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A stabilizer-free weak Galerkin finite element method on polytopal meshes

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ABSTRACT

A stabilizing/penalty term is often used in finite element methods with discontinuous approximations to enforce connection of discontinuous functions across element boundaries. Removing stabilizers from discontinuous Galerkin finite element methods will simplify formulations and reduce programming complexity significantly. The goal of this paper is to introduce a stabilizer free weak Galerkin (WG) finite element method for second order elliptic equations on polytopal meshes. This new WG method keeps a simple symmetric positive definite form and can work on polygonal/polyhedral meshes. Optimal order error estimates are established for the corresponding WG approximations in both a discrete H^1 norm and the L^2 norm. Numerical results are presented verifying the theorem.

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1. Introduction

We consider Poisson equation with a homogeneous Dirichlet boundary condition in d dimension as our model problem for the sake of clear presentation. This stabilizer free weak Galerkin method can also be used for other partial differential equations. The Poisson problem seeks an unknown function u satisfying

$$-\Delta u = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a polytopal domain in $\mathbb{R}^d$.

The weak form of the problem (1.1)–(1.2) is to find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{1.3}$$

The H^1 conforming finite element method for the problem (1.1)–(1.2) keeps the same simple form as in (1.3): find $u_h \in V_h \subset H_0^1(\Omega)$ such that

$$(\nabla u_h, \nabla v) = (f, v) \quad \forall v \in V_h, \tag{1.4}$$

where V_h is a finite dimensional subspace of $H_0^1(\Omega)$. The functions in V_h are required to be continuous, which makes the classic finite element formulation (1.4) less flexible in element constructions and in mesh generations. In contrast, finite

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element methods using discontinuous approximations have two advantages: 1. easy construction of high order elements and avoiding constructing some special elements such as C^1 conforming elements; 2. easy working on general meshes. Therefore, discontinuous finite element methods are the most active research area in the context of finite element methods for the past two decades. Discontinuous approximation was first used in finite element procedure as early as in 1970s [1–4]. Local discontinuous Galerkin methods were introduced in [5]. Then a paper [6] in 2002 provides a unified analysis of discontinuous Galerkin finite element methods for Poisson equation. More discontinuous finite element methods have been developed such as hybridizable discontinuous Galerkin method [7], mimetic finite differences method [8], hybrid high-order method [9], virtual element method [10], weak Galerkin method [11] and references therein.

One obvious disadvantage of discontinuous finite element methods is their rather complex formulations which are often necessary to enforce weak continuity of discontinuous solutions across element boundaries. Most of discontinuous finite element methods have one or more stabilizing terms to guarantee stability and convergence of the methods. Existing of stabilizing terms further complicates formulations. Complexity of discontinuous finite element methods makes them difficult to be implemented and to be analyzed. The purpose of this paper is to obtain a finite element formulation close to its original PDE weak form (1.3) for discontinuous polynomials. We believe that finite element formulations for discontinuous approximations can be as simple as follows:

$$(\nabla_w u_h, \nabla_w v) = (f, v), \tag{1.5}$$

if ∇_w, an approximation of gradient, is appropriately defined. The formulation (1.5) can be viewed as the counterpart of (1.4) for discontinuous approximations. In fact such an ultra simple formulation (1.5) has been achieved for one kind of WG method in [11], and for the conforming DG methods in [12,13]. The lowest order WG method developed in [11] has been improved in [14] for convex polygonal meshes, in which non-polynomial functions are used for computing weak gradient.

In this paper, we develop a WG finite element method that has an ultra simple formulation (1.5) and can work on polytopal meshes for any polynomial degree $k \ge 1$. The idea is to raise the degree of polynomials used to compute weak gradient ∇_w. Using higher degree polynomials in computation of weak gradient will not change the size, neither the global sparsity of the stiffness matrix. On the other side, the simple formulation of the stabilizer free WG method (1.5) will reduce programming complexity significantly. Optimal order error estimates are established for the corresponding WG approximations in both a discrete H^1 norm and the L^2 norm. Numerical results are presented verifying the theorem.

2. Weak Galerkin finite element schemes

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [15]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges or flat faces. For every element $T \in \mathcal{T}_h$, we denote by h_T its diameter and mesh size $h = \max_{T\in\mathcal{T}_h} h_T$ for $\mathcal{T}_h$.

We start by introducing weak function $v = \{v_0, v_b\}$ on element $T \in \mathcal{T}_h$ such that

$$v = \begin{cases} v_0 & \text{in } T, \\ v_b & \text{on } \partial T. \end{cases}$$

If v is continuous on Ω, then $v = \{v, v\}$.

For a given integer $k \ge 1$, let V_h be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined as follows

$$V_h = \{v = \{v_0, v_b\} :\ v_0|_T \in P_k(T),\ v_b|_e \in P_k(e),\ e \subset \partial T, T \in \mathcal{T}_h\} \tag{2.1}$$

and its subspace V_h^0 is defined as

$$V_h^0 = \{v :\ v \in V_h,\ v_b = 0 \text{ on } \partial\Omega\}. \tag{2.2}$$

We would like to emphasize that any function $v \in V_h$ has a single value v_b on each edge $e \in \mathcal{E}_h$.

For given $T \in \mathcal{T}_h$ and $v = \{v_0, v_b\} \in V_h + H^1(\Omega)$, a weak gradient $\nabla_w v \in [P_j(T)]^d$ $(j > k)$ is defined as the unique polynomial satisfying

$$(\nabla_w v, \mathbf{q})_T = -(v_0, \nabla\cdot\mathbf{q})_T + \langle v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \qquad \forall \mathbf{q} \in [P_j(T)]^d, \tag{2.3}$$

where j will be specified later.

Let Q_0 and Q_b be the two element-wise defined L^2 projections onto $P_k(T)$ and $P_k(e)$ with $e \subset \partial T$ on T respectively. Define $Q_h u = \{Q_0 u, Q_b u\} \in V_h$. Let $\mathbb{Q}_h$ be the element-wise defined L^2 projection onto $[P_j(T)]^d$ on each element T.

For simplicity, we adopt the following notations,

$$(v, w)_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} (v, w)_T = \sum_{T\in\mathcal{T}_h} \int_T v w d\mathbf{x},$$

$$\langle v, w\rangle_{\partial\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} \langle v, w\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} \int_{\partial T} v w ds.$$

Weak Galerkin Algorithm 1. A numerical approximation for (1.1)–(1.2) can be obtained by seeking $u_h = \{u_0, u_b\} \in V_h^0$ satisfying the following equation:

$$(\nabla_w u_h, \nabla_w v)_{\mathcal{T}_h} = (f,\ v_0) \quad \forall v = \{v_0, v_b\} \in V_h^0. \tag{2.4}$$

Lemma 2.1. *Let $\phi \in H^1(\Omega)$, then on any $T \in \mathcal{T}_h$,*

$$\nabla_w \phi = \mathbb{Q}_h \nabla \phi. \tag{2.5}$$

Proof. Using (2.3) and integration by parts, we have that for any $\mathbf{q} \in [P_j(T)]^d$

$$\begin{aligned}(\nabla_w \phi, \mathbf{q})_T &= -(\phi, \nabla \cdot \mathbf{q})_T + \langle \phi, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\nabla \phi, \mathbf{q})_T = (\mathbb{Q}_h \nabla \phi, \mathbf{q})_T,\end{aligned}$$

which implies the desired identity (2.5). □

3. Well posedness

For any $v \in V_h + H^1(\Omega)$, let

$$|||v|||^2 = (\nabla_w v, \nabla_w v)_{\mathcal{T}_h}. \tag{3.1}$$

We introduce a discrete H^1 semi-norm as follows:

$$\|v\|_{1,h} = \left(\sum_{T \in \mathcal{T}_h} \left(\|\nabla v_0\|_T^2 + h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 \right) \right)^{\frac{1}{2}}. \tag{3.2}$$

It is easy to see that $\|v\|_{1,h}$ defines a norm in V_h^0. The following lemma indicates that $\|\cdot\|_{1,h}$ is equivalent to the $|||\cdot|||$ in (3.1).

Lemma 3.1. *There exist two positive constants C_1 and C_2 such that for any $v = \{v_0, v_b\} \in V_h$, we have*

$$C_1 \|v\|_{1,h} \le |||v||| \le C_2 \|v\|_{1,h}. \tag{3.3}$$

Proof. For any $v = \{v_0, v_b\} \in V_h$, it follows from the definition of weak gradient (2.3) and integration by parts that

$$(\nabla_w v, \mathbf{q})_T = (\nabla v_0, \mathbf{q})_T + \langle v_b - v_0, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \mathbf{q} \in [P_j(T)]^d. \tag{3.4}$$

By letting $\mathbf{q} = \nabla_w v$ in (3.4) we arrive at

$$(\nabla_w v, \nabla_w v)_T = (\nabla v_0, \nabla_w v)_T + \langle v_b - v_0, \nabla_w v \cdot \mathbf{n} \rangle_{\partial T}.$$

From the trace inequality (4.5) and the inverse inequality we have

$$\begin{aligned}\|\nabla_w v\|_T^2 &\le \|\nabla v_0\|_T \|\nabla_w v\|_T + \|v_0 - v_b\|_{\partial T} \|\nabla_w v\|_{\partial T} \\ &\le \|\nabla v_0\|_T \|\nabla_w v\|_T + C h_T^{-1/2} \|v_0 - v_b\|_{\partial T} \|\nabla_w v\|_T,\end{aligned}$$

which implies

$$\|\nabla_w v\|_T \le C \left(\|\nabla v_0\|_T + h_T^{-1/2} \|v_0 - v_b\|_{\partial T} \right),$$

and consequently

$$|||v||| \le C_2 \|v\|_{1,h}.$$

Next we will prove $C_1 \|v\|_{1,h} \le |||v|||$. For $v \in V_h$ and $\mathbf{q} \in [P_j(T)]^d$, by (2.3) and integration by parts, we have

$$(\nabla_w v, \mathbf{q})_T = (\nabla v_0, \mathbf{q})_T + \langle v_b - v_0, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}. \tag{3.5}$$

Let n be the number of the edges/faces on a polygon/polyhedron. It has been proved in [13] that there exists $\mathbf{q}_0 \in [P_j(T)]^d$, $j = n + k - 1$, such that

$$(\nabla v_0, \mathbf{q}_0)_T = 0, \quad \langle v_b - v_0, \mathbf{q}_0 \cdot \mathbf{n} \rangle_{\partial T \setminus e} = 0, \quad \langle v_b - v_0, \mathbf{q}_0 \cdot \mathbf{n} \rangle_e = \|v_0 - v_b\|_e^2, \tag{3.6}$$

and

$$\|\mathbf{q}_0\|_T \le C h_T^{1/2} \|v_b - v_0\|_e. \tag{3.7}$$

Substituting $\mathbf{q}_0$ into (3.5), we get

$$(\nabla_w v, \mathbf{q}_0)_T = \|v_b - v_0\|_e^2. \tag{3.8}$$

It follows from Cauchy–Schwarz inequality and (3.7) that

$$\|v_b - v_0\|_e^2 \le C\|\nabla_w v\|_T \|\mathbf{q}_0\|_T \le Ch_T^{1/2}\|\nabla_w v\|_T \|v_0 - v_b\|_e,$$

which implies

$$h_T^{-1/2}\|v_0 - v_b\|_{\partial T} \le C\|\nabla_w v\|_T. \tag{3.9}$$

It follows from the trace inequality, the inverse inequality and (3.9),

$$\|\nabla v_0\|_T^2 \le \|\nabla_w v\|_T\|\nabla v_0\|_T + Ch_T^{-1/2}\|v_0 - v_b\|_{\partial T}\|\nabla v_0\|_T \le C\|\nabla_w v\|_T\|\nabla v_0\|_T.$$

Combining the above estimate and (3.9), by the definition (3.2), we prove the lower bound of (3.3) and complete the proof of the lemma. □

Lemma 3.2. *The weak Galerkin finite element scheme (2.4) has a unique solution.*

Proof. If $u_h^{(1)}$ and $u_h^{(2)}$ are two solutions of (2.4), then $\varepsilon_h = u_h^{(1)} - u_h^{(2)} \in V_h^0$ would satisfy the following equation

$$(\nabla_w \varepsilon_h, \nabla_w v) = 0, \qquad \forall v \in V_h^0.$$

Then by letting $v = \varepsilon_h$ in the above equation we arrive at

$$|||\varepsilon_h|||^2 = (\nabla_w \varepsilon_h, \nabla_w \varepsilon_h) = 0.$$

It follows from (3.3) that $\|\varepsilon_h\|_{1,h} = 0$. Since $\|\cdot\|_{1,h}$ is a norm in V_h^0, one has $\varepsilon_h = 0$. This completes the proof of the lemma. □

4. Error estimates in energy norm

Let $e_h = u - u_h$ and $\epsilon_h = Q_h u - u_h$. Next we derive an error equation that e_h satisfies.

Lemma 4.1. *For any $v \in V_h^0$, the following error equation holds true*

$$(\nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} = \ell(u, v), \tag{4.1}$$

where

$$\ell(u, v) = \langle(\nabla u - \mathbb{Q}_h \nabla u)\cdot \mathbf{n}, v_0 - v_b\rangle_{\partial T_h}.$$

Proof. For $v = \{v_0, v_b\} \in V_h^0$, testing (1.1) by v_0 and using the fact that $\sum_{T\in\mathcal{T}_h}\langle \nabla u\cdot\mathbf{n}, v_b\rangle_{\partial T} = 0$, we arrive at

$$(\nabla u, \nabla v_0)_{\mathcal{T}_h} - \langle \nabla u\cdot \mathbf{n}, v_0 - v_b\rangle_{\partial T_h} = (f, v_0). \tag{4.2}$$

It follows from integration by parts, (2.3) and (2.5) that

$$\begin{aligned}(\nabla u, \nabla v_0)_{\mathcal{T}_h} &= (\mathbb{Q}_h\nabla u, \nabla v_0)_{\mathcal{T}_h}\\ &= -(v_0, \nabla\cdot(\mathbb{Q}_h\nabla u))_{\mathcal{T}_h} + \langle v_0, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &= (\mathbb{Q}_h\nabla u, \nabla_w v)_{\mathcal{T}_h} + \langle v_0 - v_b, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &= (\nabla_w u, \nabla_w v)_{\mathcal{T}_h} + \langle v_0 - v_b, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}.\end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3) gives

$$(\nabla_w u, \nabla_w v)_{\mathcal{T}_h} = (f, v_0) + \ell(u, v). \tag{4.4}$$

The error equation follows from subtracting (2.4) from (4.4),

$$(\nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} = \ell(u, v), \quad \forall v \in V_h^0.$$

This completes the proof of the lemma. □

For any function $\varphi \in H^1(T)$, the following trace inequality holds true (see [15] for details):

$$\|\varphi\|_e^2 \le C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{4.5}$$

Lemma 4.2. *For any $w \in H^{k+1}(\Omega)$ and $v = \{v_0, v_b\} \in V_h^0$, we have*

$$|\ell(w, v)| \le Ch^k |w|_{k+1} |||v|||. \tag{4.6}$$

Proof. Using the Cauchy–Schwarz inequality, the trace inequality (4.5) and (3.3), we have

$$\begin{aligned}
|\ell(w, v)| &= \left| \sum_{T\in\mathcal{T}_h} \langle (\nabla w - \mathbb{Q}_h \nabla w)\cdot \mathbf{n}, v_0 - v_b\rangle_{\partial T} \right| \\
&\le C \sum_{T\in\mathcal{T}_h} \|(\nabla w - \mathbb{Q}_h \nabla w)\|_{\partial T} \|v_0 - v_b\|_{\partial T} \\
&\le C \left(\sum_{T\in\mathcal{T}_h} h_T \|(\nabla w - \mathbb{Q}_h \nabla w)\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\le Ch^k |w|_{k+1} |||v|||,
\end{aligned}$$

which proves the lemma. □

Lemma 4.3. *Let $w \in H^{k+1}(\Omega)$, then*

$$|||w - Q_h w||| \le Ch^k |w|_{k+1}. \tag{4.7}$$

Proof. It follows from (2.3), integration by parts, and (4.5),

$$\begin{aligned}
(\nabla_w (w - Q_h w), \mathbf{q})_T &= -(w - Q_0 w, \nabla\cdot \mathbf{q})_T + \langle w - Q_b w, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \\
&= (\nabla(w - Q_0 w), \mathbf{q})_T + \langle Q_0 w - Q_b w, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \\
&\le \|\nabla(w - Q_0 w)\|_T \|\mathbf{q}\|_T + Ch^{-1/2} \|w - Q_0 w\|_{\partial T} \|\mathbf{q}\|_T \\
&\le Ch^k |w|_{k+1,T} \|\mathbf{q}\|_T .
\end{aligned}$$

Letting $\mathbf{q} = \nabla_w (w - Q_h w)$ in the above equation and taking summation over T, we have

$$|||w - Q_h w||| \le Ch^k |w|_{k+1}.$$

We have proved the lemma. □

Theorem 4.4. *Let $u_h \in V_h^0$ be the weak Galerkin finite element solution of (2.4). Assume the exact solution $u \in H^{k+1}(\Omega)$. Then, there exists a constant C such that*

$$|||u - u_h||| \le Ch^k |u|_{k+1}. \tag{4.8}$$

Proof. It is straightforward to obtain

$$\begin{aligned}
|||e_h|||^2 &= (\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} \\
&= (\nabla_w u - \nabla_w u_h, \nabla_w e_h)_{\mathcal{T}_h} \\
&= (\nabla_w Q_h u - \nabla_w u_h, \nabla_w e_h)_{\mathcal{T}_h} + (\nabla_w u - \nabla_w Q_h u, \nabla_w e_h)_{\mathcal{T}_h} \\
&= (\nabla_w e_h, \nabla_w \epsilon_h)_{\mathcal{T}_h} + (\nabla_w u - \nabla_w Q_h u, \nabla_w e_h)_{\mathcal{T}_h}.
\end{aligned} \tag{4.9}$$

We will bound each term in (4.9). Letting $v = \epsilon_h \in V_h^0$ in (4.1) and using (4.6) and (4.7), we have

$$\begin{aligned}
|(\nabla_w e_h, \nabla_w \epsilon_h)_{\mathcal{T}_h}| &= |\ell(u, \epsilon_h)| \\
&\le Ch^k |u|_{k+1} |||\epsilon_h||| \\
&\le Ch^k |u|_{k+1} |||Q_h u - u_h||| \\
&\le Ch^k |u|_{k+1} (|||Q_h u - u||| + |||u - u_h|||) \\
&\le Ch^{2k} |u|_{k+1}^2 + \frac{1}{4} |||e_h|||^2.
\end{aligned} \tag{4.10}$$

The estimate (4.7) implies

$$\begin{aligned}
|(\nabla_w u - \nabla_w Q_h u, \nabla_w e_h)_{\mathcal{T}_h}| &\le C |||u - Q_h u||| \, |||e_h||| \\
&\le Ch^{2k} |u|_{k+1}^2 + \frac{1}{4} |||e_h|||^2.
\end{aligned} \tag{4.11}$$

Combining the estimates (4.10) and (4.11) with (4.9), we arrive at

$$|||e_h||| \le Ch^k|u|_{k+1},$$

which completes the proof. □

5. Error estimates in L^2 norm

The standard duality argument is used to obtain L^2 error estimate. Recall $e_h = \{e_0, e_b\} = u - u_h$ and $\epsilon_h = \{\epsilon_0, \epsilon_b\} = Q_h u - u_h$. The considered dual problem seeks $\Phi \in H_0^1(\Omega)$ satisfying

$$-\Delta\Phi = \epsilon_0, \quad \text{in } \Omega. \tag{5.1}$$

Assume that the following H^2-regularity holds

$$\|\Phi\|_2 \le C\|\epsilon_0\|. \tag{5.2}$$

Theorem 5.1. *Let $u_h \in V_h^0$ be the weak Galerkin finite element solution of (2.4). Assume that the exact solution $u \in H^{k+1}(\Omega)$ and (5.2) holds true. Then, there exists a constant C such that*

$$\|u - u_0\| \le Ch^{k+1}|u|_{k+1}. \tag{5.3}$$

Proof. Testing (5.1) by ϵ_0 and using the fact that $\sum_{T\in\mathcal{T}_h}\langle\nabla\Phi\cdot\mathbf{n}, \epsilon_b\rangle_{\partial T} = 0$ give

$$\begin{aligned}\|\epsilon_0\|^2 &= -(\Delta\Phi, \epsilon_0)\\ &= (\nabla\Phi,\ \nabla\epsilon_0)_{\mathcal{T}_h} - \langle\nabla\Phi\cdot\mathbf{n},\ \epsilon_0 - \epsilon_b\rangle_{\partial\mathcal{T}_h}.\end{aligned} \tag{5.4}$$

Setting $u = \Phi$ and $v = \epsilon_h$ in (4.3) yields

$$(\nabla\Phi,\ \nabla\epsilon_0)_{\mathcal{T}_h} = (\nabla_w\Phi,\ \nabla_w\epsilon_h)_{\mathcal{T}_h} + \langle\mathbb{Q}_h\nabla\Phi\cdot\mathbf{n},\ \epsilon_0 - \epsilon_b\rangle_{\partial\mathcal{T}_h}. \tag{5.5}$$

Substituting (5.5) into (5.4) gives

$$\begin{aligned}\|\epsilon_0\|^2 &= (\nabla_w\epsilon_h,\ \nabla_w\Phi)_{\mathcal{T}_h} - \langle(\nabla\Phi - \mathbb{Q}_h\nabla\Phi)\cdot\mathbf{n},\ \epsilon_0 - \epsilon_b\rangle_{\partial\mathcal{T}_h}\\ &= (\nabla_w e_h,\ \nabla_w\Phi)_{\mathcal{T}_h} + (\nabla_w(Q_h u - u),\ \nabla_w\Phi)_{\mathcal{T}_h} + \ell(\Phi, \epsilon_h)\\ &= (\nabla_w e_h,\ \nabla_w Q_h\Phi)_{\mathcal{T}_h} + (\nabla_w e_h,\ \nabla_w(\Phi - Q_h\Phi))_{\mathcal{T}_h}\\ &\quad + (\nabla_w(Q_h u - u),\ \nabla_w\Phi)_{\mathcal{T}_h} + \ell(\Phi, \epsilon_h)\\ &= \ell(u, Q_h\Phi) + (\nabla_w e_h,\ \nabla_w(\Phi - Q_h\Phi))_{\mathcal{T}_h} + (\nabla_w(Q_h u - u),\ \nabla_w\Phi)_{\mathcal{T}_h} + \ell(\Phi, \epsilon_h)\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{5.6}$$

Next we will estimate all the terms on the right hand side of (5.6). Using the Cauchy–Schwarz inequality, the trace inequality (4.5) and the definitions of Q_h and Π_h we obtain

$$\begin{aligned}I_1 &= |\ell(u, Q_h\Phi)| \le \left|\langle(\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n},\ Q_0\Phi - Q_b\Phi\rangle_{\partial\mathcal{T}_h}\right|\\ &\le \left(\sum_{T\in\mathcal{T}_h}\|(\nabla u - \mathbb{Q}_h\nabla u)\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi - Q_b\Phi\|_{\partial T}^2\right)^{1/2}\\ &\le C\left(\sum_{T\in\mathcal{T}_h} h\|(\nabla u - \mathbb{Q}_h\nabla u)\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h} h^{-1}\|Q_0\Phi - \Phi\|_{\partial T}^2\right)^{1/2}\\ &\le Ch^{k+1}|u|_{k+1}|\Phi|_2.\end{aligned}$$

It follows from (4.8) and (4.7) that

$$\begin{aligned}I_2 &= |(\nabla_w e_h,\ \nabla_w(\Phi - Q_h\Phi))_{\mathcal{T}_h}| \le C|||e_h|||\,|||\Phi - Q_h\Phi|||\\ &\le Ch^{k+1}|u|_{k+1}|\Phi|_2.\end{aligned}$$

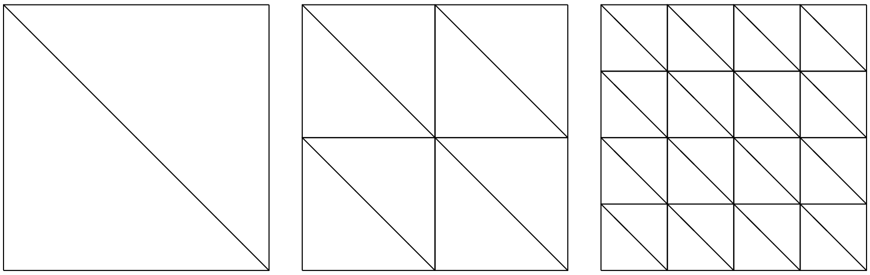

Fig. 6.1. The first three levels of grids used in the computation of Table 6.1.

To bound I_3, we define a L^2 projection element-wise onto $[P_1(T)]^d$ denoted by R_h. Then it follows from the definition of weak gradient (2.3)

$$(\nabla_w(Q_h u - u),\ R_h\nabla_w \Phi)_T = -(Q_0 u - u, \nabla\cdot R_h\nabla_w\Phi)_T + \langle Q_b u - u, R_h\nabla_w\Phi\cdot\mathbf{n}\rangle_{\partial T} = 0.$$

Using the equation above and (4.7) and the definition of R_h, we have

$$\begin{aligned} I_3 &= |(\nabla_w(Q_h u - u),\ \nabla_w\Phi)_{\mathcal{T}_h}| \\ &= |(\nabla_w(Q_h u - u),\ \nabla_w\Phi - R_h\nabla_w\Phi)_{\mathcal{T}_h}| \\ &= |(\nabla_w(Q_h u - u),\ \nabla\Phi - R_h\nabla\Phi)_{\mathcal{T}_h}| \\ &\le Ch^{k+1}|u|_{k+1}|\Phi|_2. \end{aligned}$$

It follows from (4.6), (4.7) and (4.8) that

$$\begin{aligned} I_4 &= |\ell(\Phi, \epsilon_h)| \le Ch|\Phi|_2 |||\epsilon_h||| \\ &\le Ch|\Phi|_2(|||e_h||| + |||u - Q_h u|||) \\ &\le Ch^{k+1}|u|_{k+1}\|\Phi\|_2. \end{aligned}$$

Combining all the estimates above with (5.6) yields

$$\|\epsilon_0\|^2 \le Ch^{k+1}|u|_{k+1}\|\Phi\|_2.$$

It follows from the above inequality and the regularity assumption (5.2).

$$\|\epsilon_0\| \le Ch^{k+1}|u|_{k+1}.$$

The triangle inequality implies

$$\|e_0\| \le \|\epsilon_0\| + \|u - Q_0 u\| \le Ch^{k+1}|u|_{k+1}.$$

We have completed the proof. □

6. Numerical experiments

We solve the following Poisson equation on the unit square:

$$-\Delta u = 2\pi^2 \sin\pi x \sin\pi y, \quad (x, y) \in \Omega = (0, 1)^2, \tag{6.1}$$

with the boundary condition $u = 0$ on $\partial\Omega$.

In the first computation, the level one grid consists of two unit right triangles cutting from the unit square by a forward slash. The high level grids are the half-size refinements of the previous grid. The first three levels of grids are plotted in Fig. 6.1. The error and the order of convergence are shown in Table 6.1. The numerical results confirm the convergence theory.

In Fig. 6.2, we plot the finite element solution and the discretization errors on triangular and on polygonal grids. We can see, with same number of unknowns, the solutions on triangular grids are more accurate than those on polygonal grids. This can also be seen from the two data tables.

In the next computation, we use a family of polygonal grids (with 12-side polygons) shown in Fig. 6.3. The numerical results in Table 6.2 indicate that the polynomial degree j for the weak gradient needs to be larger, which confirms the theory: j depending on the number of edges of a polygon. The convergence history confirms the theory.

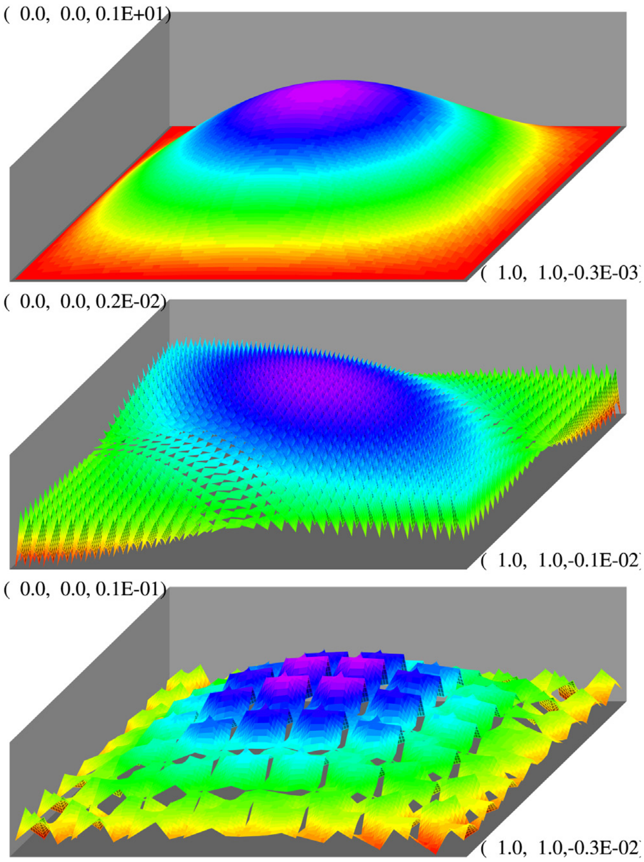

Fig. 6.2. Top: The P_1 weak Galerkin solution on the fifth level triangular grid. Middle: The error of P_1 weak Galerkin solution (dof 3008) on the fifth level triangular grid. Bottom: The error of P_1 weak Galerkin solution (dof 3456) on the fifth level 12-gon grid.

Table 6.1
Error profiles and convergence rates for (6.1) on triangular grids.

Level	$\|u_h - Q_0 u\|$	Rate	$\|\!\|u_h - u\|\!\|$	Rate
	by P_1 elements with P_1^2 weak gradient $\Rightarrow$ singular			
	by P_1 elements with P_2^2 weak gradient			
6	0.4295E−03	1.99	0.5369E−01	1.00
7	0.1075E−03	2.00	0.2684E−01	1.00
8	0.2688E−04	2.00	0.1342E−01	1.00
	by P_2 elements with P_2^2 weak gradient $\Rightarrow$ singular			
	by P_2 elements with P_3^2 weak gradient			
6	0.2383E−05	3.01	0.1013E−02	2.00
7	0.2971E−06	3.00	0.2532E−03	2.00
8	0.3709E−07	3.00	0.6330E−04	2.00
	by P_3 elements with P_3^2 weak gradient $\Rightarrow$ singular			
	by P_3 elements with P_4^2 weak gradient			
6	0.2468E−07	4.02	0.1430E−04	3.00
7	0.1532E−08	4.01	0.1789E−05	3.00
8	0.9550E−10	4.00	0.2237E−06	3.00
	by P_4 elements with P_4^2 weak gradient $\Rightarrow$ singular			
	by P_4 elements with P_5^2 weak gradient			
5	0.8154E−08	4.99	0.2441E−05	4.00
6	0.2551E−09	5.00	0.1526E−06	4.00
7	0.8257E−11	4.99	0.9539E−08	4.00

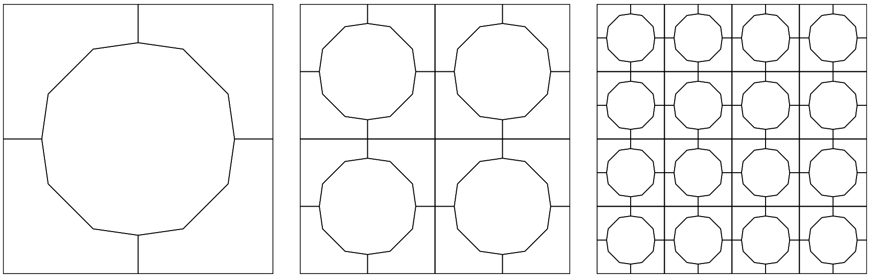

Fig. 6.3. The first three polygonal grids for the computation of Table 6.2.

Table 6.2
Error profiles and convergence rates for (6.1) on polygonal grids shown in Fig. 6.3.

Level	$\|u_h - Q_0 u\|$	Rate	$\|u_h - u\|$	Rate
	by P_1 elements with P_2^2 weak gradient $\Rightarrow$ singular			
	by P_1 elements with P_3^2 weak gradient			
5	0.9671E−03	1.98	0.1350E+00	1.00
6	0.2425E−03	2.00	0.6750E−01	1.00
7	0.6067E−04	2.00	0.3375E−01	1.00
	by P_2 elements with P_3^2 weak gradient $\Rightarrow$ singular			
	by P_2 elements with P_4^2 weak gradient			
5	0.5791E−05	3.00	0.3247E−02	2.00
6	0.7233E−06	3.00	0.8120E−03	2.00
7	0.9040E−07	3.00	0.2030E−03	2.00
	by P_3 elements with P_4^2 weak gradient $\Rightarrow$ singular			
	by P_3 elements with P_5^2 weak gradient			
4	0.8809E−06	4.00	0.3575E−03	2.99
5	0.5509E−07	4.00	0.4475E−04	3.00
6	0.3447E−08	4.00	0.5595E−05	3.00

References

[1] I. Babuška, The finite element method with penalty, Math. Comp. 27 (1973) 221–228.
[2] J. Douglas Jr., T. Dupont, Interior penalty procedures for elliptic and parabolic Galerkin methods, Comput. Methods Appl. Sci. (1976) 207–216.
[3] W. Reed, T. Hill, Triangular Mesh Methods for the Neutron Transport Equation, Technical Report la-UR-73-0479, Los Alamos Scientific Laboratory, Los Alamos, NM, 1973.
[4] M. Wheeler, An elliptic collocation-finite element method with interior penalties, SIAM J. Numer. Anal. 15 (1978) 152–161.
[5] B. Cockburn, C. Shu, The local discontinuous Galerkin finite element method for convection–diffusion systems, SIAM J. Numer. Anal. 35 (1998) 2440–2463.
[6] D. Arnold, F. Brezzi, B. Cockburn, D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal. 39 (2002) 1749–1779.
[7] B. Cockburn, J. Gopalakrishnan, R. Lazarov, Unified hybridization of discontinuous Galerkin, mixed, and conforming Galerkin methods for second order elliptic problems, SIAM J. Numer. Anal. 47 (2009) 1319-1365.
[8] K. Lipnikov, G. Manzini, F. Brezzi, A. Buffa, The mimetic finite difference method for the 3D magnetostatic field problems on polyhedral meshes, J. Comput. Phys. 230 (2011) 305–328.
[9] D. Pietro, A. Ern, Hybrid high-order methods for variable-diffusion problems on general meshes, C. R. Math. 353 (2015) 31–34.
[10] L. Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. Marini, A. Russo, Basic principles of virtual element methods, Math. Models Methods Appl. Sci. 23 (2013) 119–214.
[11] J. Wang, X. Ye, A weak Galerkin finite element method for second-order elliptic problems, J. Comput. Appl. Math. 241 (2013) 103–115.
[12] X. Ye, S. Zhang, A conforming discontinuous Galerkin finite element method, Int. J. Numer. Anal. Model. 17 (1) (2020) 110–117.
[13] X. Ye, S. Zhang, A conforming discontinuous Galerkin finite element method: Part II, Int. J. Numer. Anal. Model. 17 (2) (2020) 281–296.
[14] J. Liu, S. Tavener, Z. Wang, Lowest-order weak Galerkin finite element method for Darcy flow on convex polygonal meshes, SIAM J. Sci. Comput. 40 (2018) 1229–1252.
[15] J. Wang, X. Ye, A weak Galerkin mixed finite element method for second-order elliptic problems, Math. Comp. 83 (2014) 2101–2126.