omega_{i2}}.$$

That verifies the sign of the expression.

The proof is completed by observing that the same argument holds for all $\underline{\underline{\tau}} \in (0, \underline{\tau}]$.

(ii) By hypothesis $\frac{\omega_{i1}}{\sum_j \omega_{j1}} < 1$ for all $i \in \mathcal{I}$; so there must exist $\overline{\tau}$, with $1/\overline{\tau} \in (0, 1)$, such that

$$\forall\, i \in \mathcal{I} \quad \frac{1}{\overline{\tau}}\frac{[1/\overline{\tau} + \overline{\alpha}/(1-\overline{\alpha})]}{[1/\overline{\tau} + \underline{\alpha}/(1-\underline{\alpha})]} = \frac{1}{\overline{\tau}}\frac{[1 - A(\underline{\alpha},\overline{\tau})]}{[1 - A(\overline{\alpha},\overline{\tau})]} < 1 - \frac{\omega_{i1}}{\sum_j \omega_{j1}},$$

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad \left[1 - A(\underline{\alpha},\overline{\tau})\right]\cdot\sum_j \omega_{j1} < \overline{\tau}\cdot[1 - A(\overline{\alpha},\overline{\tau})]\cdot\sum_{j\neq i}\omega_{j1}.$$

By Lemma S.1 $A(\alpha_j, \tau_j)$ is increasing in α, and so

$$\Rightarrow \quad \forall\, i \in \mathcal{I} \quad [1 - A(\alpha_i,\overline{\tau})]\,\omega_{i1} + \sum_{j\neq i}\left[1 - A(\alpha_j,\overline{\tau})\omega_{j1}\right] < \overline{\tau}\cdot\sum_{j\neq i}\left[1 - A(\alpha_j,\overline{\tau})\right]\omega_{j1}$$

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad [1 - A(\alpha_i,\overline{\tau})]\,\omega_{i1} < (\overline{\tau}-1)\cdot\sum_{j\neq i}\left[1 - A(\alpha_j,\overline{\tau})\right]\omega_{j1}$$

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad \overline{\tau}\cdot[1 - A(\alpha_i,\overline{\tau})]\,\omega_{i1} < (\overline{\tau}-1)\cdot\left[\sum_j\left[1 - A(\alpha_j,\overline{\tau})\right]\omega_{j1}\right].$$

Since $[1 - A(\alpha_i, \tau_i)] = \frac{1-\alpha_i}{\alpha_i\tau_i}A(\alpha_i, \tau_i)$, we have

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad \frac{1-\alpha_i}{\alpha_i}\cdot A(\alpha_i,\overline{\tau})\omega_{i1} < (\overline{\tau}-1)\cdot\left[\sum_{j\neq i}\left[1 - A(\alpha_j,\overline{\tau})\right]\omega_{j1}\right].$$

By Lemma S.1 $A(\alpha_j, \tau_j)$ is increasing in τ_j, and so, for $\tau_j \le \overline{\tau}$, we must have

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad \frac{(1-\alpha_i)}{\alpha_i}\cdot A(\alpha_i,\overline{\tau})\cdot\omega_{i1} < (\overline{\tau}-1)\left[\sum_{j\neq i}[1 - A(\alpha_j,\tau_j)]\omega_{j1} + [1 - A(\alpha_i,\overline{\tau})]\omega_{i1}\right]$$

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad \frac{(1-\alpha_i)\cdot\omega_{i1}}{\sum_{j\neq i}[1 - A(\alpha_j,\tau_j)]\omega_{j1} + [1 - A(\alpha_i,\overline{\tau})]\omega_{i1}} < \frac{\alpha_i(\overline{\tau}-1)}{A(\alpha_i,\overline{\tau})}$$

$$\Leftrightarrow \quad \forall\, i \in \mathcal{I} \quad \frac{\alpha_i(1-\overline{\tau})}{A(\alpha_i,\overline{\tau})} + \frac{(1-\alpha_i)\cdot\omega_{i1}}{\sum_{j\neq i}[1 - A(\alpha_j,\tau_j)]\omega_{j1} + [1 - A(\alpha_i,\overline{\tau})]\omega_{i1}} < 0.$$

That verifies the sign of the expression.

The proof is completed by observing that the same argument holds for all $\overline{\overline{\tau}} \in [\overline{\tau}, \infty)$. ■

The next supplementary result, Lemma S.4, provides an evaluation of $\frac{\partial N_i}{\partial \tau_j}(\vec{\tau})$. It is used in proving both Lemmas 3 and 5.

Lemma S.4.

$$\frac{\partial N_i}{\partial \tau_j}(\vec{\tau}) = \frac{\frac{\alpha_j(1-\alpha_j)}{[\alpha_j\tau_j + (1-\alpha_j)]^2}}{\left[\sum_k A(\alpha_k,\tau_k)\omega_{k2}\right]^2} \cdot\left\{(1-\alpha_i)\left(f(\vec{\tau})\right)^2\cdot\omega_{j1}\omega_{i1} + \alpha_i\cdot\omega_{i2}\cdot\omega_{j2}\right\}.$$

We now state a supplementary result that prepares the groundwork for the proof of Lemma 3 where we pin down the sign of $\frac{\partial^2 v_i}{\partial \tau_i \partial \tau_i}(\vec{\tau})$ at $\vec{\tau}$ at which $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$.

Lemma S.5. *If* $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$ *and* $\frac{\partial v_j}{\partial \tau_j}(\vec{\tau}) = 0$ *then*

$$\frac{\partial N_i}{\partial \tau_j}(\vec{\tau}) = \alpha_i \left[\frac{A(\alpha_j, \tau_j)}{\tau_j}\right]^2 \left\{ \frac{\omega_{i2}\cdot\omega_{j2}}{\alpha_j \left[\sum_k A(\alpha_k, \tau_k)\omega_{k2}\right]^2} + \left[\frac{(1-\tau_i)(1-\tau_j)}{A(\alpha_i,\tau_i)\cdot A(\alpha_j,\tau_j)}\right] - \frac{1}{\left[\sum_k A(\alpha_k,\tau_k)\omega_{k2}\right]}\left\{\frac{\omega_{j2}(1-\tau_i)}{A(\alpha_i,\tau_i)} + \frac{\omega_{i2}(1-\tau_j)}{A(\alpha_j,\tau_j)}\right\}\right\}.$$

When $j = i$ the expression simplifies to

$$\frac{\partial N_i}{\partial \tau_i}(\vec{\tau}) = \frac{\left[\frac{A(\alpha_i,\tau_i)}{\tau_i}\right]^2}{\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right]}\left\{\frac{(\omega_{i2})^2}{\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right]} - 2\omega_{i2}\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}\right\} + \alpha_i\left\{\frac{(1-\tau_i)}{\tau_i}\right\}^2.$$

Our penultimate supplementary result is used in Lemmas 3 and 4.

Lemma S.6. *If* $\vec{\tau}$ *is such that* $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$ *then* $\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2} > 0$.

Proof. Since $A(\alpha_j,\tau_j) > 0$, $\omega_{j2} \geq 0$ and $\sum_{i\in\mathcal{I}}\omega_{i2} > 0$, we have (i) $\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2} \geq 0$ and (ii) $\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2} = 0$ if and only if $\omega_{j2} = 0$ for all $j \neq i$ and $\omega_{i2} > 0$.

If $\omega_{j2} = 0$ for all $j \neq i$ and $\omega_{i2} > 0$ then, using Lemma S.2,

$$sign\left\{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})\right\} = sign\left\{\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)} + \frac{(1-\alpha_i)\omega_{i1}}{\sum_j[1-A(\alpha_j,\tau_j)]\omega_{j1}} - \frac{\alpha_i}{A(\alpha_i,\tau_i)}\right\} = sign\left\{\frac{-\alpha_i\tau_i}{A(\alpha_i,\tau_i)} + \frac{(1-\alpha_i)\omega_{i1}}{\sum_j[1-A(\alpha_j,\tau_j)]\omega_{j1}}\right\}.$$

Since $\frac{\alpha_i\tau_i}{A(\alpha_i,\tau_i)} = \alpha_i\tau_i + (1-\alpha_i) = (1-\alpha_i)/\frac{(1-\alpha_i)}{\alpha_i\tau_i+(1-\alpha_i)} = \frac{1-\alpha_i}{1-A(\alpha_i,\tau_i)}$, we have

$$sign\left\{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})\right\} = sign\left\{(1-\alpha_i)\left\{-\frac{1}{1-A(\alpha_i,\tau_i)} + \frac{\omega_{i1}}{\sum_j[1-A(\alpha_j,\tau_j)]\omega_{j1}}\right\}\right\} < 0$$

unless $\omega_{j1} = 0$ for all $j \neq i$ and $\omega_{i1} > 0$ in which case the expression takes the value zero. So if $\omega_{j2} = 0$ for all $j \neq i$ and $\omega_{i2} > 0$ and $\vec{\tau}$ is such that $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$ then, necessarily, $\omega_{j1} = 0$ for all $j \neq i$ and $\omega_{i1} > 0$. But that contradicts our assumption that for every $i \in \mathcal{I}$, $\omega_i \in \mathbb{R}^2_+/\{0\}$. We conclude that if $\vec{\tau}$ is such that $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$ then $\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2} > 0$. ■

We use Lemmas S.3, S.5 and S.6 to prove Lemma 3.

Lemma 3.

$$sign\left\{\left.\frac{\partial^2 v_i}{\partial \tau_i\,\partial \tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0}\right\} = sign\left\{-2\alpha_i\tau_i\omega_{i2} - [1-2\alpha_i(1-\tau_i)]\left[\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}\right]\right\}.$$

Proof. Using Lemma 1(ii) and the definitions of the functions $M(\alpha_i,\tau_i)$ and $N_i(\vec{\tau})$ we have

$$\begin{aligned}\frac{\partial^2 v_i}{\partial \tau_i\,\partial \tau_j}(\vec{\tau}) &= \frac{\partial}{\partial \tau_j}\left\{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})\right\} \\ &= \frac{\partial}{\partial \tau_j}\left\{v_i(\vec{\tau})\cdot\frac{\partial A}{\partial \tau_i}(\alpha_i,\tau_i)\cdot[M(\alpha_i,\tau_i)+N_i(\vec{\tau})]\right\} \\ &= \frac{\partial v_i}{\partial \tau_j}(\vec{\tau})\cdot\left\{\frac{\partial A}{\partial \tau_i}(\alpha_i,\tau_i)\cdot[M(\alpha_i,\tau_i)+N_i(\vec{\tau})]\right\} \\ &\quad + v_i(\vec{\tau})\cdot\left\{\frac{\partial}{\partial \tau_j}\left[\frac{\partial A}{\partial \tau_i}(\alpha_i,\tau_i)\right]\right\}\cdot[M(\alpha_i,\tau_i)+N_i(\vec{\tau})] \\ &\quad + v_i(\vec{\tau})\cdot\frac{\partial A}{\partial \tau_i}(\alpha_i,\tau_i)\cdot\left\{\frac{\partial M}{\partial \tau_j}(\alpha_i,\tau_i)+\frac{\partial N_i}{\partial \tau_j}(\vec{\tau})\right\}.\end{aligned}$$

From Lemma S.2, $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$ if and only if $M(\alpha_i,\tau_i)+N_i(\vec{\tau}) = 0$. In addition, as noted in the proof of Lemma S.2, $v_i(\vec{\tau}) > 0$ and $\frac{\partial A}{\partial \tau_i}(\alpha_i,\tau_i) > 0$. Taking these facts into account, we have

$$sign\left\{\left.\frac{\partial^2 v_i}{\partial \tau_i\,\partial \tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0}\right\} = sign\left\{\left.\frac{\partial M}{\partial \tau_i}(\alpha_i,\tau_i)\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} + \left.\frac{\partial N_i}{\partial \tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0}\right\}.$$

From Lemmas S.3 and S.5, we have

$$\left.\frac{\partial M}{\partial \tau_i}(\alpha_i,\tau_i)\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} + \left.\frac{\partial N_i}{\partial \tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0} = -\alpha_i - \frac{1-\alpha_i}{(\tau_i)^2} + \frac{\frac{(\alpha_i)^2}{[\alpha_i\tau_i+(1-\alpha_i)]^2}}{\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right]}\left\{\frac{(\omega_{i2})^2}{\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right]} - 2\omega_{i2}\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}\right\} + \alpha_i\left\{\frac{(1-\tau_i)}{\tau_i}\right\}^2.$$

Since

$$-\alpha_i - \frac{1-\alpha_i}{(\tau_i)^2} + \alpha_i\left\{\frac{(1-\tau_i)}{\tau_i}\right\}^2 = -\frac{1-2\alpha_i+2\alpha_i\tau_i}{(\tau_i)^2} = -\frac{1-2\alpha_i(1-\tau_i)}{(\tau_i)^2},$$

it follows that

$$sign\left\{\left.\frac{\partial^2 v_i}{\partial \tau_i\,\partial \tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial \tau_i}(\vec{\tau})=0}\right\} = sign\,\{\mathcal{K}_{ii}\}$$

where

$$\mathcal{K}_{ii} = \frac{\frac{(\alpha_i)^2}{[\alpha_i\tau_i+(1-\alpha_i)]^2}}{\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right]}\left\{\frac{(\omega_{i2})^2}{\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right]} - 2\omega_{i2}\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}\right\} - \frac{1-2\alpha_i(1-\tau_i)}{(\tau_i)^2}.$$

We have

$$\mathcal{K}_{ii}\cdot(\tau_i)^2\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right]^2 = (\omega_{i2})^2\frac{(\alpha_i\tau_i)^2}{[\alpha_i\tau_i+(1-\alpha_i)]^2}$$

$$- \left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right]^2 [1-2\alpha_i(1-\tau_i)]$$
$$- 2\omega_{i2}\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}\frac{(\alpha_i\tau_i)^2}{[\alpha_i\tau_i+(1-\alpha_i)]^2}\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right],$$

which, upon introducing the notation $\mathcal{A}_{-i} := \sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}$, recalling that $A(\alpha_i,\tau_i) = \frac{\alpha_i\tau_i}{\alpha_i\tau_i+(1-\alpha_i)}$, and simplifying, can be expressed as

$$\begin{aligned} &= (\omega_{i2})^2 [A(\alpha_i,\tau_i)]^2 - 2\omega_{i2}\alpha_i(1-\tau_i)A(\alpha_i,\tau_i)[\mathcal{A}_{-i}+A(\alpha_i,\tau_i)\omega_{i2}] \\ &\quad - [\mathcal{A}_{-i}+A(\alpha_i,\tau_i)\omega_{i2}]^2[1-2\alpha_i(1-\tau_i)], \end{aligned}$$

which, by expanding the terms within brackets and collecting terms, may be expressed as

$$\begin{aligned} &= -[1-2\alpha_i(1-\tau_i)][A(\alpha_i,\tau_i)\omega_{i2}]^2 + (\omega_{i2})^2[A(\alpha_i,\tau_i)]^2 \\ &\quad - 2\omega_{i2}\alpha_i(1-\tau_i)A(\alpha_i,\tau_i)[A(\alpha_i,\tau_i)\omega_{i2}] \\ &\quad - [1-2\alpha_i(1-\tau_i)][2\mathcal{A}_{-i}\cdot A(\alpha_i,\tau_i)\omega_{i2}] \\ &\quad - 2\omega_{i2}\alpha_i(1-\tau_i)A(\alpha_i,\tau_i)[\mathcal{A}_{-i}] \\ &\quad - [1-2\alpha_i(1-\tau_i)](\mathcal{A}_{-i})^2, \\ &= [A(\alpha_i,\tau_i)]^2\{-[1-2\alpha_i(1-\tau_i)]+1-2\alpha_i(1-\tau_i)\}(\omega_{i2})^2 \\ &\quad - 2A(\alpha_i,\tau_i)\{[1-2\alpha_i(1-\tau_i)]+(1-\tau_i)\alpha_i\}[\mathcal{A}_{-i}\omega_{i2}] \\ &\quad - [1-2\alpha_i(1-\tau_i)](\mathcal{A}_{-i})^2. \end{aligned}$$

Evidently, the coefficients in the first term on the right add up to zero and the second term can be simplified to obtain

$$\mathcal{K}_{ii}\cdot(\tau_i)^2\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right]^2 = -2A(\alpha_i,\tau_i)[1-\alpha_i(1-\tau_i)] [\mathcal{A}_{-i}\omega_{i2}] - [1-2\alpha_i(1-\tau_i)](\mathcal{A}_{-i})^2,$$

which, upon recalling that $A(\alpha_i,\tau_i) = \frac{\alpha_i\tau_i}{\alpha_i\tau_i+(1-\alpha_i)}$, may be simplified to

$$= -\mathcal{A}_{-i}\{2\alpha_i\tau_i\omega_{i2} + [1-2\alpha_i(1-\tau_i)]\mathcal{A}_{-i}\}.$$

By Lemma S.6, at $\vec{\tau}$ such that $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$, $\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2} > 0$, i.e. $\mathcal{A}_{-i} > 0$, and so, at such a $\vec{\tau}$,

$$\begin{aligned} &\operatorname{sign}\left\{\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_i}(\vec{\tau})\bigg|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0}\right\} \\ &= \operatorname{sign}\left\{-2\alpha_i\tau_i\omega_{i2} - [1-2\alpha_i(1-\tau_i)]\left[\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}\right]\right\} \end{aligned}$$

where we replace $\mathcal{A}_{-i}$ by $\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}$. ■

Our last supplementary result is used in Lemma 4 to claim that $\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_i}(\vec{\tau})\Big|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0} \neq 0$.

Lemma S.7. *The two equations that follow cannot hold simultaneously:*

$$2\alpha_i\tau_i\omega_{i2} + [1-2\alpha_i(1-\tau_i)]\left[\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}\right] = 0$$
$$\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)} + \frac{(1-\alpha_i)\omega_{i1}}{\sum_j[1-A(\alpha_j,\tau_j)]\omega_{j1}} - \frac{\alpha_i\cdot\omega_{i2}}{\sum_j A(\alpha_j,\tau_j)\omega_{j2}} = 0.$$

Proof. Set $\mathcal{A}_{-i} := \sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}$, and rewrite the first equation as

$$\begin{aligned} &2\alpha_i\tau_i\omega_{i2} + [1-2\alpha_i(1-\tau_i)]\mathcal{A}_{-i} = 0 \\ &\iff \quad \alpha_i(1-\tau_i)\mathcal{A}_{-i} = \alpha_i\tau_i\omega_{i2} + \frac{\mathcal{A}_{-i}}{2}. \qquad (*) \end{aligned}$$

Now observe that

$$\begin{aligned} &\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right] - \alpha_i\omega_{i2} \\ &= \frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}[\mathcal{A}_{-i}+A(\alpha_i,\tau_i)\omega_{i2}] - \alpha_i\omega_{i2} \\ &= \frac{\alpha_i\tau_i\omega_{i2}}{A(\alpha_i,\tau_i)} + \frac{\mathcal{A}_{-i}}{2A(\alpha_i,\tau_i)} + \alpha_i(1-\tau_i)\omega_{i2} - \alpha_i\omega_{i2}, \end{aligned}$$

where we use (∗),

$$\begin{aligned} &= \frac{\alpha_i\tau_i\omega_{i2}}{A(\alpha_i,\tau_i)} + \frac{\mathcal{A}_{-i}}{2A(\alpha_i,\tau_i)} - \alpha_i\tau_i\omega_{i2} \\ &= \alpha_i\tau_i\omega_{i2}\left[\frac{1}{A(\alpha_i,\tau_i)} - 1\right] + \frac{\mathcal{A}_{-i}}{2A(\alpha_i,\tau_i)}. \end{aligned}$$

Since $1 > A(\alpha_i,\tau_i)$ and $\mathcal{A}_{-i} \geq 0$, we can conclude that

$$\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right] - \alpha_i\omega_{i2} \geq 0$$

with a strict inequality if $\omega_{i2} > 0$.

But then, since under Assumption 1, $\omega_i \neq (0,0)$ and $\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right] > 0$, we must have

$$\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right] - \alpha_i\omega_{i2} + (1-\alpha_i)\omega_{i1}\frac{\left[\sum_j A(\alpha_j,\tau_j)\omega_{j2}\right]}{\sum_j\left[1-A(\alpha_j,\tau_j)\right]\omega_{j1}} > 0. \quad ■$$

Lemma 4. $\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_i}(\vec{\tau})\Big|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0} \neq 0$. *If* $\alpha_i \leq 1/2$ *or if* $\frac{\omega_{i2}}{\sum_j \omega_{j2}} < 1$ *then* $\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_i}(\vec{\tau})\Big|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0} < 0$.

Proof. Lemmas 3, S.2 and S.7 directly imply that $\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_i}(\vec{\tau})\Big|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0} \neq 0$.

If $\alpha_i \leq 1/2$ then $0 < 1-2\alpha_i(1-\tau_i)$ for all $\tau_i > 0$, and the required result follows directly from Lemma 3.

We turn to the case where $\alpha_i > 1/2$. Let $\hat{\tau}_i > 0$ be the unique value such that

$$\hat{\tau}_i = \frac{\left[\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}\right][2\alpha_i-1]}{2\alpha_i\left[\omega_{i2}+\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}\right]},$$

which is well defined since $\sum_j \omega_{j2} > 0$ under Assumption 1.

Observe that

$$\begin{aligned} &\operatorname{sign}\left\{-2\alpha_i\tau_i\omega_{i2} - [1-2\alpha_i(1-\tau_i)]\left[\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}\right]\right\} = \\ &\operatorname{sign}\left\{\frac{\left[\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}\right][2\alpha_i-1]}{2\alpha_i\left[\omega_{i2}+\sum_{j\neq i} A(\alpha_j,\tau_j)\omega_{j2}\right]} - \tau_i\right\} = \operatorname{sign}\{\hat{\tau}_i - \tau_i\}, \end{aligned}$$

whereby we have

$$sign\left\{-2\alpha_i\tau_i\omega_{i2}-[1-2\alpha_i(1-\tau_i)]\left[\sum_{j\neq i}A(\alpha_j,\tau_j)\omega_{j2}\right]\right\}$$
$$=\begin{cases}1 & \text{if} \quad \tau_i\in\left(0,\hat{\tau}_i\right)\\ 0 & \text{if} \quad \tau_i=\hat{\tau}_i\\ -1 & \text{if} \quad \tau_i\in\left(\hat{\tau}_i,+\infty\right).\end{cases}$$

We specialize the notation for the remainder of the proof of Lemma 4: $\vec{\tau}$ refers to a profile at which $\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0$, and $\vec{\tau}^{\,i}$ denotes the ith component of the vector $\vec{\tau}$.

If $\vec{\tau}^{\,i}>\hat{\tau}_i$ then the sign of the expression in curly brackets evaluated at $\vec{\tau}^{\,i}$ is negative, and the result $\left.\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0}<0$ follows from Lemma 3.

Since $\left.\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0}\neq 0$, $\vec{\tau}^{\,i}=\hat{\tau}_i$ is ruled out.

The case that remains is where $\vec{\tau}^{\,i}<\hat{\tau}_i$. We shall show that this case leads to a contradiction and so cannot arise. For $\vec{\tau}^{\,i}<\hat{\tau}_i$ the sign of the expression in curly brackets evaluated at $\vec{\tau}^{\,i}$ is positive and, by Lemma 3, $\left.\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0}>0$. Define $\underline{\underline{\tau}}=\min\left\{\underline{\tau},\min_{j\in\mathcal{I}}\vec{\tau}^{\,j}\right\}$, where $\underline{\tau}$ is as specified in Lemma 2(i). Evidently, $\underline{\underline{\tau}}\in(0,\vec{\tau}^{\,i}]$. Since $\frac{\omega_{i2}}{\sum_j\omega_{j2}}<1$, by Lemma 2(i), $\frac{\partial v_i}{\partial\tau_i}(\underline{\underline{\tau}},\vec{\tau}_{-i})>0$. Since $\underline{\underline{\tau}}\in(0,\vec{\tau}^{\,i}]$, $\frac{\partial v_i}{\partial\tau_i}(\underline{\underline{\tau}},\vec{\tau}_{-i})>0$, $\left.\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_i}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0}>0$, and the function $\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})$ is continuous, by the Intermediate Value Theorem, it must have another "zero", so there must exist $\tilde{\tau}\in(\underline{\underline{\tau}},\vec{\tau}^{\,i})$ such that $\left.\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_i}(\tilde{\tau},\vec{\tau}_{-i})\right|_{\frac{\partial v_i}{\partial\tau_i}(\tilde{\tau},\vec{\tau}_{-i})=0}<0$. But then, by Lemma 3, the sign of the expression in curly brackets evaluated at $\tilde{\tau}$ is negative even though $\tilde{\tau}<\hat{\tau}_i$, which delivers the desired contradiction. ■

Theorem 1. *Assume that, for all $i\in\mathcal{I}$, $\frac{\omega_{i1}}{\sum_j\omega_{j1}}<1$ and $\frac{\omega_{i2}}{\sum_j\omega_{j2}}<1$. The tariff game of such an economy has an interior pure strategy Nash equilibrium.*

Proof. The proof of Theorem 1 follows from Lemmas 1–4 and Lemma S.8 below: The latter shows that when the strategy space is an interval, and player i's payoff function is (i) increasing in i's choice at the left boundary and decreasing in i's choice at the right boundary and (ii) has a negative second derivative at every point at which the first derivative is zero, the game has an interior pure strategy Nash equilibrium[20]; Lemmas 1–4 ensure that the tariff game satisfies the conditions specified in Lemma S.8. ■

Consider the following game. The set of players is $\mathcal{I}=\{1,2,\ldots,I\}$ with generic element i. S is the strategy set of each player. The choice made by player i is denoted s_i. Payoffs are given by the functions $\pi_i:S^I\rightarrow\mathbb{R}$. Let $s_{-i}:=((s_j)_{j\neq i})$ denote a profile of actions for all but agent i, and write payoffs as $\pi_i(s_i,s_{-i})$. We have

Lemma S.8. *Assume that $S:=[\underline{s},\bar{s}]\subset\mathbb{R}$, and that, for every $i\in\mathcal{I}$, the function π_i is twice continuously differentiable on $(\underline{t},\bar{t})^I$ where $[\underline{s},\bar{s}]\subset(\underline{t},\bar{t})$. Suppose that for each $i\in\mathcal{I}$ and every profile s_{-i} the following conditions hold:*

(i) $\frac{\partial\pi_i}{\partial s_i}(\underline{s},s_{-i})>0$ and $\frac{\partial\pi_i}{\partial s_i}(\bar{s},s_{-i})<0$,

(ii) if $\tilde{s}_i$ is such that $\frac{\partial\pi_i}{\partial s_i}(\tilde{s}_i,s_{-i})=0$ then $\frac{\partial}{\partial s_i}\left(\frac{\partial\pi_i}{\partial s_i}(\tilde{s}_i,s_{-i})\right)<0$.

Then there exists $(s_1^*,\ldots,s_I^*)\in S^I$, with $s_i^*\in(\underline{s},\bar{s})$ for each $i\in\mathcal{I}$, such that

for each $i\in\mathcal{I}$ $\quad s_i\in S\quad\Rightarrow\quad\pi_i(s_i^*,s_{-i}^*)\geq\pi_i(s_i,s_{-i}^*)$.[21]

Proof. Fix a profile s_{-i} and consider the problem of identifying $\hat{s}_i:=\operatorname{argmax}_{s_i\in S}\pi_i(s_i,s_{-i})$. Since S is a compact set and π_i is a continuous function of s_i, such a value $\hat{s}_i$ must exist.

Note that $\frac{\partial\pi_i}{\partial s_i}(s_i,s_{-i})$ is a continuously differentiable function of s_i. Since $S=[\underline{s},\bar{s}]$, by condition (i), continuity, and the Intermediate Value Theorem, there is a value $\tilde{s}_i\in(\underline{s},\bar{s})$ at which $\frac{\partial\pi_i}{\partial s_i}(\tilde{s}_i,s_{-i})=0$, i.e. the function has a zero in the interior of the set S. By condition (ii), and continuity of the second derivative, it can have only one zero; furthermore, since $\frac{\partial}{\partial s_i}\left(\frac{\partial\pi_i}{\partial s_i}(\tilde{s}_i,s_{-i})\right)<0$, the sufficient condition for $\tilde{s}_i$ to be a local maximum is met.

By condition (i), $\hat{s}_i\notin\{\underline{s},\bar{s}\}$, i.e. the solution to the maximization problem cannot be at either boundary point. But then $\hat{s}_i=\tilde{s}_i$ since the necessary condition for an interior point to be a maximizer is satisfied only at $\tilde{s}_i$.

We have shown that given a profile s_{-i}, i's best response always exists, is an interior point, and is a single value. But then, as we now show, the function π_i must be quasiconcave in s_i. If not then for some profile s_{-i} and some p, the upper set is not convex, i.e. there are values s_i^1, s_i^2, and s_i^3 in the set S such that $s_i^1<s_i^2<s_i^3$ and $\pi_i(s_i^1,s_{-i})\geq p$, $\pi_i(s_i^3,s_{-i})\geq p$ but $\pi_i(s_i^2,s_{-i})<p$. Since π_i is continuously differentiable, there would exist $s_i^4\in(s_i^1,s_i^3)$ such that $\frac{\partial\pi_i}{\partial s_i}(s_i^4,s_{-i})=0$ and $\frac{\partial}{\partial s_i}\left(\frac{\partial\pi_i}{\partial s_i}(s_i^4,s_{-i})\right)>0$, where the latter follows from the fact that $\pi_i(s_i^1,s_{-i})\geq p$, $\pi_i(s_i^3,s_{-i})\geq p$ but $\pi_i(s_i^2,s_{-i})<p$. But that contradicts condition (ii) in the statement of the proposition.

Since the set S is compact and convex, and for each i the payoff function is quasiconcave in s_i for a given profile s_{-i}, the existence of a pure strategy Nash equilibrium follows (see, e.g., Theorem 3 in Debreu, 1982). Interiority has already been established and is maintained since S is compact and π_i is continuous and when taken together they ensure that π_i is uniformly continuous. ■

Lemma 5. *For $i\neq j$,*

$$\text{(i)}\quad sign\left\{\left.\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_j}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0}\right\}>0;$$

$$\text{(ii)}\quad sign\left\{\frac{\partial^2 v_i}{\partial\tau_i\,\partial\tau_j}(\vec{\tau})\right\}=sign\Bigg\{\left[(1-\alpha_i)\omega_{i1}+(-\alpha_i)\big(p_2^*(\vec{\tau})\big)\omega_{i2}\right]^2$$
$$\cdot\left[\big(p_2^*(\vec{\tau})\big)^{-1}\omega_{j1}+\omega_{j2}\right]$$
$$+\left[\omega_{i1}\big(p_2^*(\vec{\tau})\big)^{-1}+\omega_{i2}\right]\cdot\big(p_2^*(\vec{\tau})\big)^2\cdot\Big[(1-\alpha_i)\big(p_2^*(\vec{\tau})\big)^{-2}$$
$$\cdot\omega_{j1}\omega_{i1}+\alpha_i\cdot\omega_{i2}\cdot\omega_{j2}\Big]$$
$$+\left[(1-\alpha_i)\omega_{i1}+(-\alpha_i)\big(p_2^*(\vec{\tau})\big)\omega_{i2}\right]\cdot\left[\big(p_2^*(\vec{\tau})\big)^{-1}\omega_{j1}+\omega_{j2}\right]$$
$$\cdot\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}\cdot\left[\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}\right]\Bigg\}.$$

[20] We state and prove the result since we were unable to find a suitable reference.

[21] Furthermore, since S is compact and π_i is continuous, all Nash equilibria are interior, $s_i^*\in(\underline{s},\bar{s})$.

Proof. Using Lemma 1(ii) and the definitions of the functions $M(\alpha_i, \tau_i)$ and $N_i(\vec{\tau})$ we have

$$\begin{aligned}\frac{\partial^2 v_i}{\partial\tau_i\partial\tau_j}(\vec{\tau}) &= \frac{\partial}{\partial\tau_j}\left\{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})\right\}\\ &= \frac{\partial}{\partial\tau_j}\left\{v_i(\vec{\tau})\cdot\frac{\partial A}{\partial\tau_i}(\alpha_i,\tau_i)\cdot[M(\alpha_i,\tau_i)+N_i(\vec{\tau})]\right\}\\ &= \frac{\partial v_i}{\partial\tau_j}(\vec{\tau})\cdot\left\{\frac{\partial A}{\partial\tau_i}(\alpha_i,\tau_i)\cdot[M(\alpha_i,\tau_i)+N_i(\vec{\tau})]\right\}\\ &\quad+v_i(\vec{\tau})\cdot\left\{\frac{\partial}{\partial\tau_j}\left[\frac{\partial A}{\partial\tau_i}(\alpha_i,\tau_i)\right]\right\}\cdot[M(\alpha_i,\tau_i)+N_i(\vec{\tau})]\\ &\quad+v_i(\vec{\tau})\cdot\frac{\partial A}{\partial\tau_i}(\alpha_i,\tau_i)\cdot\left\{\frac{\partial M}{\partial\tau_j}(\alpha_i,\tau_i)+\frac{\partial N_i}{\partial\tau_j}(\vec{\tau})\right\}.\end{aligned}$$

Since, for $i\neq j$, $\frac{\partial}{\partial\tau_j}\left\{\frac{\partial A}{\partial\tau_i}(\alpha_i,\tau_i)\right\}=0$ and $\frac{\partial M}{\partial\tau_j}(\alpha_i,\tau_i)=0$ follow easily from Lemma S.1(i) and the definition of $M(\alpha_i,\tau_i)$ respectively, we have

$$\begin{aligned}\frac{\partial^2 v_i}{\partial\tau_i\partial\tau_j}(\vec{\tau}) &= \frac{\partial A}{\partial\tau_i}(\alpha_i,\tau_i)\\ &\cdot\left\{\frac{\partial v_i}{\partial\tau_j}(\vec{\tau})\cdot[M(\alpha_i,\tau_i)+N_i(\vec{\tau})]+v_i(\vec{\tau})\cdot\frac{\partial N_i}{\partial\tau_j}(\vec{\tau})\right\}\qquad\text{for } i\neq j.\end{aligned}$$

We know from Lemma S.1 that $\frac{\partial A}{\partial\tau_i}(\alpha_i,\tau_i)>0$; it follows that

$$\begin{aligned}&sign\left\{\frac{\partial^2 v_i}{\partial\tau_i\partial\tau_j}(\vec{\tau})\right\}\\ &= sign\left\{\frac{\partial v_i}{\partial\tau_j}(\vec{\tau})\cdot[M(\alpha_i,\tau_i)+N_i(\vec{\tau})]+v_i(\vec{\tau})\cdot\frac{\partial N_i}{\partial\tau_j}(\vec{\tau})\right\}.\qquad(*)\end{aligned}$$

(i) From Lemma S.2, $\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0$ if and only if $M(\alpha_i,\tau_i)+N_i(\vec{\tau})=0$. In addition, as noted in the proof of Lemma S.2, $v_i(\vec{\tau})>0$. It follows that

$$\begin{aligned}&sign\left\{\left.\frac{\partial^2 v_i}{\partial\tau_i\partial\tau_j}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0}\right\}\\ &= sign\left\{\left.\frac{\partial N_i}{\partial\tau_j}(\vec{\tau})\right|_{\frac{\partial v_i}{\partial\tau_i}(\vec{\tau})=0}\right\}>0\qquad\text{for } i\neq j,\end{aligned}$$

since the sign of that last expression can be determined by using Lemma S.4.

(ii) Recall that $f(\vec{\tau})=\frac{1}{p_2^*(\vec{\tau})}$ and rewrite Lemma 1(i) as

$$\begin{aligned}&v_i(\vec{\tau})\\ &= (A(\alpha_i,\tau_i))^{\alpha_i}(1-A(\alpha_i,\tau_i))^{1-\alpha_i}\left[\omega_{i1}(f(\vec{\tau}))^{1-\alpha_i}+(f(\vec{\tau}))^{-\alpha_i}\omega_{i2}\right].\end{aligned}$$

We proceed to differentiate the function.

$$\begin{aligned}\frac{\partial v_i}{\partial\tau_j}(\vec{\tau}) &= (A(\alpha_i,\tau_i))^{\alpha_i}(1-A(\alpha_i,\tau_i))^{1-\alpha_i}\\ &\quad\cdot\left\{(1-\alpha_i)(f(\vec{\tau}))^{-\alpha_i}\frac{\partial f}{\partial\tau_j}(\vec{\tau})\omega_{i1}+(-\alpha_i)(f(\vec{\tau}))^{-\alpha_i-1}\frac{\partial f}{\partial\tau_j}(\vec{\tau})\omega_{i2}\right\}\\ &= (A(\alpha_i,\tau_i))^{\alpha_i}(1-A(\alpha_i,\tau_i))^{1-\alpha_i}(f(\vec{\tau}))^{-\alpha_i}\\ &\quad\cdot\left\{(1-\alpha_i)\omega_{i1}+(-\alpha_i)(f(\vec{\tau}))^{-1}\omega_{i2}\right\}\frac{\partial f}{\partial\tau_j}(\vec{\tau})\\ &= (A(\alpha_i,\tau_i))^{\alpha_i}(1-A(\alpha_i,\tau_i))^{1-\alpha_i}(f(\vec{\tau}))^{-\alpha_i}\\ &\quad\cdot\left\{(1-\alpha_i)\omega_{i1}+(-\alpha_i)(f(\vec{\tau}))^{-1}\omega_{i2}\right\}\\ &\quad\cdot\frac{\frac{\partial A}{\partial\tau_j}(\alpha_j,\tau_j)}{\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}}\left\{f(\vec{\tau})\omega_{j1}+\omega_{j2}\right\}\end{aligned}$$

where we substitute for $\frac{\partial f}{\partial\tau_j}(\vec{\tau})$ from Lemma S.1.

We can now evaluate

$$\begin{aligned}&\frac{\partial v_i}{\partial\tau_j}(\vec{\tau})\cdot[M(\alpha_i,\tau_i)+N_i(\vec{\tau})]+v_i(\vec{\tau})\cdot\frac{\partial N_i}{\partial\tau_j}(\vec{\tau})\\ &= (A(\alpha_i,\tau_i))^{\alpha_i}(1-A(\alpha_i,\tau_i))^{1-\alpha_i}(f(\vec{\tau}))^{-\alpha_i}\\ &\quad\cdot\frac{\frac{\partial A}{\partial\tau_j}(\alpha_j,\tau_j)}{\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}}\cdot\left\{(1-\alpha_i)\omega_{i1}+(-\alpha_i)(f(\vec{\tau}))^{-1}\omega_{i2}\right\}\\ &\quad\cdot\left\{f(\vec{\tau})\omega_{j1}+\omega_{j2}\right\}\cdot[M(\alpha_i,\tau_i)+N_i(\vec{\tau})]\\ &\quad+(A(\alpha_i,\tau_i))^{\alpha_i}(1-A(\alpha_i,\tau_i))^{1-\alpha_i}\left[\omega_{i1}(f(\vec{\tau}))^{1-\alpha_i}+(f(\vec{\tau}))^{-\alpha_i}\omega_{i2}\right]\\ &\quad\cdot\frac{\frac{\partial A}{\partial\tau_j}(\alpha_j,\tau_j)}{\left[\sum_k A(\alpha_k,\tau_k)\omega_{k2}\right]^2}\left\{(1-\alpha_i)(f(\vec{\tau}))^2\cdot\omega_{j1}\omega_{i1}+\alpha_i\cdot\omega_{i2}\cdot\omega_{j2}\right\},\end{aligned}$$

where we use Lemma S.4 to evaluate $\frac{\partial N_i}{\partial\tau_j}(\vec{\tau})$ and use Lemma S.1 to replace the term $\frac{\alpha_j(1-\alpha_j)}{[\alpha_j\tau_j+(1-\alpha_j)]^2}$ with $\frac{\partial A}{\partial\tau_j}(\alpha_j,\tau_j)$, and collect terms and simplify to obtain

$$\begin{aligned}&= (A(\alpha_i,\tau_i))^{\alpha_i}(1-A(\alpha_i,\tau_i))^{1-\alpha_i}(f(\vec{\tau}))^{-\alpha_i}\\ &\quad\cdot\frac{\frac{\partial A}{\partial\tau_j}(\alpha_j,\tau_j)}{[\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}]^2}\\ &\quad\cdot\Bigg\{\left[(1-\alpha_i)\omega_{i1}+(-\alpha_i)(f(\vec{\tau}))^{-1}\omega_{i2}\right]\cdot\left[f(\vec{\tau})\omega_{j1}+\omega_{j2}\right]\\ &\quad\cdot[M(\alpha_i,\tau_i)+N_i(\vec{\tau})]\cdot\left[\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}\right]\\ &\quad+[\omega_{i1}f(\vec{\tau})+\omega_{i2}]\cdot\frac{[\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}]^2}{\left[\sum_k A(\alpha_k,\tau_k)\omega_{k2}\right]^2}\\ &\quad\cdot\left[(1-\alpha_i)(f(\vec{\tau}))^2\cdot\omega_{j1}\omega_{i1}+\alpha_i\cdot\omega_{i2}\cdot\omega_{j2}\right]\Bigg\}.\end{aligned}$$

Label the term in braces in the last expression $\mathcal{L}_{ij}$. $(*)$ above together with Lemma S.1 imply that

$$sign\left\{\frac{\partial^2 v_i}{\partial\tau_i\partial\tau_j}(\vec{\tau})\right\}=sign\left\{\mathcal{L}_{ij}\right\}.\qquad(**)$$

From the definitions of the terms $M(\alpha_i,\tau_i)$ and $N_i(\vec{\tau})$ we see that

$$\begin{aligned}&[M(\alpha_i,\tau_i)+N_i(\vec{\tau})]\cdot\left[\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}\right]\\ &= \frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}\cdot\left[\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}\right]\\ &\quad+(1-\alpha_i)\omega_{i1}-\alpha_i\cdot\omega_{i2}\cdot(f(\vec{\tau}))^{-1}.\end{aligned}$$

Using the last expression, we may rewrite

$$\begin{aligned}\mathcal{L}_{ij} &= \left[(1-\alpha_i)\omega_{i1}+(-\alpha_i)(f(\vec{\tau}))^{-1}\omega_{i2}\right]\cdot\left[f(\vec{\tau})\omega_{j1}+\omega_{j2}\right]\\ &\quad\cdot\Bigg\{\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)}\cdot\left[\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}\right]\\ &\quad+(1-\alpha_i)\omega_{i1}-\alpha_i\cdot\omega_{i2}\cdot(f(\vec{\tau}))^{-1}\Bigg\}\\ &\quad+[\omega_{i1}f(\vec{\tau})+\omega_{i2}]\cdot\frac{[\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}]^2}{\left[\sum_k A(\alpha_k,\tau_k)\omega_{k2}\right]^2}\\ &\quad\cdot\left[(1-\alpha_i)(f(\vec{\tau}))^2\cdot\omega_{j1}\omega_{i1}+\alpha_i\cdot\omega_{i2}\cdot\omega_{j2}\right].\end{aligned}$$

Upon recognizing the expression for $(f(\vec{\tau}))^{-2}$ in explicit form, we obtain

$$\begin{aligned}\mathcal{L}_{ij} = &\left[(1-\alpha_i)\omega_{i1} + (-\alpha_i)(f(\vec{\tau}))^{-1}\omega_{i2}\right]^2 \cdot \left[f(\vec{\tau})\omega_{j1} + \omega_{j2}\right]\\ &+ [\omega_{i1}f(\vec{\tau}) + \omega_{i2}] \cdot (f(\vec{\tau}))^{-2}\\ &\cdot \left[(1-\alpha_i)\,(f(\vec{\tau}))^2 \cdot \omega_{j1}\omega_{i1} + \alpha_i \cdot \omega_{i2} \cdot \omega_{j2}\right]\\ &+ \left[(1-\alpha_i)\omega_{i1} + (-\alpha_i)(f(\vec{\tau}))^{-1}\omega_{i2}\right] \cdot \left[f(\vec{\tau})\omega_{j1} + \omega_{j2}\right]\\ &\cdot \frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)} \cdot \left[\sum_k [1 - A(\alpha_k,\tau_k)]\omega_{k1}\right].\end{aligned}$$

Recalling (∗∗) and that $f(\vec{\tau}) = \frac{1}{p_2^*(\vec{\tau})}$, we obtain the desired result. ■

Proposition 1. *Let $\vec{\tau}$ be such that $\tau_j = 1$ for all $j \neq i$ and $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$. Then $\tau_i > 1$ if and only if i is an exporter of good 1 when trade is free.*

Proof. Since consumption of good 1 in country i at equilibrium prices is

$$x_{i1}\left(p_2^*(\vec{\tau})\right) = A(\alpha_i,\tau_i)[\omega_{i1} + p_2^*(\vec{\tau})\,\omega_{i2}],$$

we have that, in the absence of tariffs, i.e. at the strategy profile $\vec{\tau} = (1,\dots,1)$,

$$x_{i1}\left(p_2^*(1,\dots,1)\right) = \alpha_i \cdot [\omega_{i1} + p_2^*(1,\dots,1)\,\omega_{i2}].$$

Also, from Lemma S.2,

$$\begin{aligned}sign\left\{\frac{\partial v_i}{\partial \tau_i}(1,\dots,1)\right\} &= sign\left\{\frac{1}{\sum_j[1-A(\alpha_j,1)]\omega_{j1}}\cdot\left\{(1-\alpha_i)\omega_{i1} - \frac{\alpha_i\cdot\omega_{i2}\sum_j[1-A(\alpha_j,1)]\omega_{j1}}{\sum_j A(\alpha_j,1)\omega_{j2}}\right\}\right\}\\ &= sign\left\{\frac{1}{\sum_j[1-A(\alpha_j,1)]\omega_{j1}}\cdot\left\{(1-\alpha_i)\omega_{i1} - \alpha_i\cdot\omega_{i2}\cdot p_2^*(1,\dots,1)\right\}\right\}\\ &= sign\left\{\frac{1}{\sum_j[1-A(\alpha_j,1)]\omega_{j1}}\left\{\omega_{i1} - x_{i1}\left(p_2^*(1,\dots,1)\right)\right\}\right\},\end{aligned}$$

where, under Assumption 1, $\sum_j[1-A(\alpha_j,\tau_j)]\omega_{j1} > 0$. We have shown that i is an exporter of good 1 when trade is free if and only if $\frac{\partial v_i}{\partial \tau_i}(1,\dots,1) > 0$. Since the maximizer is unique, we must have $\tau_i > 1$ as claimed. ■

Lemma 6. *Let $\vec{\tau}$ such that $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$. Then, for $i \neq j$, $sign\left\{\frac{\partial v_i}{\partial \tau_j}(\vec{\tau})\right\} = sign\{\tau_i - 1\}$.*

Proof. Since $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$, by Lemma S.2,

$$\frac{\alpha_i(1-\tau_i)}{A(\alpha_i,\tau_i)} + \frac{(1-\alpha_i)\omega_{i1}}{\sum_j[1-A(\alpha_j,\tau_j)]\omega_{j1}} - \frac{\alpha_i\cdot\omega_{i2}}{\sum_j A(\alpha_j,\tau_j)\omega_{j2}} = 0.$$

We have

$$\begin{aligned}\tau_i > 1 \quad &\Leftrightarrow \quad \frac{(1-\alpha_i)\omega_{i1}}{\sum_j[1-A(\alpha_j,\tau_j)]\omega_{j1}} > \frac{\alpha_i\cdot\omega_{i2}}{\sum_j A(\alpha_j,\tau_j)\omega_{j2}}\\ &\Leftrightarrow \quad \omega_{i1} > \alpha_i\cdot\left[\omega_{i1} + \left\{\frac{\sum_j[1-A(\alpha_j,\tau_j)]\omega_{j1}}{\sum_j A(\alpha_j,\tau_j)\omega_{j2}}\right\}\omega_{i2}\right]\\ &\Leftrightarrow \quad \omega_{i1} > \alpha_i\cdot[\omega_{i1} + p_2^*(\vec{\tau})\,\omega_{i2}].\end{aligned}$$

From the proof of Lemma 5—please refer to the beginning of the proof of (ii)—we know that, for $i \neq j$,

$$\begin{aligned}\frac{\partial v_i}{\partial \tau_j}(\vec{\tau}) = &(A(\alpha_i,\tau_i))^{\alpha_i}(1-A(\alpha_i,\tau_i))^{1-\alpha_i}(f(\vec{\tau}))^{-\alpha_i}\\ &\cdot\left\{(1-\alpha_i)\omega_{i1} + (-\alpha_i)(f(\vec{\tau}))^{-1}\omega_{i2}\right\}\\ &\cdot\frac{\frac{\partial A}{\partial \tau_j}(\alpha_j,\tau_j)}{\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}}\left\{f(\vec{\tau})\omega_{j1} + \omega_{j2}\right\}\end{aligned}$$

where $f(\vec{\tau}) = \frac{1}{p_2^*(\vec{\tau})}$. It follows that

$$\begin{aligned}\frac{\partial v_i}{\partial \tau_j}(\vec{\tau}) > 0 \quad &\Leftrightarrow \quad \omega_{i1} > \alpha_i\cdot[\omega_{i1} + p_2^*(\vec{\tau})\,\omega_{i2}]\\ &\Leftrightarrow \quad \tau_i > 1. \quad ■\end{aligned}$$

Proposition 2. *Let $\vec{\tau}$ and $\vec{\tau}\,'$ be two strategy profiles such that $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = \frac{\partial v_i}{\partial \tau_i}(\vec{\tau}\,') = 0$. If either $\tau_i > \tau_i' \geq 1$ or $\tau_i < \tau_i' \leq 1$ then $v_i(\vec{\tau}) > v_i(\vec{\tau}\,')$.*

Proof. From Lemma 6 we have

$$\frac{\partial v_i}{\partial \tau_j}(\vec{\tau}) > 0 \quad \Leftrightarrow \quad \tau_i > 1.$$

Since the best response function is strictly increasing we can conclude that, under the conditions stated, v_i increases as we move further away from $\tau_i = 1$. ■

Proposition 3. *Assume that the endowment is not a Pareto optimal allocation and consider an interior Nash equilibrium. (i) If there are countries $i, j \in \mathcal{I}$ such that $\tau_i^* > 1 > \tau_j^*$ then a Pareto improvement can be induced by forming two groups such that if i and j are in the same group then either (a) $\tau_i^* > 1$ and $\tau_j^* > 1$ or (b) $\tau_i^* < 1$ and $\tau_j^* < 1$, and with the same number of countries in each group, and moving the tariffs of both sets towards the free trade value by appropriate amounts. (ii) If all countries are on the same side of free trade then a Pareto improvement can be induced by moving the tariff rate of any one country further away from free trade.*

Proof. Let $\vec{\tau}$ be an interior Nash equilibrium profile of tariff rates so that $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$ for each $i \in \mathcal{I}$. We write τ_i instead of τ_i^*.

Define

$$\begin{aligned}\alpha_i(\vec{\tau}) &:= \frac{(A(\alpha_i,\tau_i))^{\alpha_i}(1-A(\alpha_i,\tau_i))^{1-\alpha_i}}{\sum_k[1-A(\alpha_k,\tau_k)]\omega_{k1}}(f(\vec{\tau}))^{-\alpha_i}\\ &\quad\cdot\left\{(1-\alpha_i)\omega_{i1} + (-\alpha_i)(f(\vec{\tau}))^{-1}\omega_{i2}\right\}\\ \beta_j(\vec{\tau}) &:= \frac{\partial A}{\partial \tau_j}(\alpha_j,\tau_j)\left\{f(\vec{\tau})\omega_{j1} + \omega_{j2}\right\},\end{aligned}$$

where $f(\vec{\tau}) = \frac{1}{p_2^*(\vec{\tau})}$. From the proof of Lemma 6 we know that, for $i \neq j$, $\frac{\partial v_i}{\partial \tau_j}(\vec{\tau}) = \alpha_i(\vec{\tau})\cdot\beta_j(\vec{\tau})$, that $\alpha_i(\vec{\tau}) > 0$ if and only if $\tau_i > 1$, and that $\beta_j(\vec{\tau}) > 0$.

Define $\mathcal{I}_+ := \{i \in \mathcal{I} : \alpha_i(\vec{\tau}) > 0\}$ and $\mathcal{I}_- := \{i \in \mathcal{I} : \alpha_i(\vec{\tau}) < 0\}$.

If $\mathcal{I}_+ \neq \emptyset$ and $\mathcal{I}_- \neq \emptyset$ then let $\widehat{\mathcal{I}_+} \subset \mathcal{I}_+$ and $\widehat{\mathcal{I}_-} \subset \mathcal{I}_-$ be such that $\#\widehat{\mathcal{I}_+} = \#\widehat{\mathcal{I}_-} > 0$, and define

$$\Delta\tau_i(\vec{\tau}) := \begin{cases}(-1)\frac{1}{\beta_i(\vec{\tau})} & \text{if } i \in \widehat{\mathcal{I}_+}\\ \frac{1}{\beta_i(\vec{\tau})} & \text{if } i \in \widehat{\mathcal{I}_-}\\ 0 & \text{if } i \notin \widehat{\mathcal{I}_+}\cup\widehat{\mathcal{I}_-}.\end{cases}$$

Quite generally, the change in payoff that is induced by a change in tariff rates can be calculated to be

$$\begin{aligned}\Delta v_i &= \sum_j \frac{\partial v_i}{\partial \tau_j}(\vec{\tau})\cdot\Delta\tau_j(\vec{\tau}) = \sum_{j\neq i}\frac{\partial v_i}{\partial \tau_j}(\vec{\tau})\cdot\Delta\tau_j(\vec{\tau})\\ &= \alpha_i(\vec{\tau})\cdot\left[\sum_{j\neq i}\beta_j(\vec{\tau})\cdot\Delta\tau_j(\vec{\tau})\right],\end{aligned}$$

where we use the fact that $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$ and that $\frac{\partial v_i}{\partial \tau_j}(\vec{\tau}) = \alpha_i(\vec{\tau}) \cdot \beta_j(\vec{\tau})$. Substituting the values proposed for $\Delta\tau_i(\vec{\tau})$ we have

$$\Delta v_i = \begin{cases} \alpha_i(\vec{\tau}) \cdot \left[(-1)\left(\left(\#\widehat{\mathcal{I}_+}\right) - 1\right) + \left(\#\widehat{\mathcal{I}_-}\right)\right] & \text{if } i \in \widehat{\mathcal{I}_+} \\ \alpha_i(\vec{\tau}) \cdot \left[(-1)\left(\#\widehat{\mathcal{I}_+}\right) + \left(\left(\#\widehat{\mathcal{I}_-}\right) - 1\right)\right] & \text{if } i \in \widehat{\mathcal{I}_-} \\ \alpha_i(\vec{\tau}) \cdot \left[(-1)\left(\#\widehat{\mathcal{I}_+}\right) + \left(\#\widehat{\mathcal{I}_-}\right)\right] & \text{if } i \notin \widehat{\mathcal{I}_+} \cup \widehat{\mathcal{I}_-}. \end{cases}$$

Since $\#\widehat{\mathcal{I}_+} = \#\widehat{\mathcal{I}_-} > 0$, we have

$$\Delta v_i = \begin{cases} \alpha_i(\vec{\tau}) \cdot [1] & \text{if } i \in \widehat{\mathcal{I}_+} \\ \alpha_i(\vec{\tau}) \cdot [-1] & \text{if } i \in \widehat{\mathcal{I}_-} \\ 0 & \text{if } i \notin \widehat{\mathcal{I}_+} \cup \widehat{\mathcal{I}_-}, \end{cases}$$

and it follows that $\Delta v_i > 0$ if $i \in \widehat{\mathcal{I}_+} \cup \widehat{\mathcal{I}_-}$ and that $\Delta v_i = 0$ otherwise.

If $\mathcal{I}_+ = \emptyset$ then a Pareto improvement can be induced by setting $\Delta\tau_i < 0$ for a single country (and zero otherwise); similarly, if $\mathcal{I}_- = \emptyset$ then set $\Delta\tau_i > 0$ for some country and zero for the rest. This works because the incentives of all the countries are aligned.

The case in which $\mathcal{I}_+ = \mathcal{I}_- = \emptyset$ can be ruled out since then the Nash equilibrium induces the free trade allocation which, by Proposition 5, requires that the endowment vector be a Pareto optimal allocation. ∎

Proposition 4. *(i) At every interior Nash equilibrium of the tariff game $\sum_j \tau_j^* > I - 1$ and $\sum_j \frac{1}{\tau_j^*} > I - 1$. (ii) If at an interior Nash equilibrium of the tariff game $\tau_i^* = \tau^*$ for all $i \in \mathcal{I}$, then, necessarily, $\tau_i^* = 1$ for all $i \in \mathcal{I}$.*

Proof. Let $\vec{\tau}$ be an interior Nash equilibrium profile of tariff rates. It follows $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$ for each $i \in \mathcal{I}$. We write τ_i instead of τ_i^*. By Lemma S.2, for each $i \in \mathcal{I}$,

$$\frac{\alpha_i(1-\tau_i)}{A(\alpha_i, \tau_i)} + \frac{(1-\alpha_i)\omega_{i1}}{\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}} - \frac{\alpha_i \cdot \omega_{i2}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}} = 0.$$

(i) Since $A(\alpha_i, \tau_i) = \frac{\alpha_i \tau_i}{\alpha_i \tau_i + (1-\alpha_i)}$ we have

$$\iff \quad \frac{1-\tau_i}{\tau_i} + \frac{\frac{(1-\alpha_i)\omega_{i1}}{\alpha_i\tau_i + (1-\alpha_i)}}{\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}} - \frac{\frac{\alpha_i \omega_{i2}}{\alpha_i\tau_i + (1-\alpha_i)}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}} = 0$$

$$\iff \quad \frac{1-\tau_i}{\tau_i} + \frac{[1 - A(\alpha_i, \tau_i)]\omega_{i1}}{\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}} - \frac{\frac{1}{\tau_i} A(\alpha_i, \tau_i)\omega_{i2}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}} = 0.$$

Summing over j and rearranging we have the following two implications

$$\Rightarrow \quad \sum_j \frac{1}{\tau_j} - I + 1 = \frac{\sum_j \frac{1}{\tau_j} A(\alpha_j, \tau_j)\omega_{j2}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}} > 0$$

$$I - \sum_j \tau_j - 1 = -\frac{\sum_j \tau_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}}{\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}} < 0.$$

It follows that in any interior Nash equilibrium, $\sum_j \tau_j > I - 1$ and $\sum_j \frac{1}{\tau_j} > I - 1$.

(ii) Consider an interior Nash equilibrium with $\tau_i = \tau$ for all i, i.e. one that is symmetric. We have

$$\sum_j \frac{1}{\tau_j} - I + 1 = \frac{\sum_j \frac{1}{\tau_j} A(\alpha_j, \tau_j)\omega_{j2}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}} \quad \Rightarrow \quad \frac{I}{\tau} - I + 1 = \frac{1}{\tau}.$$

Since $I > 1$, it is evident that $\tau = 1$, i.e $\tau_i = 1$ for all $i \in \mathcal{I}$. ∎

Proposition 5. *A Nash equilibrium allocation is Pareto optimal if and only if there is no trade.*

Proof. Let $\vec{\tau}$ be a Nash equilibrium profile of tariff rates. Since all Nash equilibrium profiles are interior, it follows $\frac{\partial v_i}{\partial \tau_i}(\vec{\tau}) = 0$ for each $i \in \mathcal{I}$. We write τ_i instead of τ_i^*. By Lemma S.2, for each $i \in \mathcal{I}$,

$$\frac{\alpha_i(1-\tau_i)}{A(\alpha_i, \tau_i)} + \frac{(1-\alpha_i)\omega_{i1}}{\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}} - \frac{\alpha_i \cdot \omega_{i2}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}} = 0$$

$$\iff \quad \frac{\alpha_i(1-\tau_i)}{A(\alpha_i, \tau_i)} \sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1} + (1-\alpha_i)\omega_{i1} - \frac{\alpha_i \cdot \omega_{i2} \sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1}}{\sum_j A(\alpha_j, \tau_j)\omega_{j2}} = 0 \qquad (*)$$

since, under Assumption 1, $\sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1} > 0$.

Also, i's first order condition in the domestic market for the two goods, $x_{i2} = \frac{(1-\alpha_i)}{\alpha_i \cdot \tau_i \cdot p_2^*(\vec{\tau})} x_{i1}$, must hold at $(x_{i1}, x_{i2}) = (x_{i1}(p_2^*(\vec{\tau})), x_{i2}(p_2^*(\vec{\tau})))$; the condition may also be written in the form $\tau_i \cdot p_2^*(\vec{\tau}) = \frac{(1-\alpha_i)/x_{i2}}{\alpha_i/x_{i1}}$ where $(x_{i1}, x_{i2}) = \left(x_{i1}\left(p_2^*(\vec{\tau})\right), x_{i2}\left(p_2^*(\vec{\tau})\right)\right)$.

(i) In order for the allocation at the Nash equilibrium to be Pareto optimal, marginal rates of substitution have to be equalized and so, for each pair of agents i and j, we must have

$$\frac{(1-\alpha_i)/x_{i2}\left(p_2^*(\vec{\tau})\right)}{\alpha_i/x_{i1}\left(p_2^*(\vec{\tau})\right)} = \frac{(1-\alpha_j)/x_{j2}\left(p_2^*(\vec{\tau})\right)}{\alpha_j/x_{j1}\left(p_2^*(\vec{\tau})\right)}$$

$$\iff \quad \tau_i \cdot p_2^*(\vec{\tau}) = \tau_j \cdot p_2^*(\vec{\tau})$$

and we must have $\tau_i = \tau$ for all $i \in \mathcal{I}$. But then, from Proposition 4(ii), $\tau_i = \tau = 1$ so that, at the Nash equilibrium, trade is free.

When we use $\tau_i = 1$ for all $i \in \mathcal{I}$ in $(*)$, and recall the explicit form for $p_2^*(\vec{\tau})$, we see that, for each $i \in \mathcal{I}$, we must have

$$(1-\alpha_i)\omega_{i1} - \alpha_i \cdot \omega_{i2} \cdot p_2^*(1, \ldots, 1) = 0.$$

But that is identical to i's first order condition in the domestic market for the two goods, $1 \cdot p_2^*(1 \cdots, 1) = \frac{(1-\alpha_i)/x_{i2}}{\alpha_i/x_{i1}}$, when $(x_{i1}, x_{i2}) = \omega_i$.

We have shown that, at a Nash equilibrium at which the allocation is Pareto optimal, there is no trade. That completes the proof of (i).

(ii) Suppose that the Nash equilibrium has the additional property that there is no trade. It follows that at $p_2^*(\vec{\tau})$, $\left(x_{i1}\left(p_2^*(\vec{\tau})\right), x_{i2}\left(p_2^*(\vec{\tau})\right)\right) = \omega_i$ for all $i \in \mathcal{I}$. So i's first order condition in the domestic market for the two goods, $x_{i2} = \frac{(1-\alpha_i)}{\alpha_i \tau_i p_2^*(\vec{\tau})} x_{i1}$, must hold at $(x_{i1}, x_{i2}) = \omega_i$. Since $p_2^*(\vec{\tau}) \in (0, +\infty)$, we can be certain that a no-trade outcome can occur only if $\omega_i \in \mathbb{R}^2_{++}$. To summarize, for each $i \in \mathcal{I}$, in addition to $(*)$, the following must be true:

$$p_2^*(\vec{\tau}) = \frac{1}{\tau_i} \frac{(1-\alpha_i)/\omega_{i2}}{\alpha_i/\omega_{i1}} \qquad p_2^*(\vec{\tau}) = \frac{\sum_i [1 - A(\alpha_i, \tau_i)]\omega_{i1}}{\sum_i A(\alpha_i, \tau_i)\omega_{i2}}.$$

It follows that, for all $i \in \mathcal{I}$,

$$\frac{\alpha_i(1-\tau_i)}{A(\alpha_i, \tau_i)} \sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1} + (1-\alpha_i)\omega_{i1} - \alpha_i \cdot \omega_{i2} \frac{1}{\tau_i} \frac{(1-\alpha_i)/\omega_{i2}}{\alpha_i/\omega_{i1}} = 0$$

$$\iff \quad \frac{\alpha_i(1-\tau_i)}{A(\alpha_i, \tau_i)} \sum_j [1 - A(\alpha_j, \tau_j)]\omega_{j1} + \left(1 - \frac{1}{\tau_i}\right)(1-\alpha_i)\omega_{i1} = 0,$$

which, upon recalling that $A(\alpha_i, \tau_i) = \frac{\alpha_i\tau_i}{\alpha_i\tau_i+(1-\alpha_i)}$, may be rewritten as

$$\iff \quad \left(1-\frac{1}{\tau_i}\right)\left\{-[\alpha_i\tau_i+(1-\alpha_i)]\sum_j[1-A(\alpha_j,\tau_j)]\omega_{j1} + (1-\alpha_i)\omega_{i1}\right\} = 0,$$

so that at least one of the two expressions within brackets must be zero.

It follows that if $\tau_i \neq 1$ for some $i \in \mathcal{I}$, then for all such i

$$\sum_j[1-A(\alpha_j,\tau_j)]\omega_{j1} = [1-A(\alpha_i,\tau_i)]\omega_{i1},$$

where we use the fact that $\alpha_i\tau_i+(1-\alpha_i) > 0$ and the definition of $A(\alpha_i,\tau_i)$. We have at least one equation. Also, $1-A(\alpha_i,\tau_i) > 0$ and $\omega_i \in \mathbb{R}^2_{++}$ since we have a no-trade outcome (in fact, it suffices that $\omega_{i1} > 0$ for at least two countries). It is evident that the equation(s) cannot hold.

It follows that $\left(1-\frac{1}{\tau_i}\right) = 0$ for each $i \in \mathcal{I}$, that is $\vec{\tau} = (1,\ldots,1)$ and, from i's first order condition in the domestic market for the two goods, at the interior Nash equilibrium under consideration, marginal rates of substitution are equalized across countries. This can happen only if the endowment distribution (which is the equilibrium allocation since we are considering a no trade equilibrium) is also a Pareto optimal allocation.

We have shown that, at a Nash equilibrium at which there is no trade, the allocation is Pareto optimal. That completes the proof of (ii). ■

Appendix. Robust examples of tariff games that are not supermodular

We turn to the details of the three examples presented in Section 4 in each of which the tariff game fails to be supermodular and this failure is robust; further details of the computations can be found in the Supplementary Material available online at the link in Appendix B at the end of this article. In each case we also check whether the proposed tariff rates satisfy bounds discussed in Section 2.3.

Example 1. Let there be two countries with the following parameter specification

$$\alpha_1 = 1/2 \quad \omega_{11} = 2 \quad \omega_{12} = 8/67$$
$$\alpha_2 = 1/2 \quad \omega_{21} = 4 \quad \omega_{22} = 2.$$

At $\tau_1 = 3$ we have $A(\alpha_1,\tau_1) = 3/4$, while if $\tau_2 = 1/2$ we have $A(\alpha_2,\tau_2) = 1/3$. It follows that $p_2^*(3, 1/2) = \frac{67}{16}$. It is easy to evaluate the expression in Lemma 5(ii) and check that it is negative (the verifications can be found in the Supplementary Material).

Since $\min\{\frac{8/67}{2+8/67}, \frac{2}{2+8/67}\} = \frac{8/67}{2+8/67} = \frac{8}{142}$ and $\min\{\frac{2}{2+4}, \frac{4}{2+4}\} = 1/3$, and also $\alpha_1 = \alpha_2$, for the bounds we may use any pair $(\underline{\tau}, \overline{\tau})$ such that $\underline{\tau} < \frac{8}{142}$ and $1/\overline{\tau} < 1/3$. Since $\frac{8}{142} < \frac{1}{2}$, and $\tau_1 = 3$ and $\tau_2 = 1/2$, we clearly have $\underline{\tau} < \tau_2$ and $\tau_1 < \overline{\tau}$ so that the tariff rates considered in the example are in the strategy sets induced by uniform bounds.

Example 2. Let there be I countries with the following parameter specification

$$\alpha_1 = 1/3 \quad \omega_{11} = 2 \quad \omega_{12} = 80/22$$
$$\alpha_i = 1/2 \quad \omega_{i1} = 2 \quad \omega_{i2} = 2 \qquad \text{for } i = 2,\ldots,I.$$

At $\tau_1 = 11/10$ we have $A(\alpha_1,\tau_1) = 11/31$, while if $\tau_i = 1$, $i \neq 1$, we have $A(\alpha_i,\tau_i) = 1/2$. It follows that $p_2^*(11/10, 1,\ldots,1) = 1$. Again, it is easy to evaluate the expression in Lemma 5(ii) and check that it is negative if $21 \cdot 60 < (I-1)\cdot 2$ (please refer to the Supplementary Material)—so we would require a few more than six hundred countries.

Turning to the bounds, since $\tau_i \geq 1$ for all $i \in \mathcal{I}$, we need only concern ourselves with the upper bound. We consider two different candidates, each in turn. (Where relevant, the details of the computations can be found in the Supplementary Material.)

Observe that $\underline{\alpha} = 1/3$ and $\overline{\alpha} = 1/2$. So at $\tau = 11/10$,

$$\frac{1}{\tau}\frac{[1/\tau+\overline{\alpha}/(1-\overline{\alpha})]}{[1/\tau+\underline{\alpha}/(1-\underline{\alpha})]} = (10/11)\cdot(42/31) > 1.$$

Also, since $\omega_{i1} = 2$ for all i we have $\min_{i\in\mathcal{I}} \frac{\sum_{j\neq i}\omega_{j1}}{\sum_j \omega_{j1}} = \frac{I-1}{I}$. It follows that in order for $\overline{\tau}$ to be such that the inequality

$$\frac{1}{\overline{\tau}}\frac{[1/\overline{\tau}+\overline{\alpha}/(1-\overline{\alpha})]}{[1/\overline{\tau}+\underline{\alpha}/(1-\underline{\alpha})]} < \min_{i\in\mathcal{I}} \frac{\sum_{j\neq i}\omega_{j1}}{\sum_j \omega_{j1}}$$

holds, it is necessary that $\overline{\tau} > 11/10$; in that case, from Section 2.3, such a value $\overline{\tau}$ can serve as a uniform upper bound. Therefore, $\tau_1 = 11/10$ will be within the strategy set specified by any uniform upper bound.

We turn to bounds that are tighter than the uniform bounds. One easily checks that with $\omega_{i1} = 2$ and $\tau_i = 11/10$ for all $i \in \mathcal{I}$, if $I > 600$ then

$$\frac{10}{11} < \min_{i\in\mathcal{I}} \frac{\sum_{j\neq i}\left[1-A(\alpha_j, 11/10)\right]\omega_{j1}}{\sum_j \left[1-A(\alpha_j, 11/10)\right]\omega_{j1}}.$$

It follows that there is a $\overline{\tau}$ smaller than 11/10 that satisfies

$$\frac{1}{\overline{\tau}} < \min_{i\in\mathcal{I}} \frac{\sum_{j\neq i}\left[1-A(\alpha_j, \overline{\tau})\right]\omega_{j1}}{\sum_j \left[1-A(\alpha_j, \overline{\tau})\right]\omega_{j1}}.$$

Therefore, $\tau_1 = 11/10$ is not in the strategy set induced by such an upper bound.

The wide discrepancy in the two bounds is easily explained by the fact that only one out of a large number of countries has 1/3 as its preference parameter. The violation of the second, more informative or tighter, upper bound is an artefact of the symmetry in endowments which we imposed to ease the computational pain. It may be rectified by specifying the endowment of the first good in a different way. Here is a specific example: let

$$I = 650$$
$$\omega_{i1} = 2 \quad \omega_{i2} = 80/22 \qquad \text{for } i = 1,$$
$$\omega_{i1} = 2 \quad \omega_{i2} = 2 \qquad \text{for } i = 2,$$
$$\omega_{i1} = \bar{\omega}_1 \quad \omega_{i2} = 2 \qquad \text{for } i = 3,4,\ldots,649,$$
$$\omega_{i1} = 150 \quad \omega_{i2} = 2 \qquad \text{for } i = 650,$$

where $\bar{\omega}_1$ (approximately 1.8) satisfies the equation

$$1 + (1/2)\cdot 647\cdot\bar{\omega}_1 + (1/2)\cdot 150 = (1/2)\cdot 649\cdot 2.$$

Such a specification maintains $p_2^*(11/10, 1,\ldots,1) = 1$ and does not interfere with any other element of the computation, i.e. it delivers a robust example of a tariff game that fails to be supermodular. In addition, one easily checks that

$$\min_{i\in\mathcal{I}} \frac{\sum_{j\neq i}\left[1-A(\alpha_j, 11/10)\right]\omega_{j1}}{\sum_j \left[1-A(\alpha_j, 11/10)\right]\omega_{j1}} < \frac{10}{11}.$$

From the earlier discussion we may conclude that, with the modified specification of the example, any tighter upper bound $\overline{\tau}$ must be larger than 11/10 so that now $\tau_1 = 11/10$ is in the strategy set induced by any such upper bound. That completes the desired verification and the discussion of the example.

Example 3. Let there be I countries with the following parameter specification

$$\alpha_1 = 1/2 \quad \omega_{11} = 2 \quad \omega_{12} = 10/11$$
$$\alpha_i = 1/2 \quad \omega_{i1} = 4 \quad \omega_{i2} = 2 \qquad \text{for } i = 2, \ldots, I.$$

At $\tau_1 = 11/10$ we have $A(\alpha_1, \tau_1) = 11/21$, while if $\tau_i = 1$, $i \neq 1$, we have $A(\alpha_i, \tau_i) = 1/2$. In this case $p_2^*(11/10, 1, \ldots, 1) = 2$. Again, it is easy to evaluate the expression in Lemma 5(ii) and check that it is negative if I exceeds two hundred and eleven. Since countries have identical preferences, the two approaches to the upper bound give the same result and the bound is violated. As in Example 2, the endowments can be altered to generate bounds that are satisfied.

Appendix B. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jmateco.2019.07.011.

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