Available online at www.sciencedirect.com

ScienceDirect

Journal of Economic Theory 185 (2020) 104952

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

Dominance of truthtelling and the lattice structure of Nash equilibria ☆

Olivier Bochet [a,*], Norovsambuu Tumennasan [b]

[a] *Division of Social Science, New York University Abu Dhabi, United Arab Emirates*
[b] *Department of Economics, Dalhousie University, Canada*

Received 2 August 2018; final version received 13 October 2019; accepted 18 October 2019
Available online 28 October 2019

Abstract

Truthtelling is often viewed as focal in the direct mechanisms associated with strategy-proof decision rules. Yet many direct mechanisms also admit Nash equilibria whose outcomes differ from the one under truthtelling. We study a model that has been widely discussed in the mechanism design literature (Sprumont, 1991) and whose strategy-proof and efficient rules typically suffer from the aforementioned deficit. We show that when a rule in this class satisfies the mild additional requirement of replacement monotonicity, the set of Nash equilibrium allocations of its preference revelation game is a complete lattice with respect to the order of Pareto dominance. Furthermore, the supremum of the lattice is the one obtained under truthtelling. In other words, truthtelling Pareto dominates all other Nash equilibria. For the rich subclass of weighted uniform rules, the Nash equilibrium allocations are, in addition, strictly Pareto ranked. We discuss the tightness of the result and some possible extensions.

JEL classification: C72; D11; D82

Keywords: Sequential allotment rules; Strategy-proofness; Truthtelling; Lattice; Pareto dominance

☆ The authors wish to thank an associate editor and two anonymous referees for their valuable feedback and comments.
* Corresponding author.
E-mail addresses: olivier.bochet@nyu.edu (O. Bochet), norov@dal.ca (N. Tumennasan).

https://doi.org/10.1016/j.jet.2019.104952
0022-0531/

1. Introduction

In the mechanism design literature, the single-peaked preference domain has played a central role. Most importantly, it paved a way out of the many impossibility results on the design of prior-free mechanisms. The celebrated Gibbard and Satterthwaite theorem (see Gibbard (1973) and Satterthwaite (1975)) showed the impossibility of designing efficient and strategy-proof rules that would escape the dictatorship predicament under arbitrary preferences. In contrast, within the confine of the single-peaked domain, possibility results emerge. In a pathbreaking paper, Moulin (1980) characterizes the class of generalized median voting rules when the feasible set is made of all points on a line. On the private goods front, Sprumont (1991) studies the problem of allocating a divisible and nondisposable good.[1] Sprumont (1991) characterizes a remarkable rule: the uniform rule which is uniquely characterized down by efficiency, strategy-proofness and a fairness requirement. The Sprumont model has received a great deal of attention in the mechanism design literature, from alternative characterizations of the uniform rule (see e.g. Ching (1994), Thomson (1994a,b, 1995, 1997)), to the exploration of different families of rules (Barberà et al. (1997), Moulin (1999)), or the extensions of the model and the preference domain (see e.g. Adachi (2010), Bochet et al. (2013), Massó and Neme (2004) among others).[2]

In this paper, we show an unexpected property for a rich family of rules in the Sprumont model. We consider the largest class identified in the literature, the *sequential allotment rules*, characterized in Barberà et al. (1997) by the combination of *efficiency*, *strategy-proofness* and *replacement monotonicity*. Notice that each sequential allotment rule is fully implementable in dominant strategies by its direct revelation mechanism—this can be seen for instance following the results in Mizukami and Wakayama (2007). However, with the exception of dictatorship-type rules, the sequential allotment rules admit a plethora of Nash equilibria whose outcomes differ from the rule under truthtelling in their preference revelation games. Considering a rule as a direct revelation mechanism, our main result is as follows. We show that the set of Nash equilibrium allocations of any such rule is a complete lattice *with respect to the order of Pareto dominance*.[3] Every complete lattice has a well-defined supremum and infimum. We show that the former is the allocation obtained under truthtelling, hence truthtelling Pareto dominates all the other Nash equilibrium outcomes. The infimum of the lattice may, on the other hand, be rule-specific. Nevertheless, we show that for any sequential allotment rule for which the agents' initial guaranteed levels are invariant to regime changes, the infimum of the lattice is the allocation formed by these initial guaranteed levels.[4] For instance, in the case of the uniform rule the infimum of the lattice is the equal division allocation. Finally, for the special case of the weighted uniform rules (an

[1] In the sequel, we refer to this model as the *Sprumont model*.

[2] It is not possible to pay a proper tribute to the vast literature on the Sprumont model. We refer the interested reader to Thomson (2014) for an extensive survey.

[3] Our result establishes the complete lattice structure of the set of Nash equilibrium allocations with respect to the order of Pareto dominance. Henceforth, we simply refer to this as the *lattice property*, or *lattice result*, or *lattice of the set of Nash equilibrium allocations*. This should cause no confusion.

[4] Notice that a salient feature of the model is the two different regimes generated by the sum of agents' demands. Indeed (i) if the sum of demands exceeds what is available then we say that there is overdemand, (ii) if the sum of demands falls short of the resource we say that there is underdemand. A rule may treat different agents differently when there is either underdemand or overdemand.

extension of the uniform rule to a non-symmetric treatment of agents), we show that the Nash equilibrium allocations are in fact strictly Pareto ranked.[5]

In the remaining section of the paper, we check the tightness of our results and discuss some variations of the model (or the preference domain) where the lattice result may or may not hold. We first investigate the role that replacement monotonicity plays for our result with two examples. In Example 4.1 we construct a rule that violates replacement monotonicity and whose set of Nash equilibrium allocations is not a lattice with respect to the order of Pareto dominance. This hints that replacement monotonicity is essential for the result—and it certainly is in our proof. In addition, the rule considered there is efficient and *group strategy-proof*, demonstrating that our result cannot be proved if we just impose these two properties. Example 4.3 shows that replacement monotonicity is however not necessary for the lattice structure to hold. This suggests that replacement monotonicity can be replaced with weaker requirements in our main theorem. While this remains an open question at this stage, we show that for a rule that is efficient and strategy-proof, *non-bossiness* is a necessary (but not sufficient) condition for the lattice structure to hold.

Next, we look at (i) a different preference domain for which the lattice result may hold, and (ii) a possible variant of the model. On the former, Massó and Neme (2004) show that there are efficient and strategy-proof rules in the Sprumont model for the set of (partially) single-plateaued preferences. We show that the lattice result does not hold for the extended uniform rule characterized on this domain. Regarding the latter, we consider the model of Moulin (1980). We show that for the target rules (Thomson, 1993)—the subclass of the generalized median rules that satisfy replacement monotonicity (in welfare)—the set of Nash equilibrium public good levels are strictly Pareto ranked. On the other hand, for the well-known median rule, the lattice result does not hold.

The paper is organized as follows. In Section 2 we introduce the model and the necessary definitions. In Section 3, we present our main results. In Section 4, we discuss our results and some extensions to variants of the model. We offer some concluding remarks in Section 5.

2. Model and definitions

Let $N = \{1, 2, \ldots, n\}$ be the finite set of *agents*. There is a fixed amount of a divisible resource $\Omega > 0$ to be allocated. An *allotment* for $i \in N$ is denoted by $x_i \in [0, \Omega]$. An *allocation* is a vector of allotments $x = (x_1, \ldots, x_n) \in [0, \Omega]^n$ such that $\sum_{i \in N} x_i = \Omega$. Let X be the set of all possible allocations. Each agent $i \in N$ has a preference relation R_i which is a transitive, complete, and continuous binary relation on $[0, \Omega]$. We use the usual notations P_i and I_i to denote the asymmetric and symmetric parts of R_i, respectively. We restrict our attention to the set of single-peaked preferences, denoted by $\mathcal{R}_i$. That is, there exists a "peak" function $p : \mathcal{R}_i \to [0, \Omega]$ such that whenever a pair $x_i, x_i' \in [0, \Omega]$ satisfies either $x_i' < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < x_i'$, it must be that $x_i \; P_i \; x_i'$. A preference profile is a list $R = (R_i)_{i \in N}$, and the set of single-peaked preference profiles is $\mathcal{R} \equiv \prod_{i \in N} \mathcal{R}_i$. For each $R \in \mathcal{R}$, let $p(R) \equiv (p(R_i))_{i \in N} \in [0, \Omega]^N$ be its associated peak profile. For each $R \in \mathcal{R}$, $i \in N$ and $S \subset N$, we use the following standard notations: $R_{-i} \equiv (R_j)_{j \in N \setminus \{i\}}$, $R_S \equiv (R_i)_{i \in S}$ and $R_{-S} \equiv (R_j)_{j \notin S}$. We also use $\mathcal{R}_S$ for $\prod_{i \in S} \mathcal{R}_i$.

[5] In the Sprumont model, Sakai and Wakayama (2011) show that under a set of properties—notably, envy-freeness—many rules can be lattice ordered via a relation of dominance with the uniform rule as its supremum. While the question studied in their paper is obviously different from ours, it is noteworthy to see the ranking of rules obtained there.

2.1. Pareto comparison and lattice

For a given preference profile $R \in \mathcal{R}$, we define a binary relation $\succeq_R$ on $[0, \Omega]^n$ so that for any $x, y \in [0, \Omega]^n$, $x \succeq_R y$ if and only if $x_i \; R_i \; y_i$ for each $i \in N$. Clearly, $\succeq_R$ is reflexive and transitive. Thus, $\succeq_R$ is a preorder on $[0, \Omega]^n$. On the other hand, $\succeq_R$ is not antisymmetric on $[0, \Omega]^n$, i.e., there could be some $x, y \in [0, \Omega]^n$ such that $x \neq y$ and $x \succeq_R y \succeq_R x$. Antisymmetric preorders are called partial orders. Thus, $\succeq_R$ is not a partial order on $[0, \Omega]^n$. We denote the asymmetric part of $\succeq_R$ by $\succ_R$ which is known as the Pareto dominance. Specifically, $x \succ_R y$ if and only if (i) $x \succeq_R y$ and (ii) there exists $i \in N$ such that $x_i \; P_i \; y_i$.

Fix a subset $Y \subseteq [0, \Omega]^n$. We say that a pair $(Y, \succeq_R)$ is a partially ordered set if $\succeq_R$ is a partial order on Y. If $\succeq_R$ is complete and antisymmetric on Y, then $(Y, \succeq_R)$ is a totally ordered set. Let us pick a partially ordered set $(Y, \succeq_R)$ where $Y \subseteq [0, \Omega]^n$. For any subset $Z \subseteq Y$, $y \in Y$ is an upper (lower) bound of Z if $y \succeq_R x$ ($x \succeq_R y$) for all $x \in Z$. Furthermore, $z \in Y$ is called the meet of Z if for all lower bound y of Z, $z \succeq_R y$. We reserve the notation $\bigwedge Z$ for the meet of Z. On the other hand, $z \in Y$ is the join of Z if for all upper bound y of Z, $y \succeq_R z$. The join of Z is denoted by $\bigvee Z$.

Definition 2.1 *(Lattice).* A partially ordered set $(Y, \succeq_R)$ where $Y \subseteq [0, \Omega]^n$ is a lattice if any set $Z \subset Y$ with $|Z| = 2$ has both the meet and join in Y. Furthermore, a lattice $(Y, \succeq_R)$ is complete if every subset $Z \subseteq Y$ has both $\bigwedge Z$ and $\bigvee Z$ in Y.

It is well known that any complete lattice has both a well-defined supremum and infimum, denoted by x^{sup} and x^{inf}, such that for any $x \in Y$, $\bigvee\{x, x^{\text{sup}}\} = x^{\text{sup}}$ and $\bigwedge\{x, x^{\text{inf}}\} = x^{\text{inf}}$.

For any two elements $x, y \in [0, \Omega]^n$, we construct $x \wedge y$ and $x \vee y$ in $[0, \Omega]^N$ as follows: for each $i \in N$,

$$[x \wedge y]_i = \begin{cases} x_i & \text{if } y_i \; R_i \; x_i \\ y_i & \text{otherwise,} \end{cases}$$

and

$$[x \vee y]_i = \begin{cases} x_i & \text{if } x_i \; R_i \; y_i \\ y_i & \text{otherwise.} \end{cases}$$

Observe here that it is possible that $x \wedge y \neq y \wedge x$ or $x \vee y \neq y \vee x$ for some $x, y \in [0, \Omega]^n$.

2.2. Rules and their properties

A *rule* is a function $f : \mathcal{R} \to X$ which maps each preference profile $R \in \mathcal{R}$ to an allocation $f(R) \in X$. For each $R \in \mathcal{R}$ and each $i \in N$, $f_i(R)$ stands for the resource allocated to agent i at preference profile R.

We now introduce three properties of rules frequently encountered in the literature. The first one is the well-known *Pareto efficiency* condition.

Pareto efficiency: Rule f satisfies efficiency if there exists no $R \in \mathcal{R}$ and $x \in X$ such that $x \succ_R f(R)$.

It is well-known that in the single-peaked preference domain, efficiency of f is equivalent to the following condition:

Same-sidedness: Rule f satisfies same-sidedness if $\sum_{i \in N} p(R_i) \geq \Omega$ implies $p(R_i) \geq f_i(R)$ for each $i \in N$, and $\sum_{i \in N} p(R_i) \leq \Omega$ implies $p(R_i) \leq f_i(R)$ for each $i \in N$.

The next property deals with the immunity to strategic manipulations, a requirement that is central in mechanism design. *Strategy-proofness* guarantees that agents will have no incentive to misreport their preferences.[6] A rule is said to be *manipulable* if it violates strategy-proofness.

Strategy-proofness: Rule f satisfies strategy-proofness if for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$.

The last property is *replacement monotonicity* studied in Barberà et al. (1997). It states that the allotment of a "deviator" and the other agents move in opposite directions. If agent i, the "deviator," can walk away with a bigger (resp., smaller) share of the pie, then what is left for the remaining agents has shrunk (resp., increased) compared to the original allocation. Replacement monotonicity then requires that the remaining agents be all affected in the same direction by the change in the resources available to them.

Replacement monotonicity: Rule f satisfies replacement monotonicity if for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $f_i(R'_i, R_{-i}) \geq f_i(R)$ implies that $f_j(R'_i, R_{-i}) \leq f_j(R)$ for each $j \neq i$.

Notice that replacement monotonicity does not imply that f satisfies any form of symmetry among agents such as anonymity or equal treatment of equals, two properties that have played a central role in characterizations of the uniform rule.[7][8]

In this paper, we focus on the rules that simultaneously satisfy efficiency, strategy-proofness and replacement monotonicity. These are the so-called *sequential allotment rules* characterized in Barberà et al. (1997). These rules form a rich family with, at its core, the notion of reference shares (or initial guaranteed levels). Rules in this family allow for an iterative process that is flexible enough to accommodate changes of the reference shares through the iterative procedure. In short, sequential allotment rules allow for (i) any starting reference shares, (ii) different starting reference shares for the excess demand and supply cases, and (iii) at each step of the iteration, reference shares that depend on the size of the resource that is left.

Before giving a general definition for the family of sequential allotment rules, let us introduce some necessary definitions. First let $x^L, x^H \in X$ be the (initial) minimum guaranteed levels of agents in the case of excess supply and excess demand, respectively. Fix a preference profile $R \in \mathcal{R}$. Let us define a *sequential adjustment function* $g : X \times \mathcal{R} \to X \times \mathcal{R}$. We use the notation g^t to indicate the composition of g with itself t times, with the requirement that $g^0(x, R) = (x, R)$. We use the convention that if $g^t(x, R) = (\hat{x}, R)$, for some x and R, then $g^t_1(x, R) \equiv \hat{x}$. Function

[6] If a rule is strategy-proof, then truthtelling is a (weakly) dominant strategy in the direct revelation mechanism associated to rule f.

[7] Note also that replacement monotonicity can be derived from other well-known properties. For instance, consistency and resource monotonicity, which play a prominent role in the literature, imply replacement monotonicity.

[8] Our definition of replacement monotonicity is the one used in Barberà et al. (1997). It makes reference to monotonicity in the variation of the allocation itself following a unilateral change in preferences by agent i. It is also possible to define a similar notion that makes reference to monotonicity in the variation of welfare following a unilateral change in preferences by agent i. Thomson (1997) defines this as *(one-sided) welfare domination under preference replacement*. We provide a discussion of the welfare version of replacement monotonicity in Section 4.2.

g is a sequential adjustment function with respect to the minimum guaranteed levels (x^L, x^H) if the following items are satisfied for any (x^t, R) such that $(x^t, R) = g(x^{t-1}, R) = g^t(\bar{x}, R)$ where $\bar{x} \in \{x^L, x^H\}$ for some $1 \le t \le n$:

(i) $x_i^t = p(R_i)$ if $(\Omega - \sum_{j \in N} p(R_j))(x_i^{t-1} - p(R_i)) \le 0$.
(ii) $(x_i^t - x_i^{t-1})(\Omega - \sum_{j \in N} p(R_j)) \le 0$ if $(\Omega - \sum_{j \in N} p(R_j))(x_i^{t-1} - p(R_i)) > 0$.
(iii) If $p(\tilde{R}_i) \ge p(R_i) > x_i^{t-1}$ and $\sum_{j \in N} p(R_j) \ge \Omega$, or if $p(\tilde{R}_i) \le p(R_i) < x_i^{t-1}$ and $\sum_{j \in N} p(R_j) < \Omega$ then $g(x^{t-1}, R) = g(x^{t-1}, (\tilde{R}_i, R_{-i}))$.
(iv) Let $\tilde{R}_i$ and $\hat{x}^n$ be such that $(\hat{x}^n, (\tilde{R}_i, R_{-i})) = g^n(\bar{x}, (\tilde{R}_i, R_{-i}))$. Let $(x^n, R) = g^n(\bar{x}, R)$. Then

$$\text{if } p(\tilde{R}_i) \ge p(R_i) \text{ and } \sum_{j \in N} p(R_j) \ge \Omega, \text{ then } x_k^n \ge \hat{x}_k^n \text{ for } k \ne i$$

$$\text{if } p(\tilde{R}_i) \le p(R_i) \text{ and } \sum_{j \in N} p(R_j) < \Omega, \text{ then } x_k^n \le \hat{x}_k^n \text{ for } k \ne i$$

How does one approach items (i) to (iv)? Consider for instance the over-demand case. Item (i) says that if at any stage of the iterative procedure, agent i's peak is no higher than his guaranteed share, then agent i's guaranteed level adjusts to his peak. Item (ii) says that for the agents j whose peaks are above their guaranteed shares, the latter cannot adjust downward Item (iii) says that if agent i experiences a monotonic change in his peak, when going from R_i to $\tilde{R}_i$, then the adjustment will remain unaffected Finally, item (iv) concludes by saying that if agent i experiences a monotonic change in his peak, when going from R_i to $\tilde{R}_i$, then the other agents j's actual guaranteed shares cannot decrease.

We are now ready to formally define the family of sequential allotment rules.

Sequential allotment rules: Rule f is a sequential allotment rule if there exist a pair of initial guaranteed levels $(x^L, x^H) \in X \times X$ and a sequential adjustment function $g : X \times \mathcal{R} \to X \times \mathcal{R}$ relative to x^L and x^H such that, for each $R \in \mathcal{R}$,

$$f(R) = \begin{cases} g_1^n(x^H, R) & \text{if } \sum_{j \in N} p(R_i) \ge \Omega \\ g_1^n(x^L, R) & \text{if } \sum_{j \in N} p(R_i) < \Omega \end{cases}$$

It is clear from the definition that the sequential allotment rules form a very rich class since (i) the initial guaranteed levels can differ across the two different regimes, (ii) the guaranteed levels evolve through the iterative allocation procedure implicit in the definition of the sequential allotment rules. For this reason, our main result may come as a surprise: having a common thread across the Nash equilibrium allocations for such a large family of rules is improbable. A detailed introduction to the family of sequential allotment rules can be found in the online supplement of this paper.[9] We provide below the definitions of two specific sequential allotment rules that are used extensively in the paper.

Priority rule: Rule f is a priority rule if there exists a priority ordering $\triangleright$ on N such that for each $R \in \mathcal{R}$ and each $i, j \in N$ with $i \triangleright j$, either $[f_i(R) = p(R_i)]$ or $[f_i(R) < p(R_i)$ and $f_j(R) = 0]$.

[9] The online appendix can be found at https://sites.google.com/site/obochet2/Onlineappendixlattice.pdf.

Weighted uniform rule: Rule f is a weighted uniform rule with respect to the vector $\omega^L, \omega^H \in \{\delta \in \mathbb{R}^n_+ : \delta_i > 0\ \forall i \in N;\ \sum_{i \in N} \delta_i = 1\}$ if for each $R \in \mathcal{R}$ and each $i \in N$,

$$f_i(R) = \begin{cases} \min\{p(R_i), \omega_i^H \lambda\} & \text{if } \sum_{i \in N} p(R_i) \geq \Omega \\ \max\{p(R_i), \omega_i^L \lambda\} & \text{if } \sum_{i \in N} p(R_i) < \Omega \end{cases}$$

where λ solves $\sum_{i \in N} f_i(R) = \Omega$.

An example of a weighted uniform rule is the uniform rule characterized by Sprumont (1991). The latter is obtained by setting $\omega_i^H = \omega_i^L = \frac{1}{n}$ for each $i \in N$.

We now discuss some implications of efficiency, strategy-proofness and replacement monotonicity that are used extensively in some of our proofs.

Non-bossiness: Rule f satisfies non-bossiness if for each $R \in \mathcal{R}$, each $i \in N$, and each $R_i' \in \mathcal{R}_i$, $f_i(R_i', R_{-i}) = f_i(R)$ implies $f(R_i', R_{-i}) = f(R)$.

Observe that replacement monotonicity implies non-bossiness. Indeed, pick $R \in \mathcal{R}$, $i \in N$ and $R_i' \in \mathcal{R}_i$. If $f_i(R_i', R_{-i}) = f_i(R)$, replacement monotonicity and feasibility together imply that $f_j(R_i', R_{-i}) = f_j(R)$ for all $j \neq i$. Hence $f(R_i', R_{-i}) = f(R)$ and non-bossiness holds.

Peak-onliness: Rule f satisfies peak-onliness if for each $R, \tilde{R} \in \mathcal{R}$, $p(R_i) = p(\tilde{R}_i)$ for each $i \in N$ implies that $f(R) = f(\tilde{R})$.

Observe that under strategy-proofness, efficiency and non-bossiness (implied by replacement monotonicity), rule f automatically satisfies peak-onliness. To see this, pick $R \in \mathcal{R}$, $i \in N$ and $R_i' \in \mathcal{R}_i$ such that $p(R_i') = p(R_i) = \bar{p}$. Suppose that $f_i(R_i', R_{-i}) \neq f_i(R)$. The cases where either $f_i(R_i', R_{-i}), f_i(R) < \bar{p}$ or $\bar{p} < f_i(R_i', R_{-i}), f_i(R)$ lead to a contradiction of strategy-proofness. Assume that $f_i(R_i', R_{-i}) \leq \bar{p} < f_i(R)$. Let $\tilde{R}_i \in \mathcal{R}_i$ be such that $p(\tilde{R}_i) = \bar{p}$ and $f_i(R_i', R_{-i})\ \tilde{I}_i\ f_i(R)$. By strategy-proofness either $f_i(\tilde{R}_i, R_{-i}) = f_i(R_i', R_{-i})$ or $f_i(\tilde{R}_i, R_{-i}) = f_i(R)$. In both cases, strategy-proofness is violated. Hence $f_i(R_i', R_{-i}) = f_i(R)$. By non-bossiness $f(R_i', R_{-i}) = f(R)$ and hence peak-onliness holds.

2.3. Nash equilibria of direct revelation games

The direct revelation mechanism associated with a given rule f is denoted $\Gamma = (\mathcal{R}, f)$. In such a mechanism, agents simply report a preference relation to the planner and rule f is used as outcome function. Given a preference profile $R \in \mathcal{R}$, (Γ, R) is a direct revelation game. Since all the rules studied in this paper satisfy peak-onliness, one could look at simpler direct revelation mechanisms in which agents report their peaks. For convenience, we however stay with the standard definition. From now on, we refer to f both as a rule and as a direct revelation mechanism.

Nash equilibrium: Pick rule f. Profile $\tilde{R} \in \mathcal{R}$ is a Nash equilibrium at profile $R \in \mathcal{R}$ if for each $i \in N$, $f_i(\tilde{R})\ R_i\ f_i(R_i', \tilde{R}_{-i})$ for each $R_i' \in \mathcal{R}_i$. For each $R \in \mathcal{R}$, let $NE(f, R)$ be the set of Nash equilibria of the direct revelation mechanism associated to f.

Henceforth, since we fix rule f we will simply write $NE(R)$ in place of $NE(f, R)$. For each $R \in \mathcal{R}$, let $X^{NE(R)} = \{f(\tilde{R}) : \tilde{R} \in NE(R)\} \subseteq X$ be the set of Nash equilibrium allocations at profile R. We will pair the set of Nash equilibrium allocations at profile R with the preorder $\succeq_R$ and study their properties. Our object of study is therefore $(X^{NE(R)}, \succeq_R)$ and our main result covers all the rules satisfying efficiency, strategy-proofness and replacement monotonicity.

3. Lattice structure of Nash equilibrium allocations

We first present an example which highlights some of the important features of our analysis. The example also provides some intuition for the general result that will follow.[10]

Example 3.1 *(A sequential allotment rule: meet and join of Nash equilibrium allocations).* Let $n = 4$, $\Omega = 10$ and let rule f be determined as follows. The agents are divided into two groups with the first two agents in group 1 and the last two in group 2. The rule first determines each group's allocation by initially guaranteeing 6 and 4 units to group 1 and 2 respectively (both for excess demand and supply). If both groups' demand fall on the same side of their guaranteed levels, then each group gets its initial guaranteed level and the final within-group allocation is determined, for each group, according the uniform rule. On the other hand, if the groups' demands fall on different sides of their guaranteed levels, whichever group demand is closest to its sum of peaks gets that amount while the other gets the remaining resource. For instance, if the sum of peaks is 7 for group 1 and 2 for group 2, then group 1 gets 7 while group 2 gets 3.[11] Once the allocation for the groups is determined, each group splits its allocation among its members according to the uniform rule. Let $R \in \mathcal{R}$ be such that $p(R) = (5, 1, 0, 4)$. Consider reports $\bar{R}, \tilde{R} \in \mathcal{R}$ where $p(\bar{R}) = (4, 2, 2, 2)$ and $p(\tilde{R}) = (3, 3, 1, 3)$. Note that $\bar{R}, \tilde{R} \in NE(R)$. However, observe that $f(\bar{R}) = (4, 2, 2, 2)$ is not Pareto comparable to $f(\tilde{R}) = (3, 3, 1, 3)$. At the same time, $f(\bar{R}) \wedge f(\tilde{R}) = (3, 3, 2, 2)$ and $f(\bar{R}) \vee f(\tilde{R}) = (4, 2, 1, 3)$ are not only feasible allocations but they can also be supported as Nash equilibrium allocations. Specifically, a report $\hat{R} \in \mathcal{R}$ with $p(\hat{R}) = (3, 3, 2, 1)$ gives $f(\hat{R}) = (3, 3, 2, 2)$ and is a Nash equilibrium. Finally, $\breve{R} \in \mathcal{R}$ with $p(\breve{R}) = (4, 2, 1, 3)$ gives $f(\breve{R}) = (4, 2, 1, 3)$ and is also a Nash equilibrium. In fact, the meet and join of any two Nash equilibrium allocations are also Nash equilibrium allocations, i.e., the set of Nash equilibrium allocations is a lattice, ordered by the Pareto dominance relation $\succ_R$. Importantly, if $x \in X^{NE(R)}$ then any report profile $\tilde{R} \in \mathcal{R}$ whose peak $p(\tilde{R})$ is x is itself a Nash equilibrium. $\diamond$

Before going to our main findings, we start with a result that is of independent interest: following the conclusion of Example 3.1, we show that any Nash equilibrium allocation x can be obtained with an alternative Nash equilibrium preference report with peaks at x_i for each $i \in N$.

Proposition 3.2. *Let rule f satisfy efficiency, strategy-proofness and replacement monotonicity. Let $x \in X^{NE(R)}$ for some $R \in \mathcal{R}$. Then any $\hat{R} \in \mathcal{R}$ with $p(\hat{R}) = x$ is a Nash equilibrium at profile R.*

Proof. See Appendix. $\square$

[10] The interested reader can find a more detailed description of the rule presented in Example 3.1, in the online appendix of the paper. See Example 1.4 on page 4.

[11] Notice that failing to sequentially adjust the groups guaranteed levels in this way would imply a violation of efficiency. For instance if the peaks of group 1 are (5,5), while the peaks of group 2 are (0,0), one cannot just apply the uniform rule within the groups right away. Indeed the resulting final allocation would be (3,3,2,2,), and this allocation is not efficient.

Let us provide a brief sketch of the proof. Suppose $\bar{R} \in NE(R)$ is a preference profile leading to x, i.e., $f(\bar{R}) = x$. If $\sum_i p(\bar{R}_i) = \Omega$ then efficiency and peak-onliness give the desired result. Assume without loss of generality that $\sum_i p(\bar{R}_i) > \Omega$. Consider $\hat{R}$ such that $p(\hat{R}) = f(\bar{R})$. By efficiency (and since $\sum_i f_i(\bar{R}) = \sum_i p(\hat{R}_i)$), we have that $f(\hat{R}) = p(\hat{R}) = f(\bar{R})$. We next show that $f(R_i, \bar{R}_{-i}) = f(\bar{R})$. Because $\bar{R} \in NE(R)$ and f is strategy-proof, $f(R_i, \bar{R}_{-i})\ I_i\ f(\bar{R})$ for all $i \in N$. If $f_i(\bar{R}) \leq p(R_i)$ then single-peakedness, strategy-proofness and peak-onliness give that $f_i(\bar{R}) = f(R_i, \bar{R}_{-i})$. On the other hand, if $p(R_i) < f_i(\bar{R})$ then, by efficiency, agent i should be able to decrease her allocation from $f_i(\bar{R})$, unless every $j \neq i$ has a peak at $f_j(\bar{R})$. Of course, agent i being able to decrease her allocation contradicts that f is strategy-proof. Thus, in both cases we have $f_i(R_i, \bar{R}_{-i}) = f_i(\bar{R})$ which, along with non-bossiness, gives $f(R_i, \bar{R}_{-i}) = f(\bar{R})$. We then show that agent i has no profitable deviation from $\hat{R}$. Suppose some $j \neq i$ switches her report from $\bar{R}_j$ to $\hat{R}_j$. Because $p(\hat{R}_j) = f_j(\bar{R}) = f_j(R_i, \bar{R}_{-i})$ strategy-proofness gives that $f_j(R_i, \hat{R}_j, \bar{R}_{-ij}) = f_j(R_i, \bar{R}_{-i}) = f_j(\bar{R})$. Again non-bossiness gives that $f(R_i, \hat{R}_j, \bar{R}_{-ij}) = f(\bar{R})$. Using the same argument sequentially for each agent $j \neq i$, we obtain that $f(R_i, \hat{R}_{-i}) = f(\bar{R})$. Since $f(\hat{R}) = f(\bar{R})$ and f is strategy-proof, agent i has no profitable deviation from $\hat{R}$. Since agent i is selected arbitrarily, $\hat{R}$ must be a Nash equilibrium.

We are now ready to state our main result showing that (i) the set of Nash equilibrium allocations is a complete lattice with respect to the order of Pareto dominance, and (ii) the supremum of the lattice is the allocation obtained under truthtelling. Following this, we will provide some additional results: we identify the infimum of the lattice in some special cases, and we show a much stronger result for the class of weighted uniform rules.

Theorem 3.3. *Let rule f satisfy efficiency, strategy-proofness and replacement monotonicity. For each $R \in \mathcal{R}$, $(X^{NE(R)}, \succeq_R)$ is a complete lattice whose supremum element is $f(R)$.*

Proof. See Appendix. □

Observe that Theorem 3.3 identifies the allocation obtained under truthtelling as the supremum of the lattice but it is silent on the infimum. Identifying the latter typically depends on specifics of the rule under consideration. Indeed, the initial guaranteed levels x^L and x^H play an important role in determining the infimum of the lattice of the Nash equilibrium allocations. For instance Example 1.2 in the online appendix depicts the structure of the set of Nash equilibrium allocations for the uniform rule and shows how the equal division allocation is a Nash equilibrium and is always Pareto dominated by all other Nash equilibrium allocations. The characterization in Example 1.2 hinges upon the equality between initial guaranteed levels: when $x^L = x^H$, the infimum of the lattice is always the allocation formed by the initial guaranteed levels.

Proposition 3.4. *Let f be a sequential allotment rule with $x^L = x^H = x^*$. For each $R \in \mathcal{R}$, $x^* \in X^{NE(R)}$. Moreover, for each $x \in X^{NE(R)}$ with $x \neq x^*$, x Pareto dominates x^* i.e. $x \succ_R x^*$.*

Proof. See Appendix □

When considering a set of rules for which $x^L \neq x^H$, the infimum of the lattice formed by the set of Nash equilibrium allocations will be specific to each of the rules f considered. Indeed, recall that the sequential allotment rules allow to have (i) any initial guaranteed levels, (ii) different guaranteed levels for the excess demand and supply cases, and (iii) at each step of the iteration, guaranteed levels that depend on the size of the resource that is left. Note that (iii) implies that the way the initial guaranteed levels evolve may also be rule specific. For instance, for a given f, x^H changes as a function of the remaining resource left after some agents are served. Also x^H and x^L may actually evolve differently as a function of the resources left once some agents have left the problem. As such, different preference profiles may lead to a different evolution of initial guaranteed levels. In the simpler case where the evolution of initial guaranteed levels is uniform, i.e. the case of the asymmetric weighted uniform rules described in the appendix, the initial guaranteed levels may not even be in the set of Nash equilibrium allocations for some preference profiles, as the following example shows.

Example 3.5 *(Different initial guaranteed levels).* Let $n = 3, \Omega = 6$, and $R \in \mathcal{R}$ be such that $p(R) = (1, 2, 4)$. Let $x^H = (3, 1.5, 1.5)$, $x^L = (2.9, 1.6, 1.5)$, and let f be the weighted uniform rule associated with x^L and x^H.

Case 1: Consider $\tilde{R}$ with $\Omega \leq \sum_{i \in N} p(\tilde{R}_i)$ and assume that $\tilde{R} \in NE(R)$ with $f(\tilde{R}) = (3, 1.5, 1.5) = x^H$. By Proposition 3.2, $\hat{R}$ with $p(\hat{R}) = x^H$ is a Nash equilibrium at R. Notice that at the report $(R_1, \hat{R}_{-1})$, $f_1(R_1, \hat{R}_{-1}) < 3$. Indeed, following the deviation, the initial guaranteed levels that are considered are x^L since $p(R_1) + \sum_{i \neq 1} p(\hat{R}_i) < \Omega$. Agent 1 strictly benefits from such a deviation, a contradiction with $\hat{R} \in NE(R)$.

Case 2: Consider $\tilde{R}$ with $\sum_{i \in N} p(\tilde{R}_i) < \Omega$ and assume that $\tilde{R} \in NE(R)$ with $f(\tilde{R}) = (2.9, 1.6, 1.5) = x^L$. By Proposition 3.2, $\hat{R}$ with $p(\hat{R}) = x^L$ is a Nash equilibrium at R. Notice that at the report $(R_3, \hat{R}_{-3})$, $f_3(R_3, \hat{R}_{-3}) > 1.5$. Indeed, following the deviation, the initial guaranteed levels that are considered are x^H since $p(R_3) + \sum_{i \neq 3} p(\hat{R}_i) > \Omega$. Agent 3 strictly benefits from such a deviation, a contradiction with $\hat{R} \in NE(R)$. ◇

Example 3.5 shows that whenever $x^L \neq x^H$, the initial guaranteed levels may no longer be part of the set of Nash equilibrium allocations for some preference profile R.[12] A full characterization of the lower bound of the lattice seems therefore out of reach.

We conclude with a result regarding the subclass of weighted uniform rules: the lattice formed by the Nash equilibrium allocations is in fact ranked, i.e. there are no Nash equilibrium allocations that end up not being Pareto comparable.

Proposition 3.6. *Let f be a weighted uniform rule. Then $(X^{NE(R)}, \succeq_R)$ is a totally ordered set, i.e. the Nash equilibrium allocations are all Pareto comparable.*

Proof. See Appendix. □

We now make a couple of observations. One may wonder if the set of peaks corresponding to Nash equilibria is a complete lattice, or if the set of Nash equilibrium allocations satisfies convexity. The answers to these questions turn out to be both negative as shown below.

---

[12] Of course the initial guaranteed levels may still be Nash equilibrium allocations. For instance pick $p(R) = (1, 1.4, 4)$ and let $x^H = (3, 1.6, 1.4)$, $x^L = (3, 1.4, 1.6)$. Then $\tilde{R}$ such that $\tilde{p} = (3, 1.4, 1.6)$ gives $f(\tilde{p}) = (3, 1.4, 1.6) = x^L$. One can check that $\tilde{R} \in NE(R)$. Note that x^H cannot be supported as a Nash equilibrium allocation at profile R.

Example 3.7 *(The set of peaks of Nash equilibria is not a lattice).* Let $n = 3$, $\Omega = 6$ and let f be the uniform rule. Pick $R \in \mathcal{R}$ such that $p(R) = (2, 2, 2)$. Consider in addition two additional profiles $\bar{R}, \tilde{R} \in \mathcal{R}$ with $p(\bar{R}) = (2, 2, 3)$ and $p(\tilde{R}) = (1, 2, 2)$. Observe that $f(\bar{R}) = f(\tilde{R}) = (2, 2, 2)$. Thus, $\bar{R}, \tilde{R} \in NE(R)$. However, $f(\tilde{R}_1, \bar{R}_{-1}) = (1, 2, 3)$. Clearly, both agents 1 and 3 have some profitable unilateral deviations from $(\tilde{R}_1, \bar{R}_{-1})$. Thus, $(\tilde{R}_1, \bar{R}_{-1}) \notin NE(R)$. $\diamond$

Example 3.8 *(Non-convexity of the set of Nash equilibrium allocations).* Let $n = 4$, $\Omega = 8$ and let f be the uniform rule. Pick $R \in \mathcal{R}$ such that $p(R) = (0, 1, 3, 4)$. Regardless of the choice of the sequential allotment rule f, we know that $f(R) = (0, 1, 3, 4)$ and $R \in NE(R)$. Let $\bar{R} \in \mathcal{R}$ be such that $p(\bar{R}) = (2, 2, 2, 2)$. Then $f(\bar{R}) = (2, 2, 2, 2)$ and $\bar{R} \in NE(R)$. However, the linear combination of these two allocations with equal weights is not a Nash equilibrium allocation. Indeed, allocation $(1, 1.5, 2.5, 3)$ occurs only if the agents report $\tilde{R}$ with $p(\tilde{R}) = (1, 1.5, 2.5, 3)$. However, by reverting to truthtelling agent 3 would get 2.75 which is strictly preferred to getting 2.5. $\diamond$

4. Discussion: robustness checks and extensions

4.1. On the role of replacement monotonicity

We discuss the role that replacement monotonicity plays in Theorems A.7 and 3.3.[13] We investigate this issue by means of two examples.

Example 4.1 *(Replacement monotonicity is key).* Let $n = 3$, $\Omega = 6$, and let f be described as follows. For any profile $R \in \mathcal{R}$, if $p(R_1), p(R_2) < \frac{\Omega}{n}$, then f is a priority rule with respect to ordering $1 \rhd 2 \rhd 3$. However, if at least one of the first two agents has preferences with peak at 2 or above, then f is the uniform rule.

Rule f is efficient and non-bossy: Rule f is either a priority rule or the uniform rule depending on the first two agents' peaks. Both rules are efficient implying that so is f.

Since $\Omega = 6$, observe that whenever both agents 1 and 2 report their peaks strictly below 2, each obtains his reported peak. However, whenever one of them reports a peak at 2 or above, she obtains at least 2 units of resource. Thus, no agent can force f to switch from or to the priority rule without changing her own allocation. In addition, both the priority rule and the uniform rule satisfy non-bossiness, thus f also satisfies non-bossiness.

Rule f is strategy-proof: For agent 3 this is obvious because his reported preferences alone cannot force f to switch to or from the priority rule (the same goes for the uniform rule). Given that both rules are strategy-proof, truthtelling is a dominant strategy for agent 3. Next, pick any $R \in \mathcal{R}$ and consider agent 1.

Case 1: Suppose $p(R_2) \geq 2$. Then f is the uniform rule regardless of agent 1's report. Given that the uniform rule is strategy-proof, truthtelling is a dominant strategy.

[13] In the appendix, the proof of Theorem 3.3 is split into two parts. In the first part, which we label Theorem A.7, we show that $(X^{NE(R)}, \succeq_R)$ is a lattice. In the second part, which we label Theorem 3.3, we establish the completeness of the lattice and the truthtelling allocation as the supremum of the lattice.

Case 2: Suppose $p(R_2) < 2$. If $p(R_1) < 2$, then agent 1 has no incentive to misreport her preferences because $f_1(R) = p(R_1)$. If $p(R_1) \geq 2$, then $f(R)$ is the uniform allocation at profile R. For any of her report $\tilde{R}_1$ with $p(\tilde{R}_1) \geq 2$, $f(\tilde{R}_1, R_{-1})$ is also the uniform allocation, and the uniform rule is strategy-proof. Thus, $\tilde{R}_1$ with $p(\tilde{R}_1) \geq 2$ cannot be a profitable deviation. Finally, consider $\tilde{R}_1$ with $p(\tilde{R}_1) < 2$. Now f is a priority rule. Thus, $f_1(\tilde{R}_1, R_{-1}) = p(\tilde{R}_1) < 2 \leq p(R_1)$. By reporting her peak truthfully, agent 1 is allocated at least 2 units but never more than $p(R_1)$. Thus, agent 1 has no incentive to lie. By the same reasoning, agent 2 has no incentive to lie either.

Rule f does not satisfy replacement monotonicity: Let $R \in \mathcal{R}$ such that $p(R) = (1, 2, 2)$. Because $p(R_2) \geq 2$, f allocates according to the uniform rule. Thus, $f(R) = (2, 2, 2)$. However, if agent 2 deviates to $\tilde{R}_2$ with $p(\tilde{R}_2) = 1$ then $f(\tilde{R}_2, R_{-2}) = (1, 1, 4)$ since f follows the priority rule. Following the change in the report of agent 2, it is clear that the allotments of agents 1 and 3 move in different directions.

Truthtelling does not Pareto dominate all the Nash equilibrium allocations: Consider a profile $R \in \mathcal{R}$ such that $p(R) = (1, 1, 2)$. In this case, $f(R) = (1, 1, 4)$. Consider $\tilde{R} \in \mathcal{R}$ such that $p(\tilde{R}) = (2, 2, 2)$. It is easy to see that $f(\tilde{R}) = (2, 2, 2)$ and $\tilde{R} \in NE(R)$. Clearly, the allocation under truthtelling does not Pareto dominate $f(\tilde{R})$.

The set of Nash equilibrium allocations is not a lattice ordered by the Pareto relation: For this, notice that $f(\tilde{R}) \vee f(R) = (1, 1, 2)$ and $f(\tilde{R}) \wedge f(R) = (2, 2, 4)$ are not even feasible allocations, so replacement monotonicity seems essential to guarantee the feasibility of the meet and join of Nash equilibrium allocations. $\diamond$

Remark 4.2. There are two important conclusions to draw from Example 4.1. The first one is that replacement monotonicity cannot be replaced by non-bossiness in the statement of Theorem 3.3. Next, Bochet and Tumennasan (2017) show that, under efficiency, the combination of non-bossiness and strategy-proofness is equivalent to *group strategy-proofness*.[14,15] Thus, the rule described in Example 4.1 is group strategy-proof —one can easily prove this directly. An important conclusion follows: efficiency and group strategy-proofness together do not necessarily lead to the lattice structure of the set of Nash equilibrium allocations.[16]

[14] Rule f satisfies group strategy-proofness if for each $R \in \mathcal{R}$ there does not exist $S \subseteq N$ and $\tilde{R}_S$ such that $f_i(\tilde{R}_S, R_{-S})\ R_i\ f_i(R)$ for all $i \in S$, and with $f_j(\tilde{R}_S, R_{-S})\ P_j\ f_j(R)$ for at one least $j \in S$.

[15] Bochet and Tumennasan (2017) study a new property called group resilience. They show that (i) group resilience is equivalent to strategy-proofness and non-bossiness in welfare, (ii) in our setting, under the requirement of efficiency, group-resilience is equivalent to group strategy-proofness, and (iii) non-bossiness and strategy-proofness together imply non-bossiness in welfare. By combining (i), (ii) and (iii), Bochet and Tumennasan (2017) obtain that non-bossiness and strategy-proofness together is equivalent to group strategy-proofness. We refer the reader to Bochet and Tumennasan (2017) for more details.

[16] Barberà et al. (2016) show that under some richness of the preference domain, if a rule satisfies some monotonicity condition and a weakening of non-bossiness in welfare, then strategy-proofness is equivalent to (weak) group strategy-proofness. The single-peaked domain is rich and the sequential allotment rules satisfy the joint monotonicity and respectfulness requirements identified in their paper. Strategy-proofness could therefore be replaced by (weak) group strategy-proofness in the statement of Theorem 3.3. Example 4.1 however shows that one cannot prove the Theorem using only efficiency and (weak) group strategy-proofness. Indeed a strengthening of non-bossiness is needed, in addition to the former two conditions.

One may now wonder if replacement monotonicity is necessary for the lattice result to hold for any efficient and strategy-proof rule. Our second example shows that the answer to this question is negative.

Example 4.3 *(A mixture of a group strategy-proof priority and uniform rule that violates replacement monotonicity but preserve the lattice structure).* Consider the following rule f. For any $R \in \mathcal{R}$, agent 1 always gets his peak $p(R_1)$. If $p(R_1) \leq 2$ then the remaining agents split $\Omega - p(R_1)$ according to the uniform rule. If $p(R_1) > 2$, then the remaining agents split $\Omega - p(R_1)$ using the fixed priority ordering $\triangleright$ over the set of agents $N \setminus \{1\}$. It is clear that f does not satisfy replacement monotonicity since an increase in the peak of agent 1 may generate a non-monotonic change in the allocation of agents other than her. It is also obvious that f satisfies group strategy-proofness. Finally, given what we already know about the uniform rule, it is clear that the set of Nash equilibrium is a lattice and truthtelling Pareto dominates all other Nash equilibrium allocations. $\diamond$

Under efficiency and strategy-proofness, Example 4.3 shows that replacement monotonicity is not necessary for the lattice result. Ideally, one would want to pin down a condition that is both necessary and sufficient. Unfortunately this remains an open question at this stage. We however demonstrate below that for any efficient and strategy-proof rule, non-bossiness is a necessary condition for the lattice result to hold. In conclusion, a necessary and sufficient condition lies between replacement monotonicity and non-bossiness. Recall that the rule constructed in Example 4.1 satisfies non-bossiness yet its set of Nash equilibrium allocations is not a lattice. If replacement monotonicity is to be replaced with another condition in Theorem 3.3, it should therefore be a more demanding requirement than non-bossiness.

Proposition 4.4. *Let rule f satisfy efficiency and strategy-proofness. Assume that for each $R \in \mathcal{R}$, $(X^{NE(R)}, \succeq_R)$ is a complete lattice whose supremum element is $f(R)$. Then f satisfies non-bossiness.*

Proof. See Appendix. $\square$

4.2. Alternative characterizations of Theorem 3.3

In this subsection, we discuss if some of the requirements of Theorem 3.3 can be replaced with other well-known properties. As we already pointed out in Section 2, it is well-known that efficiency is equivalent to the same-sidedness condition in the Sprumont model. Hence, we obtain the following corollary which turns out to be useful when we investigate the multi-commodity setting.

Corollary 4.5. *Let rule f satisfy same-sidedness, strategy-proofness and replacement monotonicity. For each $R \in \mathcal{R}$, $(X^{NE(R)}, \succeq_R)$ is a complete lattice whose supremum element is $f(R)$.*

We now discuss some variants of replacement monotonicity. Indeed, replacement monotonicity can be written in different ways, depending whether one is interested in a property about variations in physical terms or in welfare (see Thomson (2016) for a detailed discussion). For the latter, one possible variant is the following.[17]

Replacement monotonicity in welfare: Rule f satisfies replacement monotonicity in welfare if for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, either $f_j(R'_i, R_{-i})\ R_j\ f_j(R)$ for each $j \neq i$, or $f_j(R)\ R_j\ f_j(R'_i, R_{-i})$ for each $j \neq i$.

It turns out that replacement monotonicity in welfare is very demanding —see Thomson (1997) for a detailed discussion of the implication of replacement monotonicity in welfare and its one-sided variant.[18] For instance, the uniform rule does not satisfy this property.

Example 4.6 *(Violation of replacement monotonicity in welfare).* Let $n = 3$, $\Omega = 12$, and let f be the uniform rule. Consider R with $p(R) = (0, 0, 8)$ and $\tilde{R}_1$ with $p(\tilde{R}_1) = 8$. Then $f(R) = (2, 2, 8)$ and $f(\tilde{R}_1, R_{-1}) = (6, 0, 6)$. Clearly, $f(R)\ P_3\ f(\tilde{R}_1, R_{-1})$ but $f(\tilde{R}_1, R_{-1})\ P_2\ f(R)$. $\diamond$

The violation of replacement monotonicity in welfare is due to the regime change when going from profile R to $(\tilde{R}_1, R_{-1})$. The violation disappears if replacement monotonicity in welfare is one-sided, i.e. if it is required only for preferences changes that are not "too disruptive", i.e. the ones that do not trigger a change of regime—see Thomson (1997). Observe also that, locally, the uniform rule satisfies replacement monotonicity in welfare. We introduce the following property as a possible alternative to the one-sided formulation of replacement monotonicity in welfare. Under efficiency, both properties are equivalent to replacement monotonicity.

Local replacement monotonicity in welfare: Rule f satisfies local replacement monotonicity in welfare if for each $R \in \mathcal{R}$, there exists $\delta > 0$ such that for each $i \in N$ and each $R'_i \in \mathcal{R}_i$ such that $|p(R'_i) - p(R_i)| < \delta$, then either (i) $f_j(R'_i, R_{-i})\ R_j\ f(R)$ for each $j \in N \setminus i$, or (ii) $f_j(R)\ R_j\ f_j(R'_i, R_{-i})$ for each $j \in N \setminus i$.

We can therefore re-write Theorem 3.3 as follows.

Theorem 4.7. *Let rule f satisfy efficiency, strategy-proofness and local replacement monotonicity in welfare. For each $R \in \mathcal{R}$, $(X^{NE(R)}, \succeq_R)$ is a complete lattice whose supremum element is $f(R)$.*

4.3. Larger preference domains

One may wonder if the lattice structure of the Nash equilibrium allocations holds on a larger preference domains. This is a natural question since Massó and Neme (2004) show that there remain efficient and strategy-proof rules in the Sprumont model once the preference domain is extended to the (partially) single-plateaued preferences. We show that for the extended uniform rule characterized on this domain by Massó and Neme (2004), the lattice result is lost.

In order to make the discussion complete, let us define the single-plateaued domain. For each $i \in N$ and each $R_i \in \mathcal{R}_i$, the plateau is an interval $p(R_i) = [\underline{p}(R_i), \bar{p}(R_i)] \subseteq [0, \Omega]$. Agent

[17] We thank an anonymous referee for suggesting us to look at the welfare formulation of replacement monotonicity.

[18] Replacement monotonicity in welfare is discussed in Thomson (1997) under the moniker *welfare domination under preference replacement*. The mere difference in labeling should cause no confusion.

$i \in N$ is indifferent between the allotments in $p(R_i)$. On the other hand, if $x_i < \underline{p}(R_i)$ and $x_i < y_i \leq \bar{p}(R_i)$ or if $\bar{p}(R_i) < x_i$ and $\underline{p}(R_i) \leq y_i < x_i$ then $y_i \ P_i \ x_i$.

Extended uniform rule: Rule f is the extended uniform rule if for each $R \in \mathcal{R}$ and each $i \in N$,

$$f_i(R) = \begin{cases} \max\{\bar{p}(R_i), \lambda\} & \text{if } \sum_{i \in N} \bar{p}(R_i) < \Omega \\ \min\{\bar{p}(R_i), \underline{p}(R_i) + \lambda\} & \text{if } \sum_{i \in N} \underline{p}(R_i) < \Omega \leq \sum_{i \in N} \bar{p}(R_i) \\ \min\{\underline{p}(R_i), \lambda\} & \text{if } \Omega \leq \sum_{i \in N} \underline{p}(R_i) \end{cases}$$

where λ solves $\sum_{i \in N} f_i(R) = \Omega$.

We show that the set of Nash equilibrium allocations of the extended uniform rule is not a lattice.

Example 4.8 *(The lattice structure does not extend to a larger preference domain).* Let $n = 2$, $\Omega = 10$ and let f be the extended uniform rule. Let $R \in \mathcal{R}$ be such that $p(R_1) = [3, 4]$ and $p(R_2) = [6, 7]$. Let f be the extended uniform rule. The set of Nash equilibrium allocations is given by $X^{NE(R)} = \{x \in X : 3 \leq x_1 \leq 5\}$. Observe that $(X^{NE(R)}, \succeq_R)$ is not a lattice. In fact, it is not even a partially ordered set because $\succeq_R$ is not antisymmetric on $X^{NE(R)}$. For instance, $(3, 7) \succeq_R (4, 6) \succeq_R (3, 7)$. However, we note that truthtelling which results in the allocation $(3.5, 6.5)$ is not Pareto dominated by any other Nash equilibrium allocation. $\diamond$

4.4. Public good economies with single-peaked preferences

The interest in the study of strategy-proof rules when preferences are single-peaked notably picks up with Moulin (1980)'s seminal contribution. Moulin (1980) characterizes the class of generalized median voting rules on the basis of efficiency and strategy-proofness—see also Barberà and Jackson (1994). Thomson (1993) shows that, out of these rules, the only ones satisfying replacement monotonicity (in welfare) are the so-called *Target rules* which we define below. For simplicity let $[0, \Omega]$ be the set of possible public good levels. Preferences remain single-peaked on this interval. For any profile $R \in \mathcal{R}$, let $\min p(R) = \{x \in [0, \Omega] : \nexists i \in N \text{ with } p(R_i) < x\}$ and $\max p(R) = \{x \in [0, \Omega] : \nexists i \in N \text{ with } p(R_i) > x\}$. It is easy to see that rule f is efficient if for each $R \in \mathcal{R}$, $f(R) \in [\min p(R), \max p(R)]$.

Target rules: Rule f^a is a target rule with respect to $a \in [0, \Omega]$ if for each $R \in \mathcal{R}$, either (i) $f^a(R) = a$ if $\min p(R) \leq a \leq \max p(R)$, or (ii) $f^a(R) = \min p(R)$ if $a < \min p(R)$, or (iii) $f^a(R) = \max p(R)$ if $a > \max p(R)$.

In the family of target rules, each rule is indexed by a level $a \in [0, \Omega]$. We characterize below the structure of the set of Nash equilibrium public good levels for any given target rule. Fix $a \in [0, \Omega]$ and consider rule f^a.

Case 1: For $R \in \mathcal{R}$, $f^a(R) = a$. Then a is the unique Nash equilibrium outcome. Indeed, notice that for a not be selected it must be that agents report $\tilde{R} \in \mathcal{R}$ such that either $\max p(\tilde{R}) < a$ or

$\min p(\tilde{R}) > a$. But then $\tilde{R}$ cannot be a Nash equilibrium outcome since any agent can enforce a to be the level of public good selected.[19]

Case 2: For $R \in \mathcal{R}$, $f^a(R) \neq a$. Suppose that $f^a(R) > a$, where, by definition of the target rule, for each $i \in N$ we have $p(R_i) \geq f^a(R)$. Then any $\tilde{R} \in \mathcal{R}$ such that (i) $f^a(\tilde{R}) \in [a, f^a(R))$ and (ii) there does not exist $i \in N$, $\hat{R}_i \in \mathcal{R}_i$ such that $f^a(\hat{R}_i, \tilde{R}_{-i}) > f^a(\tilde{R})$, is a Nash equilibrium outcome. In particular any report $\tilde{R} \in \mathcal{R}$ such that for each $i, j \in N$, $p(\tilde{R}_i) = p(\tilde{R}_j) \in [a, f^a(R)]$ is a Nash equilibrium. The second case where $f^a(R) < a$ is analogous.

Therefore the set of Nash equilibrium public good levels is either a singleton (the target a), or it is an interval. For the latter, it is either the interval $[f^a(R), a]$ or $[a, f^a(R)]$. Observe that these Nash equilibrium levels are strictly Pareto ranked. To see this, consider Case 2 and suppose that $f^a(R) > a$. Then the set of Nash equilibrium levels $[a, f^a(R)]$ is a complete lattice with infimum a and supremum $f^a(R)$. For any $x, x' \in [a, f^a(R)]$, x and x' are strictly Pareto ranked. Notice that any Nash equilibrium public good level x can be supported with a preference report whose unanimous peaks are x. This observation can be seen as an analog of Proposition 3.2 that we discussed earlier. We now have the following immediate result.

Theorem 4.9. *Let rule f satisfy efficiency, strategy-proofness and replacement monotonicity in welfare, i.e. $f \equiv f^a$ for some $a \in [0, \Omega]$. For each $R \in \mathcal{R}$, $(X^{NE(R)}, \succeq_R)$ is a totally ordered set with supremum element $f^a(R)$, and infimum element a. In addition, for each $R \in \mathcal{R}$ and each $x \in X^{NE(R)}$, any $\tilde{R} \in \mathcal{R}$ with $p(\tilde{R}) = (x, ..., x)$ is a Nash equilibrium.*

The set of generalized median voting rules contain many rules that violate replacement monotonicity in welfare. An example of such a rule is the median rule itself.[20,21] One may wonder whether such rules inherit the lattice structure of the set of Nash equilibrium allocations. The answer turns out to be negative as shown in the following example.

Example 4.10 *(Replacement monotonicity in welfare is essential for the lattice structure to hold).* Let $n = 3$ and f be the median rule. Consider $R \in \mathcal{R}$ with $0 < p(R_1) < p(R_2) < p(R_3) < \Omega$ and such that there exists $x \in (0, p(R_1))$ with $x \; I_1 \; p(R_2)$. By definition of the median, $f(R) = p(R_2)$. Consider the report $\tilde{R}$ with $x < p(\tilde{R}_i) = p(\tilde{R}_j) < p(R_2)$ for each $i, j \in N$. Then $f(\tilde{R}) = p(\tilde{R}_i)$. Notice that $\tilde{R}$ is a Nash equilibrium as no unilateral deviation can alter the median. Notice also that $f(R) \; P_j \; f(\tilde{R})$ for $j = 2, 3$ while $f(\tilde{R}) \; P_1 \; f(R)$. Hence $f(R)$ and $f(\tilde{R})$ are two Nash equilibrium public good levels that are not Pareto comparable, and their meet and join are then obviously not well-defined. We conclude that the set of Nash equilibrium levels is not a complete lattice. Moreover the truthtelling public good level does not Pareto dominate all the other Nash equilibrium levels. $\diamond$

---

[19] If $\max p(\tilde{R}) < a$, then any $i \in N$ can report $\hat{R}_i > a$ for a to be efficient at $(\hat{R}_i, \tilde{R}_{-i})$. By assumption, we know that there exists an agent who would benefit by deviating and enforcing a as outcome at profile R. The reasoning is analogous if $\min p(\tilde{R}) > a$.

[20] Formally, the median rule is defined as the rule which, for each $R \in \mathcal{R}$, selects the median of the peaks and $(n-1)$ *phantoms* where (i) $\frac{(n-1)}{2}$ phantoms are positioned both at 0 and at Ω if n is odd, (ii) $\frac{(n-2)}{2}$ phantoms are positioned both at 0 and at Ω, while the remaining phantom is arbitrarily fixed at either 0 or Ω depending on the way ties are to be broken.

[21] It is easy to see that the median rule violates replacement monotonicity in welfare. Let $n = 3$ and pick $R \in \mathcal{R}$ with $p(R_1) < p(R_2) < p(R_3)$. Then $f(R) = p(R_2)$. Consider $(\tilde{R}_2, R_{-2})$ with $p(R_2) < p(\tilde{R}_2) < p(R_3)$. Then $f(\tilde{R}_2, R_{-2}) = p(\tilde{R}_2)$, and agent 3 is better-off while agent 1 is worse-off.

5. Conclusion

In this paper, we have established not only some surprising properties of truthtelling in the Sprumont model, but also an unexpected feature of the set of Nash equilibrium allocations of all the rules characterized in Barberà et al. (1997). Following our main result, the discussion we offer clarifies that replacement monotonicity, although at the heart of our characterization, does not seem to be necessary for the lattice structure to hold for some rules that are efficient and strategy-proof—see e.g. Example 4.3. Instead, under efficiency and strategy-proofness, a condition which lies "between" replacement monotonicity and non-bossiness is probably both necessary and sufficient. We could only establish that if f is efficient and strategy-proof, then non-bossiness is a necessary condition for the lattice result to hold. We leave open for future research the identification of such a condition which may be key to obtain a generalization of our results. Nevertheless, we do offer some extensions of our results provided the preference domain is confined to agents having single-peaked preferences over the set of individual outcomes—Example 4.12 makes clear that an enlargement of the domain does not seem possible.

Yet an important question is whether there are other models where the lattice structure holds. Our study of the single-plateaued domain seem to indicate that such models may be rare. We are however aware that the Nash outcomes for the Boston mechanism in the school choice literature form a complete lattice (Ergin and Sönmez, 2006). One significant difference from our result is that the allocation under truthtelling is not the supremum of the lattice. Ergin and Sönmez (2006) show that the set of Nash outcomes is equivalent to one of stable matchings which are well-known to form a complete lattice (see for instance Knuth (1976), Adachi (2000) and Hatfield and Milgrom (2005)).[22] It is not clear if there is any common thread in the two models that lead to the lattice result. We leave this question open for future research.

Another open question is whether our main result can be proved using Tarski's fixed point theorem. After all Nash equilibria are fixed points, and Tarski's fixed point theorem is concerned with the lattice structure of fixed points. In this sense, we wonder if our result is an application of this celebrated theorem.

Finally, our approach also fits with the recent literature investigating some of the additional strategic features of rules in models with single-peaked preferences. For instance, Bochet and Sakai (2010) show that the Nash equilibrium allocations of the uniform rule are Pareto dominated by the outcome obtained from truthful revelation. Also, Bochet et al. (2019) show that for a large family of manipulable rules (e.g., the proportional rule) the set of Nash equilibrium allocations is a singleton corresponding to the truthtelling allocation of the uniform rule.

Appendix A

A.1. Auxiliary lemmas

In order to prove our main result, we first need to introduce several auxiliary lemmas which will be used as facts or sometimes building blocks of some portion of the main proof. Some of these lemmas have appeared elsewhere in the literature while some are new. For all the lemmas, we assume that rule f satisfies efficiency, strategy-proofness and replacement monotonicity.

[22] Knuth attributes the result on the lattice structure of stable matchings to John Conway.

Lemma A.1. *For each $R \in \mathcal{R}$ and each $i \in N$, if $f_i(R) < p(R_i)$, then $f_i(R) = \max_{\hat{R}_i \in \mathcal{R}_i} f_i(\hat{R}_i, R_{-i})$. Similarly, if $p(R_i) < f_i(R)$, then $f_i(R) = \min_{\hat{R}_i \in \mathcal{R}_i} f_i(\hat{R}_i, R_{-i})$.*

Proof. We prove the first part only. Fix any $\hat{R}_i \neq R_i$. By strategy-proofness, we have $f_i(R)\ R_i\ f_i(\hat{R}_i, R_{-i})$. Single-peakedness then yields that $f_i(\hat{R}_i, R_{-i}) \notin (f_i(R), p(R_i)]$. If $p(R_i) < f_i(\hat{R}_i, R_{-i})$, pick $\bar{R}_i$ with $p(\bar{R}_i) = p(R_i)$ and $f(\hat{R}_i, R_{-i})\ \bar{P}_i\ f(R)$. By peak-onliness, $f(\bar{R}_i, R_{-i}) = f(R)$. By construction, $f(\hat{R}_i, R_{-i})\ \bar{P}_i\ f(R) = f(\bar{R}_i, R_{-i})$, a contradiction with the strategy-proofness of f. Thus, $f_i(\hat{R}_i, R_{-i}) \leq f_i(R)$. □

In the next lemma, we identify how a change in an agent's report affects one's own allocation. Specifically, if agent i was allocated less than his peak, then any report with a peak above his allocation does not alter the allocation. If the original and new reports have peaks respectively on the left and right of the original allocation then the new allocation can increase but never exceeds the new peak.

Lemma A.2. *For each $R \in \mathcal{R}$, each $i \in N$, and each $\tilde{R}_i \in \mathcal{R}_i$ with $p(R_i) < p(\tilde{R}_i)$, one of the following cases hold:*

(i) $f_i(R) = f_i(\tilde{R}_i, R_{-i}) \leq p(R_i) < p(\tilde{R}_i)$.
(ii) $p(R_i) \leq f_i(R) \leq f_i(\tilde{R}_i, R_{-i}) \leq p(\tilde{R}_i)$, $p(R_i) < f_i(\tilde{R}_i, R_{-i})$ *and* $f_i(R) < p(\tilde{R}_i)$.
(iii) $p(R_i) < p(\tilde{R}_i) \leq f_i(R) = f_i(\tilde{R}_i, R_{-i})$.

Proof. The lemma is a direct consequence of Lemma A.1. □

The next lemma covers the situation in which a group changes its report so that each member's new peak moves in the same direction. If no agent's unilateral change alters the allocation, then the group's report should also not lead to any changes.

Lemma A.3. *Pick $R, \bar{R} \in \mathcal{R}$. Let $S \subseteq \{i \in N : f_i(R) \leq p(\bar{R}_i)\}$ or $S \subseteq \{i \in N : p(\bar{R}_i) \leq f_i(R)\}$. If $f_i(\bar{R}_i, R_{-i}) = f_i(R)$ for each $i \in S$, then $f(\bar{R}_S, R_{-S}) = f(R)$.*

Proof. Let $S \subseteq \{i \in N : f_i(R) \leq p(\bar{R}_i)\}$. We prove this lemma by induction on the size of subsets of S.

The induction assumption: Fix any k such that $1 \leq k \leq |S| - 1$. For all $T \subset S$ with $|T| = k$, we have that $f(\bar{R}_T, R_{-T}) = f(R)$.

We know that the induction assumption is true if $k = 1$. We now show that for all $\bar{T} \subseteq S$ with $|\bar{T}| = k + 1$ it must be that $f(\bar{R}_{\bar{T}}, R_{-\bar{T}}) = f(R)$. In contrast, suppose that there exists $\bar{T}$ with $|\bar{T}| = k + 1$ such that $f(\bar{R}_{\bar{T}}, R_{-\bar{T}}) \neq f(R)$. We first show that $f_i(R) < f_i(\bar{R}_{\bar{T}}, R_{-\bar{T}})$ for each $i \in S$. Pick any $i \in \bar{T}$, and set $T = \bar{T} \setminus \{i\}$.

Because $|T| = k$, by the induction assumption, $f(\bar{R}_T, R_{-T}) = f(R)$. If $p(\bar{R}_i) = f_i(R) = f_i(\bar{R}_T, R_{-T})$ then it must be that $f_i(\bar{R}_{\bar{T}}, R_{-\bar{T}}) = f_i(\bar{R}_T, R_{-T})$ by Lemma A.2. Then by non-bossiness, $f(\bar{R}_{\bar{T}}, R_{-\bar{T}}) = f(\bar{R}_T, R_{-T}) = f(R)$. This is a contradiction. Hence, $f_i(\bar{R}_T, R_{-T}) = f_i(R) < p(\bar{R}_i)$. In this case, we would reach a contradiction if $f_i(\bar{R}_{\bar{T}}, R_{-\bar{T}}) = f_i(\bar{R}_T, R_{-T})$. Subsequently, $f_i(R) < p(\bar{R}_i)$ and $f_i(\bar{R}_{\bar{T}}, R_{-\bar{T}}) \neq f_i(\bar{R}_T, R_{-T})$. By Lemma A.2, we have

$f_i(\bar{R}_{\bar{T}}, R_{-\bar{T}}) = f_i(\bar{R}_T, R_{-T})$ unless $p(R_i) \le f_i(\bar{R}_T, R_{-T}) < f_i(\bar{R}_{\bar{T}}, R_{-\bar{T}}) \le p(\bar{R}_i)$. Consequently, we have that $p(R_i) \le f_i(\bar{R}_T, R_{-T}) < f_i(\bar{R}_{\bar{T}}, \bar{R}_{-\bar{T}}) \le p(\bar{R}_i)$. Because $f_i(\bar{R}_T, R_{-T}) < f_i(\bar{R}_{\bar{T}}, R_{-\bar{T}})$, we have $f_j(\bar{R}_{\bar{T}}, R_{-\bar{T}}) \le f_j(\bar{R}_T, R_{-T})$ for all $j \ne i$ by replacement monotonicity. Because $f(R) = f(\bar{R}_T, R_{-T})$, we obtain that $f_i(R) < f_i(\bar{R}_{\bar{T}}, R_{-\bar{T}})$ and $f_j(\bar{R}_{\bar{T}}, R_{-\bar{T}}) \le f_j(R)$ for all $j \ne i$. However, we picked agent i from S arbitrarily, which means that $f_j(R) < f_j(\tilde{R}_S, \bar{R}_{-S})$ for all $j \in S$. The last two relations are not compatible. □

The next two lemmas are concerned with Nash equilibrium reports. Suppose that one can decrease (or increase) her own Nash equilibrium allocation through a unilateral deviation. Then one's peak must have exceeded (respectively fallen short of) the original allocation.

Lemma A.4. *Pick $R, \bar{R} \in \mathcal{R}$. If $\bar{R} \in NE(R)$ and $f_i(\tilde{R}_i, \bar{R}_{-i}) < f_i(\bar{R})$ for some $i \in N$ and $\tilde{R}_i \in \mathcal{R}_i$, then $f_i(\bar{R}) \le p(R_i)$. Similarly, if $f_i(\bar{R}) < f_i(\tilde{R}_i, \bar{R}_{-i})$, then $p(R_i) \le f_i(\bar{R})$.*

Proof. Pick $R, \bar{R} \in \mathcal{R}$, $i \in N$ and $\tilde{R}_i \in \mathcal{R}_i$. Suppose that $f_i(\tilde{R}_i, \bar{R}_{-i}) < f_i(\bar{R})$ but $p(R_i) < f_i(\bar{R})$. We cannot have $\bar{R} \in NE(R)$ if $p(R_i) \le f_i(\tilde{R}_i, \bar{R}_{-i})$. Thus, let us assume that $f_i(\tilde{R}_i, \bar{R}_{-i}) < p(R_i) < f_i(\bar{R})$. Because f is strategy-proof and $\bar{R} \in NE(R)$, we obtain that $f_i(R_i, \bar{R}_i) \; I_i \; f_i(\bar{R})$. In addition, $p(R_i) < f_i(\bar{R})$ implies $f_i(R_i, \bar{R}_{-i}) < p(R_i) < f_i(\bar{R})$. This contradicts Lemma A.1. □

The next lemma states that if the peaks of one's true and new reported preferences are on the same side of the Nash equilibrium allocation then the allocation should not change. This is also true for groups.

Lemma A.5. *Pick $R, \bar{R} \in \mathcal{R}$ such that $\bar{R} \in NE(R)$, and pick any $\tilde{R} \in \mathcal{R}$. For any $S \subseteq \{i \in N : f_i(\bar{R}) \le p(\tilde{R}_i)$ & $f_i(\bar{R}) < p(R_i)\}$, it must be that $f(\tilde{R}_S, \bar{R}_S) = f(\bar{R})$. Similarly, for any $T \subseteq \{i \in N : p(\tilde{R}_i) \le f_i(\bar{R})$ & $p(R_i) < f_i(\bar{R})\}$, it must be that $f(\tilde{R}_T, \bar{R}_T) = f(\bar{R})$.*

Proof. Pick $R, \bar{R} \in \mathcal{R}$ with $\bar{R} \in NE(R)$, and any $\tilde{R} \in \mathcal{R}$. Pick any $S \subseteq \{i \in N : f_i(\bar{R}) \le p_i(\tilde{R})\}$. Let us now show that $f(\tilde{R}_i, \bar{R}_{-i}) = f(\bar{R})$ for each $i \in S$. If $p(\tilde{R}_i) = f_i(\bar{R})$, then by peak-onliness and non-bossiness, $f(\tilde{R}_i, \bar{R}_{-i}) = f(\bar{R})$. Let $f_i(\bar{R}) < p(\tilde{R}_i)$. By Lemma A.2, $f_i(\bar{R}) \le f_i(\tilde{R}_i, \bar{R}_{-i}) \le p(\tilde{R}_i)$. If $f_i(\bar{R}) < f_i(\tilde{R}_i, \bar{R}_{-i})$, then by Lemma A.4, $p(R_i) \le f_i(\bar{R})$ which contradicts that $i \in S$. Hence, $f_i(\bar{R}) = f_i(\tilde{R}_i, \bar{R}_{-i})$ which along with non-bossiness implies that $f(\bar{R}) = f(\tilde{R}_i, \bar{R}_{-i})$. Given that this is true for all $i \in S$, by Lemma A.3, we have that $f(\tilde{R}_S, \bar{R}_{-S}) = f(\bar{R})$. By adopting the above arguments slightly, we find that $f(\tilde{R}_T, \bar{R}_T) = f(\bar{R})$ for all $T \subseteq \{i \in N : p_i(\tilde{R}) \le f_i(\bar{R})\}$. □

Finally, the following lemma shows how a deviating group's and remaining agents' allocations change. Suppose a group changes its preferences so that each member's new reported preference peak exceeds one's own original allocation. Then no member gets more than one's new peak. In addition, those who are not in the group must get less than their original allocation.

Lemma A.6. *Pick $R, \bar{R} \in \mathcal{R}$. Let $S \subseteq \{i \in N : f_i(R) \le p(\bar{R}_i)\}$ and $T \subseteq \{i \in N : p(\bar{R}_i) \le f_i(R)\}$. Then $f_i(\bar{R}_S, R_{-S}) \le p(\bar{R}_S)$ for each $i \in S$ and $f_i(\bar{R}_S, R_{-S}) \le f_i(R)$ for each $i \notin S$. Similarly, $p(\bar{R}_T) \le f_i(\bar{R}_T, R_{-T})$ for each $i \in T$ and $f_i(R) \le f_i(\bar{R}_T, R_{-T})$ for each $i \notin T$.*

Proof. Pick $R, \bar{R} \in \mathcal{R}$. We here provide a proof for $f(\bar{R}_S, R_{-S})$. Pick any $i \in S$. By Lemma A.2, $p_i(R_i) \leq f_i(R) \leq f_i(\bar{R}_i, R_{-i}) \leq p(\bar{R}_i)$ or $f_i(\bar{R}_i, R_{-i}) = f_i(R)$. Consequently, $f_i(R) \leq f_i(\bar{R}_i, R_{-i})$. Then by replacement monotonicity, $f_j(\bar{R}_i, R_{-i}) \leq f_j(R)$ for each $j \neq i$. If $j \in S \setminus \{i\}$, $f_j(\bar{R}_i, R_{-i}) \leq f_j(R) \leq p(\bar{R}_j)$. Now sequentially changing the preferences of each agent $j \in S \setminus \{i\}$ from R_j to $\bar{R}_j$, we complete the proof for $f(\bar{R}_S, R_{-S})$. $\square$

A.2. Proofs of the main results

Proof of Proposition 3.2. Pick $R \in \mathcal{R}$ and $x \in X^{NE(R)}$. Let $\bar{R} \in \mathcal{R}$ be such that $\bar{R} \in NE(R)$ and $f(\bar{R}) = x$. If $\sum_{i \in N} p(\bar{R}_i) = \Omega$ then we are done because f is peak-only and efficient. So suppose that $\sum_{i \in N} p(\bar{R}_i) \neq \Omega$. Without loss of generality, let us assume that $\Omega < \sum_{i \in N} p(\bar{R}_i)$. By efficiency, $f_i(\bar{R}) \leq p(\bar{R}_i)$ for each $i \in N$. In fact, this inequality must be strict for at least one agent because $\sum_{i \in N} f_i(\bar{R}) = \Omega$. Let $S = \{i \in N : f_i(\bar{R}) < p(\bar{R}_i)\}$. We consider two cases: (a) For each agent $i \in N$, $f_i(\bar{R}) \leq p(R_i)$ and (b) There exist at least one agent $i \in N$ with $p(R_i) < f_i(\bar{R})$.

Case (a): Let $\hat{R}_S \in \mathcal{R}_S$ be such that $p(\hat{R}_i) = f_i(\bar{R})$ for each $i \in S$. By efficiency, $p(\hat{R}_i) = f_i(\bar{R})$ for each $i \in S$ and $p(\bar{R}_j) = f_j(\bar{R})$ for each $j \in N \setminus S$. Consequently, $f(\hat{R}_S, \bar{R}_{-S}) = f(\bar{R})$. We now show that $(\hat{R}_S, \bar{R}_{-S}) \in NE(R)$. Suppose otherwise. By combining this with strategy-proofness, there must exist an agent i^* with $f_{i^*}(R_{i^*}, \hat{R}_{S\setminus\{i^*\}}, \bar{R}_{N\setminus\{S\cup\{i^*\}\}})\ P_{i^*}\ f_{i^*}(\hat{R}_S, \bar{R}_{-S}) = f_{i^*}(\bar{R})$. Recall that $f_i(\bar{R}) \leq p(R_i)$ for all $i \in N$. Therefore, by single-peakedness,

$$f_{i^*}(\bar{R}) < f_{i^*}(R_{i^*}, \hat{R}_{S\setminus\{i^*\}}, \bar{R}_{N\setminus\{S\cup\{i^*\}\}}) \ \&\ f_{i^*}(\bar{R}) < p(R_{i^*}). \tag{A.1}$$

Consider now $(R_{i^*}, \bar{R}_{-i})$. Because $f_{i^*}(\bar{R}) < p(R_{i^*})$, by Lemma A.2 it must be that $f_{i^*}(R_{i^*}, \bar{R}_{-i}) \leq p(R_{i^*})$. The case $f_{i^*}(\bar{R}) < f_{i^*}(R_{i^*}, \bar{R}_{-i})$ contradicts that $\bar{R} \in NE(R)$. Thus, $f_{i^*}(\bar{R}) = f_{i^*}(R_{i^*}, \bar{R}_{-i})$ which, along with non-bossiness implies that $f(\bar{R}) = f(R_{i^*}, \bar{R}_{-i})$. Consider any $j \in S \setminus \{i^*\}$. By construction, $p(\hat{R}_j) = f_j(\bar{R}) = f_j(R_{i^*}, \bar{R}_{-i})$. Thus, by strategy-proofness, $f_j(\hat{R}_j, R_{i^*}, \bar{R}_{-j,i^*}) = f_j(\bar{R})$. Then Lemma A.3 yields that

$$f(R_{i^*}, \hat{R}_{S\setminus\{i^*\}}, \bar{R}_{N\setminus\{S\cup\{i^*\}\}}) = f(\bar{R}),$$

contradicting (A.1). Thus, $(\hat{R}_S, \bar{R}_{-S}) \in NE(R)$. Finally, the peak-onliness of f implies the claim of this step.

Case (b): Pick an agent i^* with $p(R_{i^*}) < f_{i^*}(\bar{R})$. Consider $(R_{i^*}, \bar{R}_{-i^*})$. By Lemma A.2, $p(R_{i^*}) \leq f_{i^*}(R_{i^*}, \bar{R}_{-i^*}) \leq f_{i^*}(\bar{R}) \leq p(\bar{R}_{i^*})$. The case $f_{i^*}(R_{i^*}, \bar{R}_{-i^*}) < f_{i^*}(\bar{R})$ contradicts $\bar{R} \in NE(R)$. Hence, $p(R_{i^*}) < f_{i^*}(R_{i^*}, \bar{R}_{-i^*}) = f_{i^*}(\bar{R})$ which, along with the non-bossiness of f, yields that $f(R_{i^*}, \bar{R}_{-i^*}) = f(\bar{R})$. If there exists $j \neq i^*$ with $f_j(R_{i^*}, \bar{R}_{-i^*}) = f_j(\bar{R}) < p(\bar{R}_j)$, then the allocations of agents i^* and j fall on different sides of their respective peaks at $(R_{i^*}, \bar{R}_{-i^*})$, a contradiction with efficiency. Hence, for each $j \neq i^*$, we have $p(\bar{R}_j) = f_j(\bar{R})$. Recall that $\sum_{i \in N} f_i(\bar{R}) = \Omega < \sum_{i \in N} p(\bar{R}_i)$. This condition is satisfied only if $f_{i^*}(\bar{R}) < p(\bar{R}_{i^*})$ which means that $S = \{i^*\}$. Consequently, there exists only one agent i^* with $p(R_{i^*}) < f_{i^*}(\bar{R})$ and $S = \{i^*\}$. Let now $\hat{R}_{i^*} \in \mathcal{R}$ be such that $(\hat{R}_{i^*}, \bar{R}_{-i^*}) = (\hat{R}_S, \bar{R}_{-S})$ and $p(\hat{R}_{i^*}) = f_{i^*}(\bar{R})$. We are left to show that $(\hat{R}_S, \bar{R}_{-S}) \in NE(R)$. We already know that i^* has no unilateral and profitable deviation. By following the same steps used in the proof of Case (a), one can show that no other agent has a profitable deviation. Finally, the peak-onliness of f completes the proof. $\square$

We next divide the proof of Theorem 3.3 into two parts. First we prove that for each $R \in \mathcal{R}$, $(X^{NE(R)}, \succeq_R)$ is a lattice. We label this result below as Theorem A.7. This should cause no confusion. The proof of the actual Theorem 3.3 is then devoted to show that $(X^{NE(R)}, \succeq_R)$ is not only a lattice (as shown in Theorem A.7 below), but in fact it is a complete lattice with the truthtelling allocation as supremum.

Theorem A.7. *Let rule f satisfy efficiency, strategy-proofness and replacement monotonicity. For each $R \in \mathcal{R}$, $(X^{NE(R)}, \succeq_R)$ is a lattice.*

Proof. Let $R \in \mathcal{R}$. We pick two distinct allocations $\bar{x}, \tilde{x} \in X^{NE(R)}$. By Lemma 3.2, there exist $\bar{R}, \tilde{R} \in NE(R)$ such that $f(\bar{R}) = p(\bar{R}) = \bar{x}$ and $f(\tilde{R}) = p(\tilde{R}) = \tilde{x}$. Next, let us partition N into three sets W, E and O so that $W \equiv \{i \in N : f_i(\tilde{R}) < f_i(\bar{R})\}$, $E \equiv \{i \in N : f_i(\bar{R}) < f_i(\tilde{R})\}$ and $O \equiv \{i \in N : f_i(\tilde{R}) = f_i(\bar{R})\}$.

We further decompose W into two subsets $\bar{W}$ and $\tilde{W}$ so that $\bar{W} = \{i \in W : f(\bar{R})\ R_i\ f(\tilde{R})\}$ and $\tilde{W} = \{i \in W : f(\tilde{R})\ P_i\ f(\bar{R})\}$. Similarly, we decompose E so that $\bar{E} = \{i \in E : f(\bar{R})\ R_i\ f(\tilde{R})\}$ and $\tilde{E} = \{i \in E : f(\tilde{R})\ P_i\ f(\bar{R})\}$.

In addition, let $d_i = |f_i(\tilde{R}) - f_i(\bar{R})|$. We note here that $\sum_{i \in W} d_i = \sum_{i \in E} d_i$. The proof is next divided into several steps.

Step 1. *Peak relations for partitions of agents.*
The set $\tilde{W} \neq \emptyset$ if and only if $\tilde{E} \neq \emptyset$. Similarly, the set $\bar{W} \neq \emptyset$ if and only if $\bar{E} \neq \emptyset$. In addition,

$$\sum_{i \in \bar{W}} d_i = \sum_{i \in \bar{E}} d_i \tag{A.2}$$

$$\sum_{i \in \tilde{W}} d_i = \sum_{i \in \tilde{E}} d_i \tag{A.3}$$

$$\tilde{W} = \{i \in N : p(R_i) \leq p(\tilde{R}_i) < p(\bar{R}_i)\} \tag{A.4}$$

$$\tilde{E} = \{i \in N : p(\bar{R}_i) < p(\tilde{R}_i) \leq p(R_i)\} \tag{A.5}$$

$$\bar{W} = \{i \in N : p(\tilde{R}_i) < p(\bar{R}_i) \leq p(R_i)\} \tag{A.6}$$

$$\bar{E} = \{i \in N : p(R_i) \leq p(\bar{R}_i) < p(\tilde{R}_i)\}. \tag{A.7}$$

Proof of Step 1. Suppose $\tilde{W} \neq \emptyset$. Let us now show that $\tilde{E} \neq \emptyset$. Pick any $i \in \tilde{W}$ and consider the profile $(\tilde{R}_i, \bar{R}_{-i})$. By construction, $f_i(\tilde{R}) = p(\tilde{R}_i) < p(\bar{R}_i) = f_i(\bar{R})$ and $f_i(\tilde{R})\ P_i\ f_i(\bar{R})$. Consequently, the single-peakedness of R_i implies that $p(R_i) < f_i(\bar{R})$. By Lemma A.5, $f(\tilde{R}_{\tilde{W}}, \bar{R}_{-\tilde{W}}) = f(\bar{R})$. Consider now $(\tilde{R}_{\tilde{W}}, \bar{R}_{-\tilde{W}})$ and $(\tilde{R}_W, \bar{R}_{-W})$ which differ in the preferences of those in $\bar{W}$. If $\bar{W} = \emptyset$, then $f(\tilde{R}_W, \bar{R}_{-W}) = f(\bar{R})$. If $\bar{W} \neq \emptyset$, then by definition of $\bar{W}$, we have $p(\tilde{R}_i) < f_i(\bar{R}) = f_i(\tilde{R}_{\tilde{W}}, \bar{R}_{-\tilde{W}})$. By Lemma A.6, we know that

$$p(\tilde{R}_i) \leq f_i(\tilde{R}_W, \bar{R}_{-W}) \text{ for all } i \in \bar{W} \ \ \& \ \ f_i(\bar{R}) \leq f_i(\tilde{R}_W, \bar{R}_{-W}) \text{ for all } i \notin \bar{W}.$$

If $O \cup E = \emptyset$ then $f_i(\bar{R}) \leq f_i(\tilde{R}_W, \bar{R}_{-W}) = f_i(\tilde{R})$ for all $i \in \tilde{W}$. But this cannot happen because $\tilde{W} \subseteq W$. Thus, $O \cup E \neq \emptyset$. Let $S \equiv \{i \in O \cup E : p(\bar{R}_i) \leq f_i(\tilde{R}_W, \bar{R}_{-W})\}$. If $O \neq \emptyset$, then $O \subseteq S$. Consider now $(\tilde{R}_{W \cup S}, \bar{R}_{-W \cup S})$ and $(\tilde{R}_W, \bar{R}_{-W})$. By construction, $p(\bar{R}_i) \leq p(\tilde{R}_i) \leq f_i(\tilde{R}_W, \bar{R}_{-W})$ for each $i \in S$. By Lemma A.2, we know that $f_i(\tilde{R}_{W \cup i}, \bar{R}_{-W \cup i}) = f_i(\tilde{R}_W, \bar{R}_{-W})$ for each $i \in S$. By non-bossiness, $f(\tilde{R}_{W \cup i}, \bar{R}_{-W \cup i}) = f(\tilde{R}_W, \bar{R}_{-W})$ for all $i \in S$. By Lemma A.3, we obtain that $f_i(\tilde{R}_{W \cup S}, \bar{R}_{-W \cup S}) = f_i(\tilde{R}_W, \bar{R}_{-W})$. Consequently,

$$p(\tilde{R}_i) \leq f_i(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S}) \text{ for all } i \in \bar{W} \;\;\&\;\; f_i(\bar{R}) \leq f_i(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S}) \text{ for all } i \notin \bar{W}.$$

If $W \cup S = N$, then $f_i(\bar{R}) \leq f_i(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S}) = f_i(\tilde{R})$ for all $i \in \tilde{W}$. However, this cannot happen because $\tilde{W} \subseteq W$. Thus, we find that $T \equiv N \setminus (W \cup S) = E \setminus S \neq \emptyset$.

Let us recap what we know below as it will be used later:

Fact 1. At preference profile, $(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S})$ it must be that

$$p(\bar{R}_i) = f_i(\tilde{R}_{\tilde{W}}, \bar{R}_{-\tilde{W}}) \leq f_i(\tilde{R}_W, \bar{R}_{-W}) = f_i(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S}) < p(\tilde{R}_i) \quad \text{for all } i \in T \tag{A.8}$$

$$p(\bar{R}_i) = f_i(\tilde{R}_{\tilde{W}}, \bar{R}_{-\tilde{W}}) \leq p(\tilde{R}_i) \leq f_i(\tilde{R}_W, \bar{R}_{-W}) = f_i(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S}) \quad \text{for all } i \in S \tag{A.9}$$

$$p(\tilde{R}_i) < p(\bar{R}_i) = f_i(\tilde{R}_{\tilde{W}}, \bar{R}_{-\tilde{W}}) \leq f_i(\tilde{R}_W, \bar{R}_{-W}) = f_i(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S}) \quad \text{for all } i \in \tilde{W} \tag{A.10}$$

$$p(\tilde{R}_i) < p(\bar{R}_i) = f_i(\tilde{R}_{\tilde{W}}, \bar{R}_{-\tilde{W}}) \leq f_i(\tilde{R}_W, \bar{R}_{-W}) = f_i(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S}) \quad \text{for all } i \in \bar{W} \tag{A.11}$$

Pick an agent $i \in T$. Consider now $(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S})$ and $(\tilde{R}_{-i}, \bar{R}_i)$ which differs in the preferences of those agents in $T \setminus \{i\}$. At both profiles, there is underdemand. Hence, (A.8), Lemma A.6 and efficiency yield that

$$\begin{aligned}
&f_i(\tilde{R}_{-i}, \bar{R}_i) \in [p(\bar{R}_i), f_i(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S})] \subseteq [p(\bar{R}_i), p(\tilde{R}_i)] \\
&f_j(\tilde{R}_{-i}, \bar{R}_i) = p(\tilde{R}_j) && \text{for all } j \in T \setminus \{i\} \\
&f_j(\tilde{R}_{-i}, \bar{R}_i) = [p(\tilde{R}_j), f_j(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S})] && \text{for all } j \in W \cup S
\end{aligned}$$

We know $f_i(\tilde{R}) = p(\tilde{R}_i)$. Because $\tilde{R} \in NE(R)$, $f_i(\tilde{R})\ R_i\ f_i(\tilde{R}_i, \bar{R}_{-i})$. Clearly, we cannot have that $p(R_i) \leq f_i(\tilde{R}_i, \bar{R}_i) < f_i(\tilde{R})$ by strategy-proofness. We also cannot have $f_i(\tilde{R}_i, \bar{R}_{-i}) < p(R_i) < f_i(\tilde{R})$ by Lemma A.4. Consequently,

$$f_i(\bar{R}) \leq f_i(\tilde{R}_i, \bar{R}_{-i}) < f_i(\tilde{R}) \leq p(R_i) \;\&\; f(\tilde{R})\ P_i\ f(\tilde{R}_i, \bar{R}_{-i})\ P_i\ f(\bar{R}). \tag{A.12}$$

Thus, $i \in \tilde{E}$ and given that agent i is chosen arbitrarily from T, we find that $T \subseteq \tilde{E} \neq \emptyset$.

Let us now go back and consider $(\tilde{R}_{W\cup S}, \bar{R}_{-W\cup S})$ and $\tilde{R}$ which differ in the preferences of those in T. Between these profiles, the allocations of only those in T increase. Furthermore, the allocation of each agent $i \in T$ increases by at most d_i (see (A.8)). On the other hand, the allocation of those in $\tilde{W}$ must decrease by at least $\sum_{i\in\tilde{W}} d_i$ (see (A.11)). Thus, we find that $\sum_{i\in\tilde{W}} d_i \leq \sum_{i\in T} d_i$. Given that $T \subseteq \tilde{E}$, we have $\sum_{i\in\tilde{W}} d_i \leq \sum_{i\in\tilde{E}} d_i$. The same logic used to find the above inequality also yields that $\sum_{i\in\tilde{E}} d_i \leq \sum_{i\in\tilde{W}} d_i$. Consequently,

$$\sum_{i\in\tilde{E}} d_i = \sum_{i\in T} d_i = \sum_{i\in\tilde{W}} d_i.$$

Thus, we obtain (A.3). Because $T \subseteq \tilde{E} \subseteq E$, the above equation implies that $T = \tilde{E}$. Then because (A.12) must be satisfied for all $i \in T = \tilde{E}$ we obtain (A.5). The rest of the step can be proved similarly.

Step 2. *Allocations for various configurations of profiles $\tilde{R}$ and $\bar{R}$.*
We must have that

$$f_i(\tilde{R}_{-\tilde{E}}, \bar{R}_{\tilde{E}}) = f_i(\tilde{R}_{\bar{W}}, \bar{R}_{-\bar{W}}) = p_i(\bar{R}_i) = f_i(\tilde{R}_{\bar{E}}, \bar{R}_{-\bar{E}}) = f_i(\tilde{R}_{-\tilde{W}}, \bar{R}_{\tilde{W}}) \quad \text{for all } i \in \tilde{W} \cup \tilde{E}$$

$$f_i(\tilde{R}_{-\tilde{E}}, \bar{R}_{\tilde{E}}) = f_i(\tilde{R}_{\bar{W}}, \bar{R}_{-\bar{W}}) = p_i(\tilde{R}_i) = f_i(\tilde{R}_{\bar{E}}, \bar{R}_{-\bar{E}}) = f_i(\tilde{R}_{-\tilde{W}}, \bar{R}_{\tilde{W}}) \quad \text{for all } i \in \bar{W} \cup \bar{E}$$

$$f_i(\tilde{R}_{-\tilde{E}}, \bar{R}_{\tilde{E}}) = f_i(\tilde{R}_{\bar{W}}, \bar{R}_{-\bar{W}}) = p_i(\tilde{R}_i) = f_i(\tilde{R}_{\bar{E}}, \bar{R}_{-\bar{E}}) = f_i(\tilde{R}_{-\tilde{W}}, \bar{R}_{\tilde{W}}) \quad \text{for all } i \in O$$

Proof of Step 2. Recall Fact 1 in Step 1 and that $\tilde{E} = T$. Thus, between profiles $(\tilde{R}_{-\tilde{E}}, \bar{R}_{\tilde{E}})$ and $\tilde{R}$, the allocations of only those in $\tilde{E}$ increase collectively by $\sum_{i\in\tilde{E}}(p(\tilde{R}_i) - f_i(\tilde{R}_{-\tilde{E}}, \bar{R}_{\tilde{E}})) \leq \sum_{i\in\tilde{E}} d_i$. At the same time, the allocations of those in $\tilde{W}$ must decrease collectively by $\sum_{i\in\tilde{W}}(f_i(\tilde{R}_{-\tilde{E}}, \bar{R}_{\tilde{E}}) - p(\tilde{R}_i)) \geq \sum_{i\in\tilde{W}} d_i$ in order to have $f_i(\tilde{R}) = p(\tilde{R}_i)$ for each agent i. Because $\sum_{i\in\tilde{E}} d_i = \sum_{i\in\tilde{W}} d_i$, we find that

$$f_i(\tilde{R}_{-\tilde{E}}, \bar{R}_{\tilde{E}}) = p_i(\bar{R}_i) \text{ for all } i \in \tilde{W} \;\; \& \;\; f_i(\tilde{R}_{-\tilde{E}}, \bar{R}_{\tilde{E}}) = p_i(\bar{R}_i) \text{ for all } i \in \tilde{E}.$$

In addition, because the increase in the allocations of those in $\tilde{E}$ cancels the decrease in the allocations of those in $\tilde{W}$, we must have that

$$f_i(\tilde{R}_{-\tilde{E}}, \bar{R}_{\tilde{E}}) = p_i(\tilde{R}_i) \text{ for all } i \in \bar{W} \cup O \cup \bar{E}.$$

Recall from Step 1, that $f(\tilde{R}_W, \bar{R}_{-W}) = f(\tilde{R}_{-\tilde{E}}, \bar{R}_{\tilde{E}})$. Consider now $(\tilde{R}_{\bar{W}}, \bar{R}_{-\bar{W}})$. By definition, $p(\tilde{R}_i) < p(\bar{R}_i)$ for each $i \in W$. Hence, by Lemma A.6 we know that $f_i(\tilde{R}_{\bar{W}}, \bar{R}_{-\bar{W}}) \geq f_i(\bar{R}) = p(\bar{R}_i) > p(\tilde{R}_i)$ for each $i \in \tilde{W}$. Then by Lemma A.2 and non-bossiness, for each $i \in \tilde{W}$,

$$f_i(\tilde{R}_{\bar{W}\cup i}, \bar{R}_{-(\bar{W}\cup i)}) = f_i(\tilde{R}_{\bar{W}}, \bar{R}_{-\bar{W}}).$$

Then by Lemma A.3, we have that

$$f(\tilde{R}_W, \bar{R}_{-W}) = f(\tilde{R}_{\bar{W}}, \bar{R}_{-\bar{W}}).$$

Given that $f(\tilde{R}_W, \bar{R}_{-W}) = f(\tilde{R}_{-\tilde{E}}, \bar{R}_{\tilde{E}}))$ we complete the proof for the first two equalities in the equation given in Step 2. The last two equalities are proved similarly.

Step 3. *The set of Nash equilibrium allocations is a partially ordered set.*
For all $R \in \mathcal{R}$, $(X^{NE(R)}, \succeq_R)$ is a partially ordered set.

Proof of Step 3. One needs to show that $\succeq_R$ is antisymmetric on $X^{NE(R)}$. Because $\tilde{x} \neq \bar{x}$ are chosen arbitrarily, we only need to show that there exists at least one agent who is not indifferent between $\tilde{x} = f(\tilde{R})$ and $\bar{x} = f(\bar{R})$. This immediately follows from Step 1 and the single-peakedness of preferences.

We are now ready to prove that the set of Nash equilibrium allocations is a lattice. To prove this, we only need to show that $\bar{x} \wedge \tilde{x} = f(\bar{R}) \wedge f(\tilde{R}) \in X^{NE(R)}$ and $\bar{x} \vee \tilde{x} = f(\bar{R}) \vee f(\tilde{R}) \in X^{NE(R)}$. This is obvious if $f(\bar{R})$ and $f(\tilde{R})$ are Pareto comparable. Thus, we complete the proof once we prove the following statement.

Step 4. *Concluding: meets and joins of non-comparable Nash equilibrium allocations.*
If $f(\bar{R})$ and $f(\tilde{R})$ are not Pareto comparable, then $f(\tilde{R}_{\tilde{W}\cup\tilde{E}}, \bar{R}_{\bar{W}\cup\bar{E}}, \bar{R}_O) = f(\bar{R}) \vee f(\tilde{R})$ and $f(\tilde{R}_{\bar{W}\cup\bar{E}}, \bar{R}_{\tilde{W}\cup\tilde{E}}, \bar{R}_O) = f(\bar{R}) \wedge f(\tilde{R})$. In addition, $(\tilde{R}_{\tilde{W}\cup\tilde{E}}, \bar{R}_{\bar{W}\cup\bar{E}}, \bar{R}_O) \in NE(R)$ and $(\tilde{R}_{\bar{W}\cup\bar{E}}, \bar{R}_{\tilde{W}\cup\tilde{E}}, \bar{R}_O) \in NE(R)$.

Proof of Step 4. Let $\hat{R} \in \mathcal{R}$ be such that $\hat{R} = (\tilde{R}_{\tilde{W}\cup\tilde{E}}, \bar{R}_{\bar{E}\cup\bar{W}}, \bar{R}_O)$. By Step 1,

$$\sum_{i\in\tilde{W}\cup\tilde{E}} p(\tilde{R}_i) = \sum_{i\in\tilde{W}\cup\tilde{E}} f_i(\tilde{R}) = \sum_{i\in\tilde{W}\cup\tilde{E}} f_i(\bar{R}) = \sum_{i\in\tilde{W}\cup\tilde{E}} p(\bar{R}_i)$$

and

$$\sum_{i\in\bar{W}\cup\bar{E}} p(\tilde{R}_i) = \sum_{i\in\bar{W}\cup\bar{E}} f_i(\tilde{R}) = \sum_{i\in\bar{W}\cup\bar{E}} f_i(\bar{R}) = \sum_{i\in\bar{W}\cup\bar{E}} p(\bar{R}_i).$$

We also know that for all $i \in O$, $p(\tilde{R}_i) = f_i(\tilde{R}) = f_i(\bar{R}) = p(\bar{R}_i)$ and $\sum_{i\in N} p(\tilde{R}_i) = \sum_{i\in N} p(\bar{R}_i) = \Omega$. Subsequently, $\sum_{i\in N} p(\hat{R}_i) = \Omega$.

By efficiency, we have $f_i(\hat{R}) = p(\hat{R}_i)$ for all $i \in N$. In addition,

$$f_i(\hat{R}) = \begin{cases} f_i(\tilde{R}) & \text{if } i \in \tilde{E} \cup \tilde{W} \\ f_i(\bar{R}) & \text{if } i \in \bar{E} \cup \bar{W} \\ f_i(\tilde{R}) = f_i(\bar{R}) & \text{if } i \in O. \end{cases}$$

We are left to show that $\hat{R} \in NE(R)$. Suppose that $\hat{R} \notin NE(R)$. Then by strategy-proofness, there must exist $i \in N$ such that $f_i(R_i, \hat{R}_{-i})\ P_i\ f_i(\hat{R})$. Of course, it must be that $p(R_i) \neq f_i(\hat{R})$. Without loss of generality, let us assume that $f_i(\hat{R}) < p(R_i)$. We know also that $p(R_j) \leq p(\tilde{R}_j) = f_j(\hat{R})$ for each $j \in \tilde{W}$ and $p(R_j) \leq p(\bar{R}_j) = f_j(\hat{R})$ for each $j \in \bar{E}$ by step 1. Hence, if $f_i(\hat{R}) < p(R_i)$, then $i \in \tilde{E} \cup \bar{W} \cup O$. Suppose that $i \in \tilde{E} \cup O$ and let $V = \tilde{E} \cup \{i\}$. Because $p(\bar{R}_j) < p(\tilde{R}_j) \leq p(R_j)$ for each $j \in \tilde{E}$ and $p(\tilde{R}_j) = p(\bar{R}_j)$ for each $j \in O$, we must have that $p(\bar{R}_i) = f_i(\bar{R}) < p(R_i)$.

Because $f_i(\hat{R}) < p(R_i)$ (by assumption) and $f_i(R_i, \hat{R}_{-i})\ P_i\ f_i(\hat{R})$, we must have that $p(\tilde{R}_i) = f_i(\hat{R}) < f_i(R_i, \hat{R}_{-i})$. By replacement monotonicity, for each $j \neq i$ we have

$$f_j(R_i, \hat{R}_{-i}) \leq f_j(\hat{R}).$$

We now claim that the above inequality can be strict only for agents in $V \setminus \{i\}$. Because $f_i(\bar{R}) < p(R_i)$, $f_j(\bar{R}) < p(\tilde{R}_j)$ for each $j \in T \setminus \{i\}$ and $\bar{R} \in NE(R)$, we must have that $f(R_i, \tilde{R}_{T\setminus\{i\}}, \bar{R}_T) = f(\bar{R})$ by Lemma A.5. Consider now $(R_i, \tilde{R}_{T\setminus\{i\}}, \bar{R}_{-T})$ and $(R_i, \hat{R}_{-i})$ which differ in the preferences of $\tilde{W}$. We know that $p(R_j) \leq p(\bar{R}_j) = f_j(\bar{R}) = f_j(R_i, \tilde{R}_{T\setminus\{i\}}, \bar{R}_T)$ for each $j \in \tilde{W}$. By Lemma A.6, it must be that,

$$p(\tilde{R}_j) \leq f_j(R_i, \hat{R}_{-i}) \text{ for all } i \in \tilde{W} \ \ \&$$
$$p(\bar{R}_j) = f_j(R_i, \tilde{R}_{T\setminus i}, \bar{R}_{-T}) \leq f_j(R_i, \hat{R}_{-i}) \text{ for all } j \notin \tilde{W}.$$

Because $(R_i, \hat{R}_{-i})$ is a report for which there is overdemand,

$$f_i(R_i, \hat{R}_{-i}) \leq p(R_i)$$
$$f_j(R_i, \hat{R}_{-i}) \leq p(\tilde{R}_i) \text{ for all } j \in \tilde{W} \cup \tilde{E} \setminus i$$
$$f_j(R_i, \hat{R}_{-i}) \leq p(\bar{R}_j) \text{ for all } j \in O \cup \bar{W} \cup \bar{E} \setminus \{i\}.$$

By combining the above equations, we find that

$$\begin{aligned} &f_i(R_i, \hat{R}_{-i}) \in [p(\bar{R}_i), p(R_i)] \\ &f_j(R_i, \hat{R}_{-i}) \in [p(\bar{R}_i), p(\tilde{R}_j)] && \text{for all } j \in \tilde{W} \cup \tilde{E} \setminus i \\ &f_j(R_i, \hat{R}_{-i}) = p(\tilde{R}_i) && \text{for all } j \in \tilde{W} \\ &f_j(R_i, \hat{R}_{-i}) = p(\bar{R}_j) && \text{for all } j \in O \cup \bar{W} \cup \bar{E} \setminus \{i\}, \end{aligned}$$

and

$$\sum_{j \in T} f_j(R_i, \hat{R}_{-i}) = \sum_{j \in T} p(\bar{R}_i).$$

The last equality comes from feasibility. However, we already know that $f_i(\hat{R}) = p(\tilde{R}_i) < f_i(R_i, \hat{R}_{-i})$ and $f_j(R_i, \hat{R}_{-i}) \leq f_j(\hat{R}) = p(\tilde{R}_j)$ for all $j \in \tilde{E} \setminus \{i\}$. By combining this with the results above, we prove the claim. Observe here that there must exist at least one agent $i^* \in \tilde{E} \setminus \{i\}$ for whom $f_{i^*}(R_i, \hat{R}_{-i}) < f_{i^*}(\hat{R}) = p(\tilde{R}_{i^*})$.

Consider now $(R_i, \hat{R}_{-i})$ and $(R_i, \bar{R}_{\bar{W}}, \tilde{R}_{-(\bar{W} \cup i)})$ which differ in the preferences of those agents in $\bar{E}$. For each $j \in \bar{E}$, $f_j(R_i, \hat{R}_{-i}) = p(\bar{R}_j) < p(\tilde{R}_j)$. From Lemma A.6, $f_j(R_i, \hat{R}_{-i}) \leq f_j(\bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W}}) \leq p(\tilde{R}_j)$ for all $j \in \bar{E}$, and $f_j(\bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W}}) \leq f_j(R_i, \hat{R}_{-i})$ for all $j \notin \bar{E}$. From Step 2, $f(\bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W}}) = f(\tilde{R})$. The cases in which $f_i(R_i, \bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W} \cup i}) < f_i(\bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W}}) = p(\tilde{R}_i) < p(R_i)$ are incompatible with strategy-proofness.

Suppose that $f_i(R_i, \bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W} \cup i}) = f_i(\bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W}}) = p(\tilde{R}_i)$. Then by non-bossiness we have that $f(R_i, \bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W} \cup i}) = f(\bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W}})$. This means that $f_{i^*}(R_i, \hat{R}_{-i}) < p(\tilde{R}_{i^*}) = f_{i^*}(R_i, \bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W} \cup i}) = f(\bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W}})$ which cannot occur as pointed out earlier. Finally suppose that $f_i(\bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W}}) < f_i(R_i, \bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W} \cup i})$. In this case, consider $(R_i, \bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W} \cup i})$ and $(R_i, \tilde{R}_{-i})$ which differ in the preferences of those agents in $\bar{W}$. Because $p(\tilde{R}_j) < p(\bar{R}_j)$ for each $j \in \bar{W}$ and $i \notin \bar{W}$, by Lemma A.2 and replacement monotonicity, we have that $f_i(\tilde{R}) = p(\tilde{R}_i) < f_i(R_i, \bar{R}_{\bar{W}}, \tilde{R}_{-\bar{W} \cup i}) \leq f_i(R_i, \tilde{R}_{-i})$. Then by Lemma A.4, we must have $p(R_i) \leq f_i(\tilde{R})$ which contradicts $f_i(\tilde{R}) = f_i(\tilde{R}) < p(R_i)$.

We will reach a similar contradiction if $i \in \bar{E}$. This completes the proof that $(\tilde{R}_{\tilde{W} \cup \tilde{E}}, \bar{R}_{\bar{W} \cup \bar{E}}, \bar{R}_O) \in NE(R)$.

Let now $\hat{R} \in \mathcal{R}$ be such that $\hat{R} = (\bar{R}_{\tilde{W} \cup \tilde{E}}, \tilde{R}_{\bar{W} \cup \bar{E}}, \bar{R}_O)$. Similarly to the previous case, we have that $f_i(\hat{R}) = p(\hat{R}_i)$ for all $i \in N$ and

$$f_i(\hat{R}) = \begin{cases} f_i(\bar{R}) & \text{if } i \in \tilde{E} \cup \tilde{W} \\ f_i(\tilde{R}) & \text{if } i \in \bar{E} \cup \bar{W} \\ f_i(\tilde{R}) = f_i(\bar{R}) & \text{if } i \in O. \end{cases}$$

We are left to show that $\hat{R} \in NE(R)$. Suppose that $\hat{R} \notin NE(R)$. Then by strategy-proofness, there must exist $i \in N$ such that $f_i(R_i, \hat{R}_{-i})\ P_i\ f_i(\hat{R})$. Of course, it must be that $p(R_i) \neq f_i(\hat{R})$. Without loss of generality, let us assume that $f_i(\hat{R}) < p(R_i)$. We know also that $p(R_j) \leq p(\tilde{R}_j) < p(\bar{R}_i) = f_j(\hat{R})$ for each $j \in \tilde{W}$ and $p(R_j) \leq p(\bar{R}_j) < p(\tilde{R}_j) = f_j(\hat{R})$ for each $j \in \bar{E}$. Hence, if $f_i(\hat{R}) < p(R_i)$, then $i \in \tilde{E} \cup \bar{W} \cup O$. Suppose that $i \in \tilde{E} \cup O$ and let $V = \tilde{E} \cup \{i\}$. Because $p(\bar{R}_j) < p(\tilde{R}_j) \leq p(R_j)$ for each $j \in \tilde{E}$ and $p(\tilde{R}_j) = p(\bar{R}_j)$ for each $j \in O$, we must have that $f_i(\tilde{R}) < p(R_i)$.

Consider now $(R_i, \tilde{R}_{-i})$. By Lemma A.2, we must have that $p(\tilde{R}_i) = f_i(\tilde{R}) \le f_i(R_i, \tilde{R}_{-i}) \le p(R_i)$. If $f_i(\tilde{R}) < f_i(R_i, \tilde{R}_{-i})$, then $f_i(R_i, \tilde{R}_{-i})\ P_i\ f_i(\tilde{R})$, contradicting that $\tilde{R} \in NE(R)$. Thus, $f_i(\tilde{R}) = f_i(R_i, \tilde{R}_{-i})$ which along with non-bossiness gives that $f(\tilde{R}) = f(R_i, \tilde{R}_{-i})$. Consider now $(R_i, \tilde{R}_{-i})$ and $(R_i, \bar{R}_{\tilde{W}}, \tilde{R}_{-(\tilde{W}\cup i)})$ which differ in the preferences of those in $\tilde{W}$. By definition of $\tilde{W}$, $p(\tilde{R}_j) = f_j(\tilde{R}) = f_j(R_i, \bar{R}_{\tilde{W}}, \tilde{R}_{-(\tilde{W}\cup i)}) < p(R_i)$ for each $j \in \tilde{W}$. By Lemma A.6, we find that $f_i(R_i, \bar{R}_{\tilde{W}}, \tilde{R}_{-(\tilde{W}\cup i)}) \le f_i(R_i, \tilde{R}_{-i}) = p(\tilde{R}_i) < p(R_i)$. By Lemma A.2, we obtain that $f_i(R_i, \bar{R}_{\tilde{W}}, \tilde{R}_{-(\tilde{W}\cup i)}) = f_i(\bar{R}_{\tilde{W}}, \tilde{R}_{-\tilde{W}})$. By non-bossiness, $f(R_i, \bar{R}_{\tilde{W}}, \tilde{R}_{-(\tilde{W}\cup i)}) = f(\bar{R}_{\tilde{W}}, \tilde{R}_{-\tilde{W}})$. We know from Step 2, $f(\bar{R}_{\tilde{W}}, \tilde{R}_{-\tilde{W}}) = f(\hat{R})$. Finally, consider $(R_i, \bar{R}_{\tilde{W}}, \tilde{R}_{-(\tilde{W}\cup i)})$ and $(R_i, \hat{R}_{-i})$ which differ in the preferences of those in $V \setminus \{i\}$. For each $j \in T \setminus \{i\}$, we have that $p(\bar{R}_j) = f_j(R_i, \bar{R}_{\tilde{W}}, \tilde{R}_{-(\tilde{W}\cup i)}) = f_j(\bar{R}_{\tilde{W}}, \tilde{R}_{-\tilde{W}})$. Then by Lemma A.2 and non-bossiness, $f(R_i, \bar{R}_j, \bar{R}_{\tilde{W}}, \tilde{R}_{-(\tilde{W}\cup i\cup j)}) = f(R_i, \bar{R}_{\tilde{W}}, \tilde{R}_{-(\tilde{W}\cup i)})$ for all $j \in T \setminus \{i\}$. By Lemma A.3, we have that $f(R_i, \hat{R}_{-i}) = f(R_i, \bar{R}_{\tilde{W}}, \tilde{R}_{-(\tilde{W}\cup i)}) = f(\hat{R})$, a contradiction. Thus, $\hat{R} = (\bar{R}_{\tilde{W}\cup\tilde{E}}, \tilde{R}_{\bar{W}\cup\bar{E}}, \bar{R}_O) \in NE(R)$. □

We now prove our main Theorem establishing that for each $R \in \mathcal{R}$, $(X^{NE(R)}, \succeq_R)$ is a complete lattice. Since we just showed that $(X^{NE(R)}, \succeq_R)$ is a lattice, we only need to establish the completeness of the lattice and the fact that its supremum is well-identified as the allocation under truthtelling.

Proof of Theorem 3.3. We prove this theorem in three steps.

Step 1: For each $R \in \mathcal{R}$, the supremum of $(X^{NE(R)}, \succeq_R)$ is $x^{\sup} = f(R)$.

Proof of Step 1: Let $R \in \mathcal{R}$. Contrary to the statement assume that for some $i \in N$ and $x \in X^{NE(R)}$, $x\ P_i\ f_i(R)$. By Proposition 3.2, there exists $\bar{R} \in NE(R)$ such that $\sum_{i\in N} p(\bar{R}_i) = \Omega$ and $f(\bar{R}) = x$. By efficiency, $p(\bar{R}_i) = f_i(\bar{R})$ for all $i \in N$. If $\sum_{i\in N} p(R_i) = \Omega$, then again by efficiency, $f_i(R) = p(R_i)$ for each $i \in N$. Thus, $f_i(R)\ R_i\ f_i(\bar{R})$ for each $i \in N$, which is a contradiction. Without loss of generality assume that $\sum_{i\in N} p(R_i) < \Omega$.

Let $T \equiv \{i \in N : p(R_i) \le f_i(\bar{R})\}$. It must be that $T \ne \emptyset$ because $\sum p(R_i) < \Omega = \sum f_i(\bar{R})$ and $p(\bar{R}_i) = f_i(\bar{R})$ for all i. In addition, $\sum_{i\in T} p(R_i) + \sum_{i\in N\setminus T} p(\bar{R}_i) < \Omega$. In fact, for each S with $T \subseteq S$, we have that $\sum_{i\in S} p(R_i) + \sum_{i\in N\setminus S} p(\bar{R}_i) < \Omega$. Fix any $i \in T$. If $p(R_i) = f_i(\bar{R})$, then $f_i(R_i, \bar{R}_{-i}) = f_i(\bar{R})$. If $p(R_i) < f_i(\bar{R})$, then by Lemma A.2 we obtain that $f_i(R_i, \bar{R}_{-i}) = f_i(\bar{R})$. Consequently, Lemma A.3 yields $f(R_T, \bar{R}_{-T}) = f(\bar{R})$. If $T = N$, we are done and $f(\bar{R}) = f(R)$. Suppose that $T \ne N$. Consider now $(R_T, \bar{R}_{-T})$ and R. We know that for each $i \in N \setminus T$, $f_i(R_T, \bar{R}_{-T}) = f_i(\bar{R}) < p(R_i)$. Hence, by Lemma A.6, we know that

$$f_i(R) \le p(R_i) \text{ for all } i \in N \setminus T \ \ \&\ \ f_i(R) \le f_i(R_T, \bar{R}_{-T}) \text{ for all } i \in T.$$

By efficiency, it is the case that $p(R_i) \le f_i(R)$ for all $i \in N$. By combining this with the above two conditions we find that

$$\begin{aligned} &f_i(R) = p(R_i) && \text{for all } i \in N \setminus T \\ &f_i(R) \in [p(R_i), f(R_T, \bar{R}_{-T})] = [p(R_i), f(\bar{R})] && \text{for all } i \in T. \end{aligned}$$

Consequently, $f(\bar{R}) \ne f(R)$, $f_i(R)\ P_i\ f_i(\bar{R})$ for each $i \in N \setminus T$ and $f_i(R)\ R_i\ f_i(\bar{R})$ for all $i \in T$. Thus, $f(R) \succ_R f(\bar{R}) = x$.

Before we can conclude the proof that $(X^{NE}, \succeq_R)$ is a complete lattice, we need to show that the set of Nash equilibrium allocations is a closed set.

Step 2. For each $R \in \mathcal{R}$, $X^{NE(R)}$ is a closed set.

Proof of Step 2. Let $R \in \mathcal{R}$ and pick any convergent sequence $\{x^t\} \to \tilde{x}$ where $x^t \in X^{NE(R)}$. We need to show that $\tilde{x} \in X^{NE(R)}$. Let $\{R^t\}$ be a sequence of preference profiles with $p(R_i^t) = x_i^t$. By Proposition 3.2, we know that $R^t \in NE(R)$ for each $t \in \mathbb{N}$. Let $\tilde{R} \in \mathcal{R}$ be a preference profile with $p(\tilde{R}_i) = \tilde{x}_i$ for all $i \in N$. We now show that $\tilde{R} \in NE(R)$. Suppose otherwise. Then by strategy-proofness, there exists $i \in N$ such that $f_i(R_i, \tilde{R}_{-i})\ P_i\ f_i(\tilde{R})$. Without loss of generality, assume that $f_i(\tilde{R}) = p(\tilde{R}_i) < f_i(R_i, \tilde{R}_{-i})$. By Lemma A.2, $p(\tilde{R}_i) = f_i(\tilde{R}) < f_i(R_i, \tilde{R}_{-i}) < p(R_i)$. Because $f(R^t) \to_{t\to\infty} f(\tilde{R})$, there exists $\bar{t}$ such that $f_i(R^t) < p(R_i)$ for all $t \geq \bar{t}$. For any $t \geq \bar{t}$, consider $f_i(R_i, R_{-i}^t)$. By Lemma A.2, $p(R_i^t) = f_i(R^n) \leq f_i(R_i, R_{-i}^t) \leq p(R_i)$. Because $R^t \in NE(R)$ and $p(R_i^t) < p(R_i)$, we must have $f_i(R^n) = f_i(R_i, R_{-i}^t)$. As a result, $f_i(R_i, R_{-i}^t) < f_i(R_i, \tilde{R}_{-i})$. In addition, by non-bossiness, $f(R^t) = f(R_i, R_{-i}^t)$. Consequently,

$$\lim_{t\to\infty} f(R_i, R_{-i}^t) = \lim_{t\to\infty} f(R^t) = f(\tilde{R}). \tag{A.13}$$

Let $\epsilon^t \equiv \max_{j\neq i} |p(\tilde{R}_j) - p(R_j^t)|$. Let us now reach $(R_i, \tilde{R}_{-i})$ from (R_i, R_{-i}^t) by sequentially changing the agents' preferences. At any step of this process, by Lemma A.2, the allocation of the agent whose preference is modified changes at most by ϵ^t. Hence, by replacement monotonicity, i's allocation changes at most by ϵ^t at any step or by $(n-1)\epsilon^t$ as a result of this whole process. However, $\epsilon^t \to_{t\to\infty} 0$. Hence,

$$\lim_{t\to\infty} f_i(R_i, R_{-i}^t) = f_i(R_i, \tilde{R}_{-i}) > p(\tilde{R}_i) = f_i(\tilde{R}).$$

This contradicts (A.13).

We are now ready to conclude.

Step 3. For each $R \in \mathcal{R}$, $(X^{NE}(R), \succeq_R)$ is a complete lattice.

Proof of Step 3. Pick any $Y \subseteq X^{NE(R)}$. We need to show that both the meet and join of Y exists on $X^{NE(R)}$. We only show this for the meet. Denote the closure of Y by $cl(Y)$. By Step 2, $cl(Y) \subseteq X^{NE(R)}$. In addition, because X is bounded so is $X^{NE(R)}$. Consequently, $cl(Y)$ is compact. Because R_i is continuous for each $i \in N$, there must exist $y^i \in cl(Y)$ such that $x\ R_i\ y^i$ for any $x \in cl(Y)$. Since N is finite, $\bigwedge_i y^i \in X^{NE(R)}$. Clearly, $\bigwedge_i y^i = \bigwedge cl(Y)$. We are now left to show that $\bigwedge_i y^i$ is the meet of Y on $X^{NE(R)}$. This is obvious if $y^i \in Y$ for all $i \in N$. Suppose this is not the case. Because $Y \subset cl(Y)$, $\bigwedge_i y^i$ is a lower bound of Y on $X^{NE(R)}$. Thus, if $\bigwedge_i y^i$ is not the meet of Y then there exists another lower bound of Y in $X^{NE(R)}$, say y, that Pareto dominates $\bigwedge_i y^i$. Fix an agent j for whom $y\ P_j\ \bigwedge_i y^i\ I_j\ y^j$. By construction, if $y^j \in Y$, then $y\ P_j\ y^j$. Thus, y is not a lower bound. As a result, $y^j \notin Y$. However, because $y^j \in cl(Y)$ there exists an allocation $x \in Y$ which is arbitrarily close y^j. By continuity of R_j, $y\ P_j\ x$. This contradicts that y is a lower bound of Y on $X^{NE(R)}$. Hence, $\bigwedge_i y^i = \bigwedge Y$ □

We now prove that when the initial guaranteed levels are invariant to regime changes, the infimum of the lattice is the equal division allocation.

Proof of Proposition 3.4. Let $R, \bar{R} \in \mathcal{R}$ be such that $p(\bar{R}_i) = x_i^*$ for each $i \in N$. We first show that $\bar{R} \in NE(R)$. Clearly, $f(\bar{R}) = x^*$. Under any sequential allotment rule, every agent $i \in N$ who reports $\bar{R}_i$ will get allocated exactly x_i^* regardless of the others' reports. Thus, if some agent i^* reports some other $\tilde{R}_{i^*}$ then $f_{i^*}(\tilde{R}_{i^*}, \bar{R}_{-i^*}) = x_i^*$. Thus, $\bar{R} \in NE(R)$.

Consider any $\tilde{R} \in NE(R)$. We now show that as long as $f(\tilde{R}) \neq f(\bar{R})$, $f(\tilde{R})$ Pareto dominates $f(\bar{R})$. We already mentioned that any agent $i \in N$ can obtain the initial guaranteed level x_i^* regardless of the others' reports. Hence, for each $i \in N$, $f_i(\tilde{R})\ R_i\ f_i(\bar{R})$. Because $f(\tilde{R}) \neq f(\bar{R})$, there must exist an agent j for whom $p(\tilde{R}_j) < x_j^* = p(\bar{R}_j)$. Because $f_j(\bar{R}_j, \tilde{R}_{-j}) = x_j^* < f_j(\tilde{R})$ and $\tilde{R} \in NE(R)$, by Lemma A.4, we have that $p(R_j) \leq f_j(\tilde{R})$. Consequently, $f_j(\tilde{R})\ P_j\ f_j(\bar{R})$. By combining this with $f_i(\tilde{R})\ R_i\ f_i(\bar{R}_i, \tilde{R}_{-i}) = f_i(\bar{R})$ for each $i \in N$, we obtain that $f(\tilde{R}) \succ_R f(\bar{R})$. Hence $x^* = x^{inf}$. □

We now prove that for the class of weighted uniform rules, for each $R \in \mathcal{R}$ the set $(X^{NE(R}, \succeq_R)$ is totally ordered.

Proof of Proposition 3.6. Let f be the weighted uniform rule with respect to weights w^H and w^L. Consider $R \in \mathcal{R}$. In contrast to the theorem, suppose that there exist two Pareto incomparable allocations $\bar{x}, \tilde{x} \in X^{NE(R)}$. Let $\bar{R}, \tilde{R} \in \mathcal{R}$ be such that for each $i \in N$, $p(\bar{R}_i) = \bar{x}_i$ and $p(\tilde{R}_i) = \tilde{x}_i$. By Proposition 3.2, $\bar{R}, \tilde{R} \in NE(R)$. As in the proof of Theorem A.7, define $\tilde{W}$, $\bar{W}$, $\bar{E}$, $\tilde{E}$ and O. For $\tilde{x}$ and $\bar{x}$ be Pareto incomparable, the first four sets must be nonempty by Step 1 in the proof of Theorem A.7. Because $\tilde{x} \neq \bar{x}$, at least one of the following two cases must be satisfied: $\min_{j \in N}\left\{\frac{f_j(\bar{R})}{w_j^L}\right\} \neq \max_{j \in N}\left\{\frac{f_j(\bar{R})}{w_j^L}\right\}$ and $\min_{j \in N}\left\{\frac{f_j(\tilde{R})}{w_j^L}\right\} \neq \max_{j \in N}\left\{\frac{f_j(\tilde{R})}{w_j^L}\right\}$. Assume without loss of generality that the former is satisfied. Fix $i^* \in \tilde{W}$. By Step 1 of the proof of Theorem A.7, we have that $p(R_{i^*}) < p(\bar{R}_{i^*}) < f_{i^*}(\bar{R})$. We claim that it cannot be $\min_{j \in N}\left\{\frac{f_j(\bar{R})}{w_j^L}\right\} < \frac{f_{i^*}(\bar{R})}{w_{i^*}^L}$. Otherwise, because f is a weighted uniform rule, i^* can decrease its allocation slightly by reporting a preference with a slightly smaller peak. Thus, the single-peakedness of R_{i^*} and $\bar{R} \in NE(R)$ give that $f_{i^*}(\bar{R}) \leq p(R_{i^*})$, a contradiction. Thus, we have that

$$\frac{f_{i^*}(\bar{R})}{w_{i^*}^L} = \min_{j \in N}\left\{\frac{f_j(\bar{R})}{w_j^L}\right\}. \tag{A.14}$$

Fix $j^* \in \bar{E}$. If $\min_{j \in N}\left\{\frac{f_j(\tilde{R})}{w_j^L}\right\} \neq \max_{j \in N}\left\{\frac{f_j(\tilde{R})}{w_j^L}\right\}$, then the same logic as above gives that

$$\frac{f_{j^*}(\tilde{R})}{w_{j^*}^L} = \min_{j \in N}\left\{\frac{f_j(\tilde{R})}{w_j^L}\right\}. \tag{A.15}$$

Of course, the relation above is true if $\min_{j \in N}\left\{\frac{f_j(\tilde{R})}{w_j^L}\right\} = \max_{j \in N}\left\{\frac{f_j(\tilde{R})}{w_j^L}\right\}$.

By (A.14) and (A.15), we have that $\frac{f_{i^*}(\bar{R})}{w_{i^*}^L} \leq \frac{f_{j^*}(\bar{R})}{w_{j^*}^L}$ and $\frac{f_{j^*}(\tilde{R})}{w_{j^*}^L} \leq \frac{f_{i^*}(\tilde{R})}{w_{i^*}^L}$. At the same time, by the definition of $\tilde{W}$ and $\bar{E}$, we have that $f_{i^*}(\tilde{R}) < f_{i^*}(\bar{R})$ and $f_{j^*}(\bar{R}) < f_{j^*}(\tilde{R})$. These two inequalities are not compatible with the preceding two inequalities. Thus, either $\tilde{W} = \emptyset$ or $\bar{E} = \emptyset$. This is the contradiction we are looking for. □

Proof of Proposition 4.4. Suppose by contradiction that f violates non-bossiness. Then there exist $i \in N$, $R \in \mathcal{R}$ and $\bar{R}_i \in \mathcal{R}_i$ such that $f_i(R) = f_i(\bar{R}_i, R_{-i})$ and $f(R) \neq f(\bar{R}_i, R_{-i})$. By efficiency, either (i) there exists $k \neq i$ such that $f_k(\bar{R}_i, R_{-i})\ P_k\ f_k(R)$ or (ii) $f_j(\bar{R}_i, R_{-i})\ I_j\ f_j(R)$ for all $j \neq i$. Strategy-proofness and the fact that $f_i(R) = f_i(\bar{R}_i, R_{-i})$ together imply that $\bar{R}_i$ is i's best response to $(\bar{R}_i, R_{-i})$. By strategy-proofness, for each $j \neq i$ the true report R_j is a best response to $(\bar{R}_i, R_{-i})$. Hence, $(\bar{R}_i, R_{-i}) \in NE(R)$. Because $f(R)$ is the supremum of the complete lattice $(X^{NE(R)}, \succeq_R)$, we have $f_j(R)\ R_j\ f_j(\bar{R}_i, R_{-i})$ for all $j \neq i$. Hence, we can dispose of case (i) immediately. In addition, in case (ii), we have two suprema which is a contradiction. □

Here we simply show that efficiency can be replaced by same-sidedness for the complete lattice result, i.e., we show that it is equivalent to efficiency in the Sprumont model.

Proof of Corollary 4.5. We only need to show that same-sidedness implies efficiency. Suppose otherwise. Consequently, there exist $R \in \mathcal{R}$ and $x \in X$ such that $x \succ_R f(R)$. Clearly, $x \neq f(R)$. If $\sum_{i \in N} p(R_i) = \Omega$ then same-sidedness would require that $f(R) = p(R)$. Hence, $f(R) \succeq_R x$, a contradiction. Thus, $\sum_{i \in N} p(R_i) \neq \Omega$. Without loss of generality, assume $\sum_{i \in N} p(R_i) < \Omega$. By same-sidedness, $p(R_i) \leq f_i(R)$ for all $i \in N$. Because $x \neq f(x)$, there exists $j \in N$ with $f_j(R) < x_j$. By single-peakedness, $f_j(R)\ P_j\ x$, a contradiction. □

References

Adachi, H., 2000. On a characterization of stable matchings. Econ. Lett. 68 (1), 43–49.

Adachi, T., 2010. The uniform rule with several commodities: a generalization of Sprumont's characterization. J. Math. Econ. 46 (1), 952–964.

Barberà, S., Berga, D., Moreno, B., 2016. Group strategy-proofness in private good economies. Am. Econ. Rev. 106 (4), 1073–1099.

Barberà, S., Jackson, M., 1994. A characterization of strategy-proof social choice functions for economies with public goods. Soc. Choice Welf. 11, 241–252.

Barberà, S., Jackson, M., Neme, A., 1997. Strategy-proof allotment rules. Games Econ. Behav. 18 (1), 1–21.

Bochet, O., Ilkilic, R., Moulin, H., 2013. Egalitarianism under earmark constraints. J. Econ. Theory 148 (2), 535–562.

Bochet, O., Sakai, T., 2010. Secure implementation in allotment economies. Games Econ. Behav. 68 (1), 35–49.

Bochet, O., Sakai, T., Thomson, W., 2019. Preference Manipulations Lead to the Uniform Rule. Working paper. NYU Abu Dhabi.

Bochet, O., Tumennasan, N., 2017. Prevalence of Truthtelling and Implementation. Working Paper.

Ching, S., 1994. An alternative characterization of the uniform rule. Soc. Choice Welf. 11 (2), 131–136.

Ergin, H., Sönmez, T., 2006. Games of school choice under the Boston mechanism. J. Public Econ. 90 (1–2).

Gibbard, A., 1973. Manipulation of voting schemes: a general result. Econometrica 41 (4), 587–601.

Hatfield, J., Milgrom, P., 2005. Matching with contracts. Am. Econ. Rev. 95 (4), 913–935.

Knuth, D.E., 1976. Mariages Stables. Les Presses de I'Universite de Montreal, Montreal.

Massó, J., Neme, A., 2004. A maximal domain of preferences for strategy-proof, efficient, and simple rules in the division problem. Soc. Choice Welf. 23 (2), 187–206.

Mizukami, H., Wakayama, T., 2007. Dominant strategy implementation in economic environments. Games Econ. Behav. 60 (2), 307–325.

Moulin, H., 1980. On strategy-proofness and single peakedness. Public Choice 35 (4), 437–455.

Moulin, H., 1999. Rationing a commodity along fixed paths. J. Econ. Theory 84 (1), 41–72.

Sakai, A., Wakayama, T., 2011. Uniform, Equal Division, and Other Envy-Free Rules Between the Two. Technical report.

Satterthwaite, M., 1975. Strategy-proofness and arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. J. Econ. Theory 10 (2), 187–217.

Sprumont, Y., 1991. The division problem with single-peaked preferences: a characterization of the uniform allocation rule. Econometrica 59 (2), 509–519.

Thomson, W., 1993. The replacement principle in public good economies with single-peaked preferences. Econ. Lett. 42 (1), 31–36.
Thomson, W., 1994a. Resource-monotonic solutions to the problem of fair division when preferences are single-peaked. Soc. Choice Welf. 11 (3), 205–223.
Thomson, W., 1994b. Consistent solutions to the problem of fair division when preferences are single-peaked. J. Econ. Theory 63 (2), 219–245.
Thomson, W., 1995. Population-monotonic solutions to the problem of fair division when preferences are single-peaked. Econ. Theory 5 (2), 229–246.
Thomson, W., 1997. The replacement principle in economies with single-peaked preferences. J. Econ. Theory 76 (1), 145–168.
Thomson, W., 2014. Fully Allocating a Commodity Among Agents with Single-Peaked Preferences. Technical report.
Thomson, W., 2016. Non-bossiness. Soc. Choice Welf. 47 (3), 665–696.