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Asymmetric budget constraints in a first-price auction ☆

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Abstract

I solve a first-price auction for two bidders with asymmetric budget distributions and known valuations for one object. I show that in any equilibrium, the expected utilities and bid distributions of both bidders are unique. If budgets are sufficiently low, the bidders will bid their entire budget in any equilibrium. For sufficiently high budgets, mass points in the equilibrium strategies arise. A less restrictive budget distribution could make both bidders strictly worse off. If the budget distribution of one bidder is dominated by the budget distribution of the other bidder in the reverse-hazard-rate order, the weaker bidder will bid more aggressively than the stronger bidder. In contrast to existing results for symmetric budget distributions, with asymmetric budget distributions, a second-price auction can yield a strictly higher revenue than a first-price auction. Under an additional assumption, I derive the unique equilibrium utilities and bid distributions of both bidders in an all-pay auction.

JEL classification: C72; D44; D82

Keywords: Budget constraints; Asymmetric bidders; First-price auctions; All-pay auctions

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1. Introduction

Auctions are a widely used method of allocating objects, property rights and procurement contracts. If bidders in an auction are budget constrained, this will influence their bidding strategies, break the revenue equivalence of standard auctions, and lower revenues. Budget constraints can arise due to credit limits and imperfect capital markets, such that bidders' willingness to pay might exceed their ability to pay.

The existing research on standard auctions with budget constrained bidders concentrates on identical budget distributions. Yet, there are scenarios where bidders have asymmetric budget distributions. In a narrow market with a few players, e.g., a telecommunications sector, bidders hold noisy information about the other bidders and their budgets. This information might stem from previous interactions or from publicly available information, such as annual budget reports. Moreover, an auctioneer can contribute to this asymmetry by revealing the identities of the participants before the auction via a participation register.

In the spectrum auction of the U.S. Federal Communications Commission, 30 bidders registered for the auction (Salant, 1997). Assessing the budget constraint of rival bidders was a major part of the preparation before the auction (Salant, 1997). GTE was one of the largest telecommunication firms in the U.S. It is reasonable to expect that the expectations of GTE about the budget of a smaller bidder, such as Poka Lambro, differed from the expectation of the smaller bidder about the financial resources of GTE.

The contribution of this paper is to solve the first-price auction for bidders with asymmetric budget distributions. I develop a solution technique that builds on an indirect utility approach by Che and Gale (1996). I provide a closed-form expression for the expected utilities and bid distributions of the bidders, which are unique in any equilibrium.[1]

In my model, two bidders are competing for one object in a first-price auction. Their valuations are common knowledge and might differ. Each bidder has a private budget constraint that is drawn independently from a bidder-specific distribution. Budget constraints are hard, that is, no bidder can bid above his budget.[2] First, budget constraints directly limit the ability to bid. Second, budgets have an indirect strategic effect: if a bidder is budget constrained, the necessary bid to outbid him might be lower than without a budget constraint. Then, the constrained bidder anticipates this inference and incorporates this into his bidding strategy, and so forth. The extent of these strategic effects varies with the asymmetry in budget distributions.

Che and Gale (1996) solve the first-price auction for bidders with identical budget distributions and the same common value for the object. Equilibrium utility in their model always equals some exogenous *lower bound* on utility. This lower bound is the highest utility a bidder can achieve if the other bidder always bids his entire budget and, thus, minimizes the winning probability at any bid. They restrict attention to monotonic bidding strategies and symmetric equilibria, hence, mass points cannot arise in their setup.

In my model, I allow for asymmetric budget distributions and different values. I do not restrict attention to symmetric or monotonic equilibria. In contrast to the symmetric setup in Che and Gale (1996), the equilibrium utility no longer always equals its lower bound. Mass points arise in equilibrium. Bidders would like to deviate and bid at the mass point or slightly above to increase their winning probability, but cannot afford such deviations due to their budget constraints. I show that each bidder places at most one mass point.

[1] I do not restrict attention to symmetric equilibria, nor to monotonic bidding strategies.

[2] See, e.g., Zheng (2001) for a model with soft budget constraints where bidders can borrow.

If the reverse hazard rate of both bidders is above a threshold, bidders bid the entire budget in any equilibrium. Then, equilibrium utility equals the lower bound utility, and bidding the entire budget on this interval constitutes a fix point.

If either of the two reverse hazard rates drops below the threshold, equilibrium utilities either jump up due to a mass point, or are constant on some indifference regions. In indifference regions, the other bidder's bid distribution makes a bidder indifferent between any bid in this interval. Equilibrium utilities can strictly exceed their lower bound.

I show that asymmetric budget distributions break the revenue dominance of the first-price auction over the second-price auction. For the special case of reverse hazard rate dominance in budget distributions, a weak bidder bids more aggressively than a strong bidder. This is in line with the literature on asymmetrically distributed valuations (Maskin and Riley, 2000), where the weaker bidder (with regards to the valuation distribution) bids more aggressively. Similarly, if budget distributions are identical, but one bidder values the object more, he bids more aggressively.

I find the necessary and sufficient conditions for a bidder to derive a higher utility than the other bidder at every budget realization. Both bidders can be strictly harmed if one bidder's budget constraint is relaxed. A first-price auction might allocate the object inefficiently, even if the bidder with the highest value for the object has also the highest budget realization. Finally, I apply my technique to derive a closed-form equilibrium for an all-pay auction.

Related Literature: Che and Gale (1996, 1998, 2000) are among the first to derive the equilibrium for auctions with budget constrained bidders. They show that revenue equivalence no longer holds when bidders are budget constrained. Research on budget constraints in standard 1-object auctions (see, e.g., Che and Gale, 1996, 1998; Kotowski, forthcoming; Kotowski and Li, 2014) considers symmetric budget distributions. Literature on asymmetrically budget constrained bidders is scarce. Malakhov and Vohra (2008) derive the optimal auction with two bidders, where only one is constrained. Some work (e.g., Benoît and Krishna, 2001; Dobzinski et al., 2012; Boulatov and Severinov, 2018) considers asymmetric, but publicly known budget realizations. In this work, I merge the assumption of asymmetric budgets into a framework that allows for private information about budget realization.

Closest to my framework is Che and Gale (1996). They considered many bidders with an identical commonly known valuation for the object. Budget realizations are private and independent draws from the same distribution. My model generalizes their model in two directions: first, in my model, budgets are drawn from asymmetric distributions. Second, the valuations for the object may differ between bidders. This allows me to capture the effect of valuation heterogeneity on the bidding strategies. In contrast to Che and Gale (1996), I do not restrict attention to symmetric and monotonic equilibria, but I impose log-concavity on the budget distribution and consider only two bidders.

The analysis in this paper relates to asymmetric auctions, in which the valuations of bidders are drawn from non-identical distributions, and bidders do not have budget constraints (see the seminal contribution of Maskin and Riley, 2000). Analytical solutions exist for only a few particular distributions, e.g., Maskin and Riley (2000) and Kaplan and Zamir (2012) for uniform distributions, and Plum (1992) and Cheng (2006) for power distributions. Asymmetric auctions have been approached by perturbation analysis (e.g., Fibich and Gavious, 2003; Fibich et al., 2004; Lebrun, 2009). For two bidders with asymmetrically drawn valuations from the same support, no general closed-form solution is known. The first-price and second-price auctions no longer yield the same revenue under asymmetric value distributions, with the revenue ranking depending on the asymmetry of the value distributions (Maskin and Riley, 2000; Cantillon, 2008; Gavious and Minchuk, 2014).

If bidders are asymmetric not in valuations but in budgets, my results apply. In contrast to asymmetry in valuations, a closed-form solution exists for asymmetric budget distributions. A unique equilibrium utility and bid distribution exist for all log-concave budget distributions with the same support, without assuming any stochastic dominance order.

The paper is organized as follows. Section 2 introduces the model. The characterization of the equilibrium in a first-price auction follows in Section 3, using a lower bound on the utility (Section 3.1). In Section 3.2, I derive the unique equilibrium utility. Section 3.3 establishes the uniqueness of the bid distributions, and Section 3.4 the existence of an equilibrium. Section 4 discusses the implications for symmetric bidders, bidding aggression, welfare, and efficiency. In Section 5, I extend my results to compare the revenue in a first-price and a second-price auction, analyze information disclosure about budget types, and solve an all-pay auction. I conclude in Section 6. All omitted proofs are in the Appendix.

2. Model

An auctioneer (she) sells one object with zero value for her in a first-price auction (FPA) with no reserve price and an equal tie-breaking rule. There are 2 risk-neutral bidders, indexed by $i \in \{1, 2\}$. Bidder i has a valuation v_i for the object. The valuation tuple $\{v_1, v_2\}$ is common knowledge for the bidders.

Each bidder (he) has a private budget w_i, which is drawn independently from a distribution with a continuous and differentiable cumulative distribution function $F_i(w)$ and probability density function $f_i(w)$. Both distribution functions $\{F_i(w)\}_{i=1,2}$ have full and common support on $[\underline{w}, \overline{w}]$, are atom-less and common knowledge. Both bidders are budget constrained with non-zero probability, $\min\{v_1, v_2\} > \underline{w}$.[3]

Assumption 1. $F_1(w)$ and $F_2(w)$ satisfy log-concavity on $(\underline{w}, \overline{w})$.[4]

The bidding strategy of bidder i maps his budget w into a distribution over feasible bids in $[0, w]$. Let b_i be a random variable denoting the placed bid of bidder i. Let $b_i(w)$ be a bid in the bidding support of bidder i with budget w. Bidders have hard[5] budget constraints: they cannot bid above their budget. A feasible bidding strategy satisfies $b_i(w) \leq w$ for all bids of any budget type w. If a bidder i wins the object by bidding b_i, his utility is $v_i - b_i$.

Example 1. Bidder 1 and 2 have the same valuation $v := v_1 = v_2$ for the object. Their budget distributions are $F_1(w) = w^2$ and $F_2(w) = w$ for $w \in [0, 1]$.

In Example 1, bidder 1 is stronger than bidder 2 in the sense of first order stochastic dominance (FOSD). I use this example in the following to depict my solution technique.[6]

3 If one bidder is unconstrained, $v_i \leq \underline{w}$, the game effectively reduces to Bertrand competition.

4 See Bagnoli and Bergstrom (2005) for many commonly used distributions that satisfy log-concavity.

5 An equivalent formulation is to impose fines on overbidding and to forbid renegotiation. See Footnote 2 in Che and Gale (1996).

6 I do not impose any stochastic order between $F_1(w)$ and $F_2(w)$ in the general model.

3. Equilibrium of the first price auction

Let $G_i(x) = \Pr(b_i \leq x)$ be the cumulative distribution function of bidder i's bid, that is, the probability of bidder i bidding below or equal to x. A *feasibility constraint* holds as a necessity of the hard budget constraints: $\forall x \in [0, \overline{w}]$, $G_i(x) \geq F_i(x)$. Bidder i with a budget below x bids weakly below x. Moreover, bidder i with a budget strictly above x might shade his bid down below x, yielding the weak inequality in the feasibility constraint.[7]

Let $U_i(w)$ be the expected utility that bidder i with budget w obtains in some equilibrium:

$$U_i(w) = \max_{0 \leq b_i \leq w} \{(v_i - b_i)[\Pr(b_j < b_i)] + \frac{1}{2}(v_i - b_i)\Pr(b_j = b_i)\}. \tag{1}$$

In my model, equilibrium strategies may contain mass points and the probability of a tie (the second summand) is therefore non-negligible. I find a unique equilibrium utility U_i via an indirect utility approach, using a lower bound on the equilibrium utility.

3.1. Lower bound

Consider the lowest feasible bound on the equilibrium utility of bidder 1 with budget w, called the *lower bound* utility $\underline{U}_1(w)$.[8] It is achieved if bidder 2 always bids his entire budget and, hence, minimizes the winning probability of bidder 1 at any bid. Then, bidder 1 with bid b_1 wins with the lowest feasible probability $G_2(b_1) = F_2(b_1)$.[9]

Lemma 1. *Let bidder j bid his entire budget, and F_j be log-concave. Then, the unique best response bid for bidder $i \neq j$ with budget w is*

$$\arg\max_{b_i \leq w} (v_i - b_i)F_j(b_i) = \begin{cases} w & \text{if } w < m_i, \\ m_i & \text{if } w \geq m_i, \end{cases} \tag{2}$$

with some unique $m_i \in (\underline{w}, \overline{w}]$. The lower bound $\underline{U}_i(w)$ for $i \in \{1, 2\}$ is continuous, strictly increasing for $w < m_i$ and constant for $w \in [m_i, \overline{w}]$.

In what follows, I assume without loss of generality that $m_1 \leq m_2$.[10] Bid m_i is the unconstrained best response of bidder i to bidder j bidding $G_j = F_j$. Either bidder i can afford to bid m_i (if $w \geq m_i$), or he bids his entire budget to bid as close as possible to m_i (if $w < m_i$). The resulting lower bound utility is

$$\underline{U}_i(w) = \max_{b_i \leq w} (v_i - b_i)F_j(b_i) = \begin{cases} (v_i - w)F_j(w) & \text{if } w < m_i, \\ (v_i - m_i)F_j(m_i) & \text{if } w \geq m_i. \end{cases} \tag{3}$$

The marginal utility of an increase in bid b_i is non-negative if the gain in the probability of winning offsets the higher payment in case of a win. This occurs if and only if

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[7] If bidders always bid their entire budget, the feasibility constraint holds with equality at every x.

[8] The lower bound utility $\underline{U}_i(w)$ is a generalization of the lower bound utility in Che and Gale (1996) to asymmetric budget distributions and different valuations.

[9] Under any other feasible strategy for bidder 2, bidder 1 with bid b_1 wins with a weakly higher probability.

[10] If $m_1 = m_2$, I label bidders without loss such that $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$.

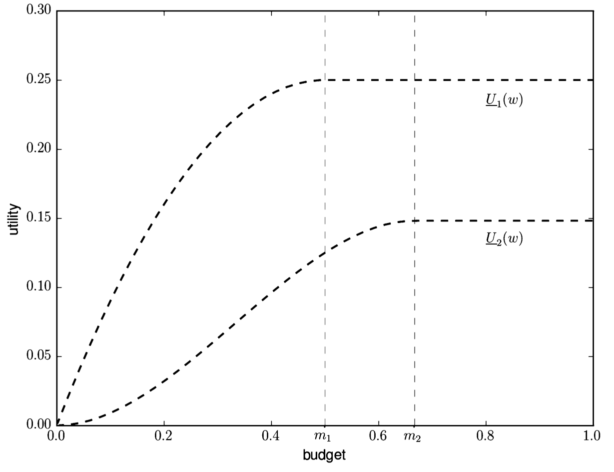

Fig. 1. Lower bound utilities for Example 1 with $v = 1$.

$$\frac{f_j(b_i)}{F_j(b_i)} \geq \frac{1}{v_i - b_i}, \tag{4}$$

where the reverse hazard rate (RHR) $\frac{f_j(b_i)}{F_j(b_i)}$ is monotonically decreasing by log-concavity, and the right hand side is strictly increasing in b_i. Inequality (4) holds with equality for $b_i = m_i$. Any bid above m_i yields a strictly lower payoff than bidding m_i for bidder i. The unconstrained best response m_i to $G_j = F_j$ is increasing in (and always below) v_i. If v_i is sufficiently high, bidding the entire budget may be the best response for every budget.

Fig. 1 shows the lower bound utilities for Example 1 with $v = 1$. $\underline{U}_1(w)$ is strictly increasing for $w < m_1 = 1/2$, and $\underline{U}_2(w)$ is strictly increasing for $w < m_2 = 2/3$.

3.2. Equilibrium utility

In the following, I derive four properties that any candidate equilibrium U_i satisfies. Together, these properties rule out all but a single candidate for the shape of the equilibrium utility.

Lemma 2. *Let $U_i(w)$ be strictly increasing on some open interval (w', w''). Then, bidder i with any budget realization $w \in (w', w'')$ always bids his entire budget, and $G_i(w) = F_i(w)$.*

A budget is not payoff-relevant unless it constrains the bid. If a bidder achieves a strictly higher utility with a higher budget than a lower budget, then the lower budget bidder cannot afford the bid of the higher budget bidder. If the equilibrium utility is strictly increasing in the budget, bidders bid their entire budget: this is the only feasible bid which cannot be mimicked by any lower budget type.[11]

The following lemma shows that whenever the utility is strictly increasing, the lower bound utility binds.

[11] This has been noted in Footnote 7 by Che and Gale (1996).

Lemma 3. *Let $U_i(w)$ be strictly increasing for $w \in (w', w'')$. Then, for all $w \in (w', w'')$ the lower bound binds: $U_i(w) = \underline{U}_i(w)$.*

Let the equilibrium utility of bidder 1 be strictly increasing in some interval. Thus, bidder 1 exhausts his entire budget on this interval (Lemma 2). Either bidder 2 would not want to bid within this interval at all (if the interval is above m_2), or he would also want to exhaust his entire budget. The former leads to a contradiction, as bidder 1 would never want to be the only one bidding in an open interval. The latter leads to both bidders exhausting their entire budget, and by definition receiving no more than their lower bound utility.

Lemma 4. *In any equilibrium, bidders with a budget $w \in (\underline{w}, m_1)$ bid their entire budget. For all $w \in [\underline{w}, m_1)$, the lower bound binds: $U_i(w) = \underline{U}_i(w)$.*

Assume bidder 2 bids his entire budget on $(\underline{w}, m_1)$. Then, by definition, it is a best response for bidder 1 to also bid his budget on $(\underline{w}, m_1)$. Lemma 4 establishes that this is the unique best response correspondence in any equilibrium.[12]

The following result further narrows down the set of candidate equilibria.

Lemma 5. *For $i \neq j$, the following holds in any equilibrium:*

1. *U_i has at most one discontinuity. If it arises, it occurs at m_j and $U_j(m_j) = \underline{U}_j(m_j)$.*
2. *U_1 is constant on (m_1, m_2) and constant on $(m_2, \overline{w}]$.[13] U_2 is constant on $(m_1, \overline{w}]$.*
3. *$U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2$.*

Discontinuities can only occur due to mass points in bidding distributions, and only at m_1 (in bidder 1's strategy) and m_2 (in bidder 2's strategy). For a sketch of the argument, let bidder 1 place a mass point at some bid $x > m_1$. For bidder 2, a bid at x yields lower utility than bidding infinitesimally above (due to a discrete jump in winning probability). Hence, bidder 2 with a budget above x never bids at or below x, and bidder 2 with a budget below x cannot afford x. Thus, $G_2(x) = F_2(x)$. But then, bidder 1 has a strictly higher payoff from bidding m_1 instead of x. Similarly, bidder 2 cannot place a mass point above m_2. Furthermore, bidder 2 cannot place a mass point on $x \in [m_1, m_2)$: bidder 1 would best respond with $G_1(x) = F_1(x)$. But then, as $(v_2 - b)F_1(b)$ is strictly increasing below m_2, bidder 2 would be strictly better off bidding slightly above the mass point x.

By Statement 2. of the Lemma 5, U_i cannot be strictly increasing on some open interval above m_1. For example, bidder 1 only bids $b_1 > m_1$ if it yields a sufficiently high winning probability $G_2(b_1) > F_2(b_1)$ (if not, bidding $b_1 = m_1$ is strictly better by Lemma 1). If bidder 1's utility U_1 is strictly increasing on some open interval above m_1, bidder 1 bids his entire budget on this interval, and $F_1 = G_1$ by Lemma 2. But then, bidder 2 would not want to bid in this interval at all if it is in $(m_2, \overline{w})$ (bidding $b_2 = m_2$ is strictly better), or would also want to bid his full budget if it is in (m_1, m_2), resulting in $F_2 = G_2$ which does not give bidder 1 enough winning probability to make a bid $b_1 > m_1$ worthwhile.

[12] Lemma 4 does not specify the bid of the lowest budget type $\underline{w}$, while it determines his utility $U_i(\underline{w}) = 0$. As a budget $\underline{w}$ is a zero-probability event, $b_i(\underline{w})$ has no impact on G_i.

[13] U_1 can take two different values on these intervals.

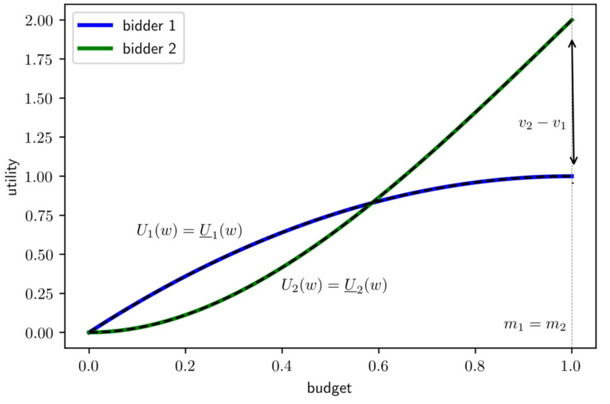

Fig. 2. Case (C1) with $v_1 = 2$, $v_2 = 3$, and $m_1 = m_2 = \overline{w} = 1$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

The utilities of bidders with budget $\overline{w}$ are exactly $v_1 - v_2$ apart, as both bidders share the same supremum bid that wins with a probability of one.[14] In order to achieve this distance $(v_1 - v_2)$, Lemma 5 allows U_1 to jump once at m_2 (U_2 to jump once at m_1), but not increase continuously. A jump in U_i at m_j determines the utility of bidder $j \neq i$ to $\underline{U}_j(m_j)$ for an interval of higher budget levels. There is one unique way to allocate discontinuities such that the required utility difference $U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2$ is satisfied. To show this, I differentiate between two cases:

$(C1)$ $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$.
$(C2)$ $\underline{U}_1(m_1) - \underline{U}_2(m_2) < v_1 - v_2$.

This bifurcation determines who can have a discontinuity in equilibrium: only bidder 1 in Case (C1), and possibly both bidders in Case (C2).

In the following figures, lower bound utilities are depicted by dashed black lines, and the equilibrium utility of bidder 1 (bidder 2) by a solid blue (green) line. Where no lower bound utility is visible, it is because it coincides with the equilibrium utility. Fig. 2 illustrates the case $m_1 = m_2 = \overline{w}$ for the budget distributions in Example 1, with $v_1 = 2$ and $v_2 = 3$. By Lemma 4, $U_i(w) = \underline{U}_i(w)$ for all $w \in (\underline{w}, \overline{w} = m_1)$.

Fig. 3 sketches Case (C1) for $m_1 = m_2$ and $\underline{U}_1(m_1) - \underline{U}_2(m_2) = v_1 - v_2$. The lower bounds bind for every budget, and there are no discontinuities: any mass point makes one U_i jump up and distorts the utilities of the bidders away from the correct distance $v_1 - v_2$.

Fig. 4 illustrates Case (C1) with $v_1 - v_2 = 0.05$, $w \in [0, 1]$ with $m_1 = 0.5$ and $m_2 = 0.\overline{6}$. Thus, $\underline{U}_1(m_1) - \underline{U}_2(m_2) > v_1 - v_2 = 0.05 = U_1(1) - U_2(1)$. The lower bounds and the equilibrium utilities coincide for $w < m_1$. By Lemma 5, bidder 2's utility is constant on $(m_1, \overline{w}]$ and can only have a jump discontinuity at m_1. For sufficiently high budgets, the distance between the dashed black lines is larger than the necessary distance $(v_1 - v_2)$ between the solid green and blue lines. Thus, at least one utility has to lie strictly above the lower bound. Bidder 2's utility

[14] See Lemma A.1 in the Appendix for further details. This is reminiscent of Bertrand competition with unconstrained bidders. If both bidders have unlimited budget and $v_1 > v_2$, bidder 1 wins by bidding v_2 and bidder 2 randomizes in some non-empty interval below v_2 (Blume, 2003). If $v_1 = v_2$, both unconstrained bidders have zero payoff. The difference in payoffs of unconstrained bidders is $v_1 - v_2$.

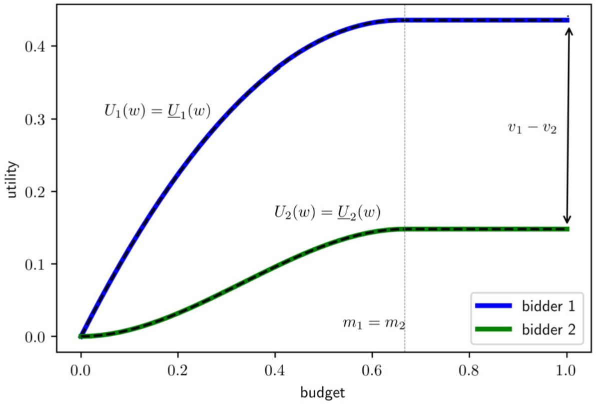

Fig. 3. Case (C1) with $\underline{U}_1(m_1) - \underline{U}_2(m_2) = v_1 - v_2$.

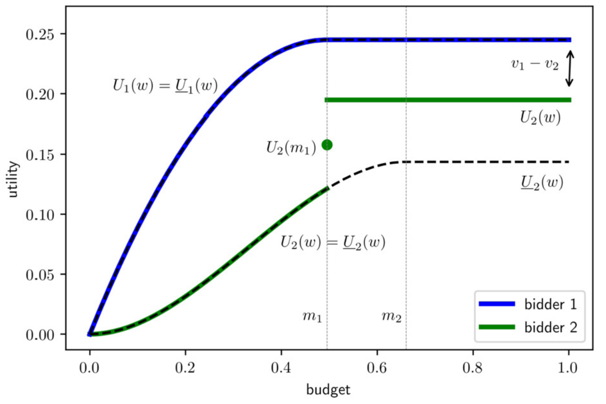

Fig. 4. Case (C1) with $\underline{U}_1 > \underline{U}_2$.

jumps at m_1 by such a magnitude so that the difference between $U_1(w > m_1)$ and $U_2(w > m_1)$ amounts to $v_1 - v_2 = 0.05$. Why can there be no discontinuity in U_1? By Lemma 5, U_1 can only jump at m_2, and then bidder 2's utility would be $\underline{U}_2(m_2) = U_2(m_2) = U_2(1)$. But then, $U_1(\overline{w}) - U_2(\overline{w}) > \underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$.

Fig. 5 shows another example for Case (C1) with $\underline{U}_1 < \underline{U}_2$. Bidder 1 achieves exactly his lower bound utility for every budget level. Bidder 2's utility U_2 jumps at m_1 above the lower bound, in the only feasible way to satisfy $U_1(\overline{w}) - U_2(\overline{w}) = (v_1 - v_2)$.

Fig. 6 shows an example for the Case (C2). The difference between $U_1(1) - U_2(1)$, that has to amount to $(v_1 - v_2)$, is larger than $\underline{U}_1(m_1) - \underline{U}_2(m_2)$. This cannot be achieved by a mass point of bidder 1 alone, as then $U_1(1) = \underline{U}_1(m_1)$ (Lemma 5) and $U_2(w > m_1) \geq \underline{U}_2(m_2)$. In equilibrium, bidder 2 places an atom m_2 to elevate $U_1(w > m_2)$ to its final level to satisfy $U_1(1) - U_2(1) = v_1 - v_2$. Fig. 7 illustrates the utilities in Case (C2) with $\underline{U}_1 < \underline{U}_2$. Bidder 2's utility jumps at m_1, such that it is constant on $(m_1, \overline{w}]$. Then, bidder 1's utility jumps at m_2 (requiring $U_2(m_2) = \underline{U}_2(m_2)$) to achieve $U_1(1) - U_2(1) = v_1 - v_2$.

In summary, if $m_1 = \overline{w}$, Lemma 4 characterizes effectively the entire equilibrium utility and no discontinuities arise. If $m_1 < \overline{w}$ and Case (C1) holds, then bidder 1 places a mass point of such

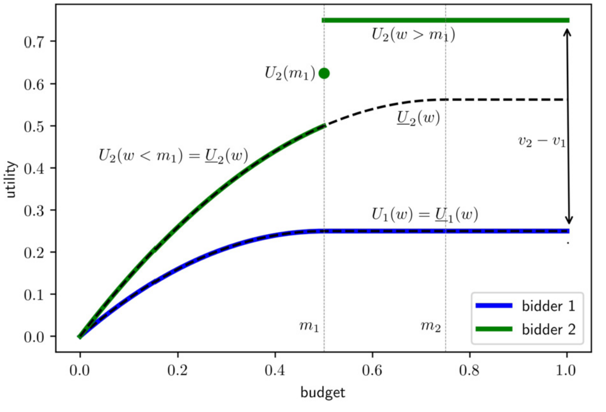

Fig. 5. Case (C1) with $\underline{U}_1 < \underline{U}_2$.

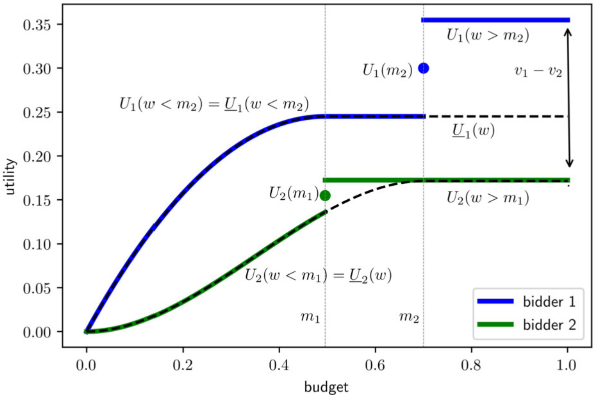

Fig. 6. Case (C2) for $\underline{U}_1 > \underline{U}_2$.

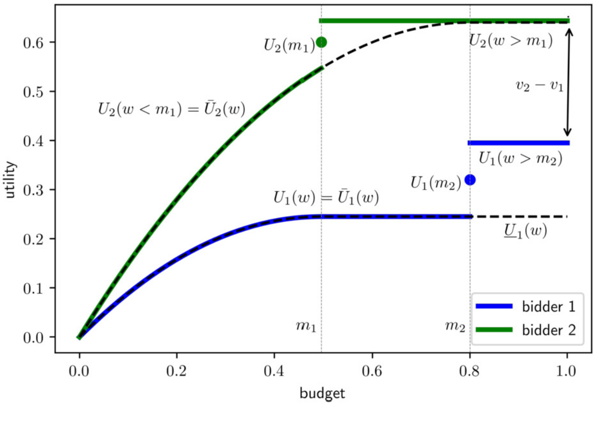

Fig. 7. Case (C2) with $\underline{U}_1 < \underline{U}_2$.

magnitude to elevate bidder 2's utility to its final level ($v_1 - v_2$ below his own utility). In Case (C2), bidders 1 and 2 place mass points at m_1 and m_2 to achieve the required distance $v_1 - v_2$. This is formalized in the next result.

Theorem 1. *Let (C1) hold. In any equilibrium, utilities are*

$$U_1(w) = \underline{U}_1(w) \text{ for all } w, \tag{5}$$

$$U_2(w) = \begin{cases} \underline{U}_2(w) \text{ if } w < m_1, \\ \frac{1}{2}\left(\underline{U}_2(m_1) + \underline{U}_1(m_1) - (v_1 - v_2)\right) \text{ if } w = m_1, \\ \underline{U}_1(m_1) - (v_1 - v_2) \text{ otherwise.} \end{cases} \tag{6}$$

Let (C2) hold. In any equilibrium, utilities are

$$U_1(w) = \begin{cases} \underline{U}_1(w) \text{ if } w < m_2, \\ \frac{1}{2}\left(\underline{U}_1(m_2) + \underline{U}_2(m_2) + (v_1 - v_2)\right) \text{ if } w = m_2, \\ \underline{U}_2(m_2) + (v_1 - v_2) \text{ otherwise.} \end{cases} \tag{7}$$

$$U_2(w) = \begin{cases} \underline{U}_2(w) \text{ if } w < m_1, \\ \frac{1}{2}\left(\underline{U}_2(m_1) + \underline{U}_2(m_2)\right) \text{ if } w = m_1, \\ \underline{U}_2(m_2) \text{ otherwise.} \end{cases} \tag{8}$$

As the above theorem shows, a unique equilibrium utility can be recovered by computing m_1 and m_2, the lower bound utilities $\underline{U}_1$ and $\underline{U}_2$, and $v_1 - v_2$.

3.3. Equilibrium bid distributions

The next result shows the unique bid distributions and supremum bids in any equilibrium.

Theorem 2. *In any equilibrium, the supremum bid of both bidders is*

$$\overline{b} = \begin{cases} v_1 - (v_1 - m_1)F_2(m_1) & \text{if } (C1), \\ v_2 - (v_2 - m_2)F_1(m_2) & \text{if } (C2), \end{cases} \tag{9}$$

and the cumulative bid distributions satisfy

$$G_1(b) = \begin{cases} F_1(b) & \text{if } b < m_1, \\ \frac{v_2 - \overline{b}}{v_2 - b} & \text{if } b \in [m_1, \overline{b}], \end{cases} \qquad G_2(b) = \begin{cases} F_2(b) & \text{if } b < m_1, \\ \frac{(v_1 - m_1)F_2(m_1)}{v_1 - b} & \text{if } b \in [m_1, m_2), \\ \frac{v_1 - \overline{b}}{v_1 - b} & \text{if } b \in [m_2, \overline{b}]. \end{cases} \tag{10}$$

Equilibrium bid distributions are unique. In any equilibrium, $G_i = F_i$ for bids below m_1. Above m_1, bidders place bids on a non-empty interval to make each other indifferent. For example, in Case (C1), both bidders allocate their bidding mass on $\left(m_1, \overline{b}\right)$ in such a way that any bid in this interval yields the same expected payoff. The equilibrium bid distributions might require mass points, as the following corollary of Theorem 2 shows.

Corollary 1. *Each bidder has at most one mass point. Bidder 1 has a mass point at m_1, unless $m_1 = m_2$ and Case (C1) holds with equality (i.e., $\underline{U}_1(m_1) - \underline{U}_2(m_2) = v_1 - v_2$). Bidder 2 has a mass point at m_2 if and only if (C2) holds.*

3.4. Equilibrium existence

To establish existence of an equilibrium, I derive pure strategy weakly monotonic bidding functions that are feasible and optimal for the bidders.

Theorem 3. *A pure strategy weakly monotonic equilibrium exists in the FPA.*

The proof is by construction. If (C1) holds, the following monotonic bidding functions constitute an equilibrium. (The bidding functions for (C2) are in the Appendix.)

$$b_1(w) = \begin{cases} w & \text{if} \quad w \in [\underline{w}, m_1), \\ m_1 & \text{if} \quad w \in [m_1, F_1^{-1}(\frac{v_2-\overline{b}}{v_2-m_1})], \\ v_2 - \frac{v_2-\overline{b}}{F_1(w)} & \text{otherwise.} \end{cases} \tag{11}$$

$$b_2(w) = \begin{cases} w & \text{if} \quad w \in [\underline{w}, m_1), \\ v_1 - \frac{v_1-\overline{b}}{F_2(w)} & \text{otherwise.} \end{cases} \tag{12}$$

These bidding functions are feasible (i.e., $b_i(w) \le w$) and aggregate into G_1 and G_2 in Theorem 2. They are also optimal for the bidders. Bidder 2 with $w \in [\underline{w}, m_1)$ would prefer to bid more than his budget, but cannot afford it. Bidder 2 with budget m_1, who bids at the mass point m_1 of bidder 1, prefers to increase his bid and get a jump in winning probability. However, he cannot afford this upward deviation as he is already bidding his entire budget.

Consider Example 1 with $v = 1$. Then, $m_1 = 1/2$, $m_2 = 2/3$ and (C1) applies. The following pure strategy monotonic bidding functions constitute an equilibrium:

$$b_1(w) = \begin{cases} w & \text{if} \quad w < \frac{1}{2}, \\ \frac{1}{2} & \text{if} \quad w \in [\frac{1}{2}, \frac{1}{\sqrt{2}}], \\ 1 - \frac{1}{4w^2} & \text{otherwise,} \end{cases} \quad \text{and} \quad b_2(w) = \begin{cases} w & \text{if} \quad w < \frac{1}{2}, \\ 1 - \frac{1}{4w} & \text{otherwise.} \end{cases}$$

Fig. 8 illustrates these two bidding functions. The blue dashed (green dotted) line is the bidding function of bidder 1 (bidder 2). Bidders place their entire budget if $w < 1/2$. Bidder 1 places a mass point on $m_1 = 1/2$. The highest bid $\overline{b} = 3/4$ wins with a probability of one and yields the same payoff $v - \overline{b} = 1/4$ to both bidders.

Fig. 9 shows the corresponding equilibrium utility of bidder 1 (blue line) and bidder 2 (green line). Utility is strictly increasing below $m_1 = 1/2$. Bidder 1's mass point at m_1 raises bidder 2's utility to the same level as his own (for budgets above 1/2). Bidder 2, with a budget at or below the mass point, cannot deviate up as his budget constraint binds.

Note that the constancy in equilibrium utility does not correspond to constancy in bids. The bidding function of bidder 2 makes bidder 1 indifferent between all bids in $[\frac{1}{2}, \frac{3}{4}]$, including his mass point m_1. Similarly, bidder 2 is indifferent between any bid in $(\frac{1}{2}, \frac{3}{4}]$.

4. Discussion of the results

4.1. Symmetric bidders

Let $F(w) := F_1(w) = F_2(w)$ log-concave and $v := v_1 = v_2$. Then, $\underline{U}(w) := \underline{U}_1(w) = \underline{U}_2(w)$ and $m := m_1 = m_2$. Case (C1) applies. The following holds in any (possibly asymmetric and non-monotonic) equilibrium as a direct corollary of Theorems 1 and 2:

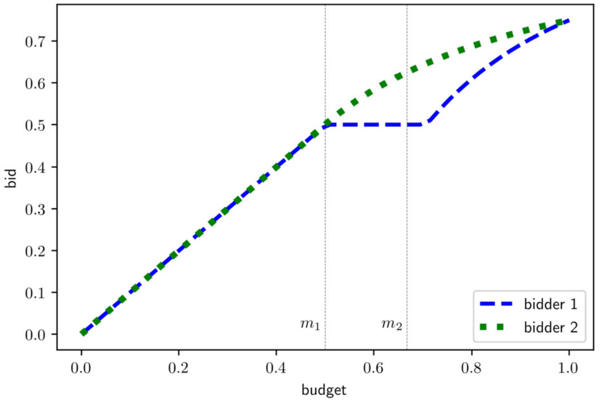

Fig. 8. Bidding functions in (C1) for $v = 1$.

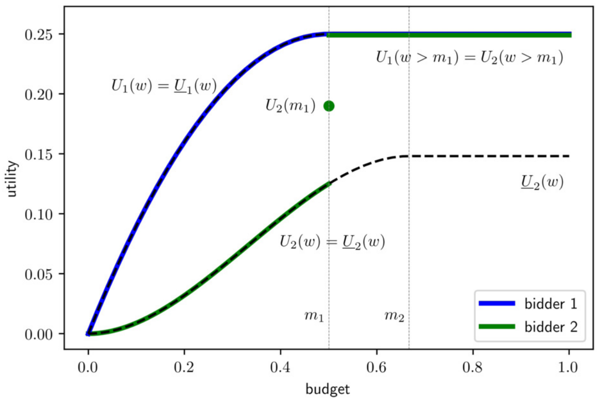

Fig. 9. Utilities in (C1) for $v = 1$.

Corollary 2. *Let $v := v_1 = v_2$ and $F(w) := F_1(w) = F_2(w)$ satisfy Assumption 1. In any equilibrium,*

1. *lower bound utilities bind, i.e., for all w, $U_1(w) = U_2(w) =: \underline{U}(w)$,*
2. *bid distributions are* $G_1(b) = G_2(b) = \begin{cases} F(b) & \text{if } b \in [\underline{w}, m), \\ \frac{(v-m)F(m)}{v-b} & \text{if } b \in [m, v-(v-m)F(m)]. \end{cases}$

Bidding distributions contain no mass points in any equilibrium. Fig. 10 shows an example with $v = 1$ and $F(w) = w$ for $w \in [0, 1]$. The equilibrium utilities and lower bound utilities coincide for both bidders and are strictly increasing below $m = 0.5$.

For equal budget distributions $F(w)$ and identical valuations v, Che and Gale (1996) show that the lower bound binds in any *symmetric* equilibrium of the FPA, $U_i(w) = \underline{U}(w)$ for all w. Che and Gale (1996) derive a symmetric equilibrium in strictly increasing bidding strategies

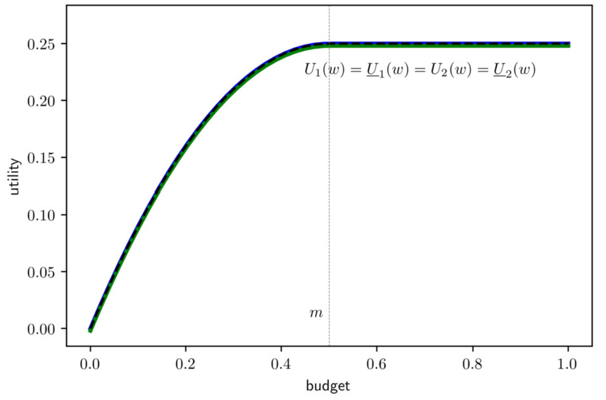

Fig. 10. Equilibrium utilities with $v = 1$ and $F(w) = w$.

(see their Lemma 1). For the case of two bidders with log-concave budget distributions,[15] these strategies coincide with the bid functions in Equations (11) and (12) in this paper and aggregate into the bid distributions in Statement 2 of Corollary 2.

What other equilibria can exist within the symmetric framework? If $m = \overline{w}$, then, by Lemma 4, bidders with $w \in (\underline{w}, \overline{w})$ bid their entire budget in any equilibrium. Hence, if $m = \overline{w}$, there exists a unique[16] equilibrium as in Che and Gale (1996).

If $m < \overline{w}$, there also exists a variety of asymmetric, mixed or non-monotonic equilibria apart from the symmetric increasing equilibrium in Che and Gale (1996). Bidders with a budget $w > m$ are indifferent between any bid in $[m, \overline{b}]$ and can play any strategy in equilibrium as long as it aggregates into the same $G_i(b)$ in Corollary 2. Only G_i is payoff relevant, not the specific bidding function that leads to G_i.[17] All these additional equilibria are payoff-equivalent as $U_i(w) = \underline{U}(w)$. However, these equilibria need not be allocation-equivalent: let $v = 1$, $F(w) = w$ for $w \in [0, 1]$, $w_1 = 3/4$ and $w_2 = 4/5$. Bidder 1 loses in the strictly monotonic equilibrium in Che and Gale (1996) but wins in an equilibrium where both bidders follow the non-monotonic bidding function in Footnote 17.

4.2. *Bidding aggression*

The monotonic pure strategy bidding functions in Section 3.4 allow a direct comparison in bidding behavior: Which bidder bids more aggressively if both have the same budget? There are two channels of interest. First, how does bidding aggression depend on the budget distribution? Second, how does bidding aggression depend on the valuation for the object?

As Lemma 4 shows, bidders with a budget in $[\underline{w}, m_1)$ bid their full budget and are equally aggressive, irrespectively of any order statistic assumption on their budget distributions.

15 Che and Gale (1996) allow for $n \geq 2$ bidders and do not impose log-concavity on $F(w)$.

16 It is unique up to the behavior of the lowest budget bidder, who will lose for any feasible bid.

17 For example, with $v = 1$ and $F(w) = w$, bidder 1 could bid according to the following non-monotonic feasible bidding function, which aggregates into the bid distribution G_1 in Statement 2. of Corollary 2,

$$b_1(w) = \begin{cases} w & \text{if } w \in [0, \frac{3}{4}], \\ \frac{1}{8}\left(\sqrt{16w^2 + 8w - 15} - 4w + 7\right) & \text{if } w \in (\frac{3}{4}, 1]. \end{cases}$$

Definition 1. F_i dominates F_j in terms of RHRs ($F_i \geq_{RHR} F_j$) if for all $x \in (\underline{w}, \overline{w})$,

$$\frac{f_i(x)}{F_i(x)} \geq \frac{f_j(x)}{F_j(x)}.$$

Next, let both bidders have the same v to isolate the differences in bidding aggression that are only due to differences in the budget distributions and not to heterogeneous valuations.

Proposition 1. *Let $v := v_1 = v_2$ and $F_i \geq_{RHR} F_j$. Then, $b_i(w) \leq b_j(w)$ for all $w \in [\underline{w}, \overline{w}]$.*

Proof. As bidder i RHR-dominates bidder j, it holds that $i = 1$ because $m_i \leq m_j$. As RHR-dominance implies FOSD, it holds that $\underline{U}_1(m_1) \geq \underline{U}_2(m_2)$ and Case (C1). The highest bid is $\overline{b} = v - (v - m_1)F_2(m_1)$ and it can be easily checked that the bidding strategies in Equations (11) and (12) imply $b_1(w) \leq b_2(w)$ for all $w \in [\underline{w}, \overline{w}]$. □

Fig. 8 depicts the bidding functions for equal values $v = 1$ and RHR-dominance in budget distributions for Example 1. Maskin and Riley (2000) target a related question for asymmetrically distributed valuations and bidders with unconstrained liquidity. They consider a variant of the RHR-dominance on valuation distributions and show that if both bidders have the same valuation, the RHR-dominated bidder bids more aggressively. This is in line with the findings of this paper: the RHR-weaker bidder bids more aggressively.

Next, I compare bidders with identical budget distributions $F(w) := F_1(w) = F_2(w)$, but different valuations $v_i > v_j$.

Proposition 2. *Let $v_i > v_j$ and $F_i(w) = F_j(w)$. Then, $b_i(w) \geq b_j(w)$ for all $w \in [\underline{w}, \overline{w}]$.*

Proof. Inequality (4) is satisfied for bidder i whenever it is satisfied for bidder j, because $\frac{1}{v_i - w} < \frac{1}{v_j - w}$ for all $w \in (\underline{w}, \min\{v_j, \overline{w}\})$. Therefore, $j = 1$ and $v_2 > v_1$. Let $F(w) := F_1(w) = F_2(w)$. Then, $\underline{U}_1(m_1) = \underline{U}_1(m_2) \geq (v_1 - m_2)F(m_2)$, and hence

$$\underline{U}_2(m_2) - \underline{U}_1(m_1) \leq (v_2 - m_2)F(m_2) - (v_1 - m_2)F(m_2) \leq v_2 - v_1.$$

Thus, Case (C1) applies. Using the pure monotonic bidding strategies in Equations (11) and (12), it immediately follows that $b_2(w) \geq b_1(w)$ for $w \in [\underline{w}, F^{-1}\left(\frac{v_2 - \overline{b}}{v_2 - m_1}\right)]$. For higher w, as $v_1 < v_2$, and it holds that $b_1(w) = v_2 - \frac{v_2 - \overline{b}}{F(w)} \leq v_1 - \frac{v_1 - \overline{b}}{F(w)} = b_2(w)$. □

Let $F(w) = w$, $v_1 = 1$ and $v_2 = 1.2$. Then, bidders exhaust their entire budget below $m_1 = 1/2$. Bidder 2 follows a strictly increasing bidding function. Bidder 1 places a mass point at $m_1 = 1/2$ if his budget is in $[0.5, 0.643)$, and follows an increasing bidding function thereafter. The bidder who values the object more bids more aggressively. Fig. 11 sketches the payoffs, and Fig. 12 sketches the bidding functions for this framework (see Appendix).

4.3. Bidder welfare

When does one bidder have a higher utility level than the other bidder at any budget realization? The following result provides necessary and sufficient conditions.

Proposition 3. *The following statements are equivalent:*

1. *For all w, $U_i(w) \geq U_j(w)$.*
2. *For all w, $\underline{U}_i(w) \geq \underline{U}_j(w)$ and $v_i \geq v_j$.*

For which primitives of the model $\{v_1, v_2, F_1, F_2\}$ is bidder i's lower bound utility $\underline{U}_i$ larger than bidder j's lower bound utility $\underline{U}_j$ at every budget realization w? If $v_1 = v_2$, FOSD is a sufficient, but not a necessary condition for $\underline{U}_i \geq \underline{U}_j$. Let $m_1 < \overline{w}$. Below m_1, bidders always exhaust their budget, and the condition $\underline{U}_i(w) \geq \underline{U}_j(w)$ is equivalent to $F_i(w) \leq F_j(w)$. For budget realizations higher than m_1, the condition $\underline{U}_i(w) \geq \underline{U}_j(w)$ is weaker than FOSD. This stresses that the precise shape of the budget distribution matters only for sufficiently low budget realizations, for which Inequality (4) holds.

Below are the two special cases from Section 4.2 that are sufficient for a bidder to derive the highest utility at every budget realization.

Observation 1. Let $v_i = v_j$, and $F_i \geq_{RHR} F_j$. Then, $i = 1$, (C1) holds, and $\underline{U}_1(w) \geq \underline{U}_2(w)$ for all w.

Let $v_i > v_j$, and $F_i = F_j$. Then, $i = 2$, (C1) holds, and $\underline{U}_2(w) \geq \underline{U}_1(w)$ for all w.

If a bidder has either a higher valuation or a higher budget distribution (in the sense of the RHR-dominance), he enjoys a higher lower bound utility: at every budget level, he is either more likely to win (RHR-dominance) or values the event of winning more (higher valuation) when bidding against a naive bidder.

A related question is how a change in the budget distribution affects the payoff. Is bidder $k \in \{1, 2\}$ better off with a higher (RHR-dominant) budget distribution $\hat{F}_k$ instead of F_k?[18]

As in Equation (1), let $U_i(w)$ be the equilibrium utility of bidder i with F_k, and $\hat{U}_i(w)$ his equilibrium utility with $\hat{F}_k$. Let $\overline{b}$ be the supremum bid with distribution F_k, and $\hat{\overline{b}}$ the supremum bid with distribution $\hat{F}_k$.

Proposition 4. *Let $\hat{F}_k \geq_{RHR} F_k$ for some $k \in \{1, 2\}$ and fix $\{v_1, v_2, F_{3-k}\}$. Then, $\hat{\overline{b}} \geq \overline{b}$. For both $i \in \{1, 2\}$, $\hat{G}_i$ is FOSD over G_i and for every w, $\hat{U}_i(w) \leq U_i(w)$. Both bidders can be strictly worse off under $\hat{F}_k$ than under F_k.*

The private budget constraints shield bidders from overbidding each other as in Bertrand competition, until at least one surplus is zero (Blume, 2003). A less restrictive budget distribution strengthens a bidder's competitive position, and bidders react by bidding higher and lowering the winning probabilities of every bid.

4.4. Efficiency

If $v_1 \neq v_2$, is the winner of the FPA the bidder with the highest valuation of the object? It is straightforward to see that this is not the case: a bidder with the lowest budget $\underline{w}$ always loses, irrespective of his valuation.

[18] My findings rely on both bidders knowing the budget distribution F_k or $\hat{F}_k$ of bidder k.

A weaker requirement on efficiency is the following: Does a bidder i who has a higher valuation $v_i > v_j$ *and* a higher budget realization $w_i > w_j$ win? In the following, I show that this weaker statement is also not true in general, but can hold under additional assumptions on the budget distributions F_1 and F_2.

Let $m_1 = \overline{w}$. Then, $b_i(w) = w$ for all $w > \underline{w}$. If a bidder with the highest value has the highest budget, he wins with probability one.

Let $m_1 < \overline{w}$ and consider the monotonic equilibrium in Section 3.4. If $v_i > v_j$ and $F := F_1 = F_2$, then $i = 2$ (i.e., $v_2 > v_1$) and Case (C1).[19] Proposition 2 establishes that $b_2(w) \geq b_1(w)$: bidder 2 with the higher valuation wins if he has the highest budget.

The finding that the highest valuation bidder wins if he has a higher budget cannot be extended to arbitrary distributions. For example, let $v_1 = 1.2$, $v_2 = 1$ and the budget distributions stem from Example 1. A quick computation reveals that Case (C1) holds, as $m_1 = 3/5$, $m_2 = 2/3$, $\underline{U}_1(m_1) = 9/25$ and $\underline{U}_2(m_2) = 4/27$. Bidder 1 with a budget in $[0.6, \sqrt{0.4}]$ bids at the mass point on 0.6 and loses against bidder 2 who has a budget above m_1. Hence, although $v_1 > v_2$ and $w_1 > w_2$, bidder 1 loses with a probability of one for all $w_1 \in (0.6, \sqrt{0.4})$ if $w_2 > m_1$. The stronger bidder bids less aggressively and admits a mass point. This is inefficient if the stronger bidder has a higher valuation.

5. Extensions

5.1. Revenue comparison

Revenue equivalence between standard auctions does not hold when bidders are budget constrained (e.g., Che and Gale, 1996, 1998, 2006). If budgets are drawn from an identical distribution, Che and Gale (1996) showed that the FPA yields a higher revenue than the second-price auction (SPA). I show that this revenue ranking does not hold under asymmetric budget distributions: the SPA can yield a strictly higher revenue than the FPA.

Proposition 5. *Let $v := v_1 = v_2$. In a SPA, it is a weakly dominant strategy to bid $b_i(w) = \min\{v, w\}$, $\forall i \in \{1, 2\}, \forall w \in [\underline{w}, \overline{w}]$.*

Proof. Consider bidder i with a budget w, who faces a bid b_j from bidder j. Let $w < v$. Then, bidding $b_i > w$ is infeasible. Bidding $b_i < w$ instead of $b_i = w$ is not profitable: it only changes the outcome if $w > b_j \geq b_i$. In this case $b_i < w$ loses, while bidding $b_i = w$ yields a strictly positive payoff $v - b_j$. Let $v \leq w$. Then, the standard argument of the SPA applies: bidding $b_i < v$ or $b_i > v$ (with $b_i \leq w$) yields a weakly lower payoff than bidding $b_i = w$. □

If both bidders have a budget above the object value, the auctioneer gets v. Whenever at least one bidder has a budget below v, the payoff of the seller is the lower budget. Let $x := \min\{v, \overline{w}\}$. The expected revenue of the auctioneer in the SPA, using Proposition 5, is

$$\Pi^{SPA} = \int_{\underline{w}}^{x} w\,[f_2(w)(1 - F_1(w)) + f_1(w)(1 - F_2(w))]\,dw + x(1 - F_1(x))(1 - F_2(x)). \tag{13}$$

[19] For details, see the proof of Lemma 2.

Next, consider the revenue in a FPA. The bidders share the same valuation v, the auctioneer's valuation is zero, and the object is always sold. Total generated surplus equals v. The revenue of the seller is the object value minus the expected utilities of the bidders,

$$\Pi^{FPA} = v - \int_{\underline{w}}^{\overline{w}} U_1(w) f_1(w) dw - \int_{\underline{w}}^{\overline{w}} U_2(w) f_2(w) dw. \tag{14}$$

Proposition 6. *Let $v := v_1 = v_2$, and budgets be drawn with log-concave distribution functions $F_1(w)$ and $F_2(w)$. Then, the SPA can yield a strictly higher revenue than the FPA.*

Proof. Let $w \in [0, 1]$, $F_1(w) = w^9$, $F_2(w) = w^{\frac{1}{9}}$, and $v = 0.2$. Then, $m_1 = \frac{1}{50}$, and $m_2 = \frac{9}{50}$. Plugging this into Equation (13) yields $\Pi^{SPA} \approx 0.05$.

Next, consider the FPA. The ex ante utilities of the bidders can be computed from the equilibrium utilities for Case (C1) in Equations (11) and (12).

$$EU_1 = \int_{\underline{w}}^{m_1} (v - w) F_2(w) f_1(w) dw + (v - m_1) F_2(m_1)(1 - F_1(m_1)) \approx 0.117,$$

$$EU_2 = \int_{\underline{w}}^{m_1} (v - w) F_1(w) f_2(w) dw + (v - m_1) F_2(m_1)(1 - F_2(m_1)) \approx 0.041.$$

Plugging this into Equation (14) yields $\Pi^{FPA} \approx 0.042 < \Pi^{SPA}$. $\square$

In the literature on standard auctions without budget constraints, asymmetrically distributed valuations break revenue equivalence between standard auctions (Maskin and Riley, 2000). No general revenue ranking exists: for some particular distributions, revenue in a FPA is higher than in a SPA (see, e.g., Maskin and Riley, 2000). This ranking can be reversed (Gavious and Minchuk, 2014). With asymmetric budget constraints, I show that the revenue ranking $\Pi^{FPA} \geq \Pi^{SPA}$ no longer holds. It remains unresolved under which conditions on asymmetric budget distributions the FPA yields a higher revenue than the SPA.

5.2. Information disclosure

Let both bidders value the object equally, $v := v_1 = v_2$. Bidders are ex ante symmetric: their budget is drawn from the same prior distribution. In the following, the auctioneer can publishing a participation register, so that bidders can look up annual budget reports and make inferences about the budget distribution of each other.[20] I show that the auctioneer can never gain by disclosing noisy information about the budgets.

Let S be the finite set of budget type distributions, with each $s \in S$ corresponding to a log-concave budget distribution function $F_s(w)$ with equal full support $[\underline{w}, \overline{w}]$. The term *type* in this section refers to the budget distribution type s, not the budget realization w. The budget distribution types s_1 and s_2 of bidders 1 and 2 are drawn independently and identically, with a probability

[20] In many auctions, bidders remain anonymous and place bids by phone. In a narrow market with few participants, e.g., the telecommunication market, anonymity might not be implementable.

$p_s > 0$ for type $s \in S$, with $\sum_{s\in S} p_s = 1$. Let the expected budget distribution $F(w) := \sum_{s\in S} p_s F_s(w)$ also satisfies log-concavity.

Before the start of the auction, the auctioneer decides whether she wants to publish a participation register. Then, bidders arrive and budget types $F_i \in \{F_s\}_{s\in S}$ are drawn for $i = \{1, 2\}$. Bidders know their own type, but not the type of the other bidder. The auctioneer publicly announces both types, if she committed to doing so. Then, budgets are drawn and observed only by the respective bidder. Finally, a FPA takes place.

Proposition 7. *Let the budget distribution of the two bidders, s_1 and s_2, be drawn i.i.d. from a finite set of distributions S. Then, the revenue is weakly lower if the auctioneer discloses the budget distribution types s_1 and s_2 than under no disclosure.*

The auction generates a total surplus of v, consisting of the auctioneer's revenue and the bidders' expected utilities. A higher expected utility for the bidders corresponds to a lower payoff for the auctioneer. Under no disclosure, the bidders have identical expectations about each other's budget distribution: the lower bound on the equilibrium utility binds for every budget w (Corollary 2). Under disclosure of budget types, the lower bound utility is weakly higher, as bidders can make their bid conditional on the budget type of the other bidder. Under asymmetry, a bidder can achieve an equilibrium utility strictly above his lower bound utility (Theorem 1). Thus, under information disclosure, bidders are better off than under no disclosure. This leaves a smaller share of the total surplus for the auctioneer.

In this section, I analyzed a specific information disclosure rule: no disclosure or full disclosure of budget distribution types that satisfy log-concavity and full support. Enabling the auctioneer to create types with different support (e.g., by allowing a monotone partition into a low-budget and a high-budget interval) or to send private and potentially correlated messages about budgets might yield further insights about optimal disclosure policy.

5.3. All-pay auction

In this section, I apply my results to the all-pay auction. Similar to the FPA, the lower bound of bidder i with budget w, who faces a bidder j bidding his entire budget, is

$$\underline{U}_i^a(w) := \max_{0\le b_i\le w} v_i F_j(b_i) - b_i.$$

For the FPA, Assumption 1 is sufficient to guarantee that the lower bound utility $\underline{U}_i$ is strictly increasing below $m_i > \underline{w}$, and constant thereafter. For the all-pay auction, Assumption 1 is not sufficient to guarantee these properties of the lower bound utility $\underline{U}_i^a$. For example, with $v \le 1$ and $F_2(w) = w$ for $w \in [0, 1]$, the lower bound utility of bidder 1 is $\underline{U}_1^a(w) = 0$ for all w. To use similar tools to the ones developed in previous sections, I impose the following assumption.

Assumption 2. Let $\underline{w} = 0$. For any $i \in \{1, 2\}$, $v_i F_j(b) - b$ has a unique global maximum at $m_i^a > \underline{w}$, and is strictly increasing in b below m_i^a.

For example, $v_i \ge 1$, $w \in [0, 1]$ and F_i strictly concave for $i = 1, 2$ satisfies Assumption 2. The following result sums up the equilibrium bid distribution in an all-pay auction.

Theorem 4. *Let Assumption 2 hold. The all-pay auction has an equilibrium. In any equilibrium, the supremum bid for both bidders is*

$$\overline{b}^a = \begin{cases} v_1(1 - F_2(m_1^a)) + m_1^a & \text{if} \quad \underline{U}_1^a(m_1^a) - \underline{U}_2^a(m_2^a) \geq v_1 - v_2, \\ v_2(1 - F_1(m_2^a)) + m_2^a & \text{if} \quad \underline{U}_1^a(m_1^a) - \underline{U}_2^a(m_2^a) < v_1 - v_2. \end{cases} \tag{15}$$

In any equilibrium, equilibrium utilities are unique. Bid distributions are

$$G_1^a(b) = \begin{cases} F_1(b) & \text{if } b \in [0, m_1^a), \\ \frac{v_2 - \overline{b}^a + b}{v_2} & \text{if } b \in [m_1^a, \overline{b}^a], \end{cases} \qquad G_2^a(b) = \begin{cases} F_2(b) & \text{if } b \in [0, m_1^a), \\ F_2(m_1^a) + \frac{b - m_1^a}{v_1} & \text{if } b \in [m_1^a, m_2^a), \\ \frac{v_1 - \overline{b}^a + b}{v_1} & \text{if } b \in [m_2^a, \overline{b}^a]. \end{cases} \tag{16}$$

As in the FPA, bid distributions are unique. Bidder 1 bids with a uniform distribution $(m_1^a, \overline{b}^a]$, and bidder 2 bids uniformly on (m_1^a, m_2^a) and $(m_2^a, \overline{b}^a]$. Similar to the FPA, equilibrium utilities in the all-pay auction are unique. They are given also by Theorem 1, after substituting $\underline{U}_i^a$ for $\underline{U}_i$, and m_i^a for m_i. In the proof in the Appendix, I construct monotonic bidding strategies (Equations (22) and (23)) to establish the existence of an equilibrium.

6. Concluding remarks

I derived the unique equilibrium utilities and bid distributions for two bidders with asymmetric budget distributions, who compete for one object. I allow for any asymmetry in the budget distributions, as long as they satisfy log-concavity and common full support.

Che and Gale (1996) showed that in a symmetric equilibrium with identically distributed budgets, the equilibrium utilities of the bidders equal a lower bound on utility. I have extended the framework of Che and Gale (1996) in two directions: I have allowed for different valuations of the object, and have introduced asymmetric budget constraints. In this framework, the lower bound does not necessarily bind. However, the equilibrium utilities in a FPA can still be recovered from the lower bound.

Mass points can be part of an equilibrium because budget constraints are hard, and bidders cannot outbid their budget. Due to the tie-breaking rule, bidding below a mass point yields a strictly lower utility than bidding exactly at a mass point. Furthermore, bidding at a mass point of the other bidder yields strictly lower utility than bidding above a mass point. The incentives to increase the available budget are particularly strong around mass points. For example, if a bidder with a budget slightly below a mass point could borrow to increase his budget, he could derive a discrete jump in surplus by bidding at the mass point. This might influence the initial budget distribution if the budget is determined endogenously before the start of the auction.[21] Finding an equilibrium with asymmetric budget distributions and allowing bidders to borrow (see, e.g., Zheng, 2001, for soft budgets with a borrowing market) might be an interesting question for future research.

[21] I am grateful to an anonymous referee for pointing this out.

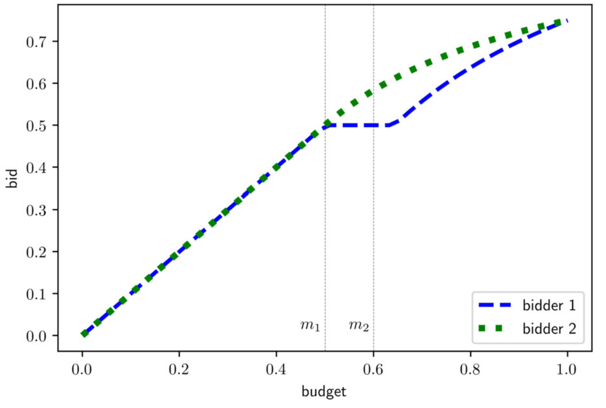

Fig. 11. Bidding functions in (C1) for $v_1 = 1$, $v_2 = 1.2$ and $F(w) = w$.

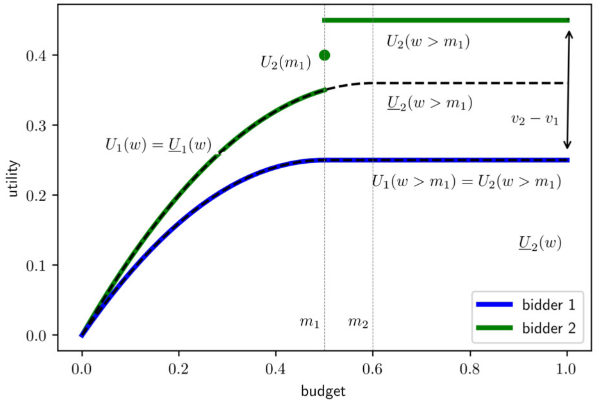

Fig. 12. Utilities in (C1) for $v_1 = 1$, $v_2 = 1.2$ and $F(w) = w$.

Appendix A

A.1. Auxiliary lemmas

Let $\overline{b}_i := \inf\{b : G_i(b) = 1\}$ and $\underline{b}_i := \sup\{b : G_i(b) = 0\}$.

Lemma A.1. *The following holds in any equilibrium:*

(i) $\overline{b} := \overline{b}_1 = \overline{b}_2$ *and* $\underline{b} := \underline{b}_1 = \underline{b}_2$,
(ii) $\overline{b} < min\{v_1, v_2\}$ *and there is no mass point on* $\overline{b}$.

Proof. Part (i): Assume by contradiction that (without loss) $\overline{b}_1 < \overline{b}_2$. Any bid $b_2 \in (\overline{b}_1, \overline{b}_2]$ wins with probability one and yields a payoff $(v_2 - b_2)$. Hence, lowering any bid in this interval yields a strictly profitable deviation via a strictly lower payment.

Assume by contradiction that $\underline{b}_1 < \underline{b}_2$. Then, any bid $b_1 \in [\underline{b}_1, \underline{b}_2)$ of bidder 1 yields a zero payoff. Bidding $\underline{w} + \epsilon$ for some ϵ sufficiently small yields a strictly profitable deviation as such a bid wins with strictly positive probability.

Part (ii): Without loss, let $v_1 \leq v_2$. A bid $b_1 \geq v_1$ yields a non-positive payoff to bidder 1. A bid $b_1 = v_1 - \epsilon$ yields a utility arbitrarily close to 0 for ϵ sufficiently small. A lower bid $b_1 = m_1 < v_1$ yields a strictly higher payoff. Hence, $\overline{b}_1 = \overline{b}_2 < min\{v_1, v_2\}$.

A mass point at $\overline{b} = \overline{w}$ is infeasible, as only bidders with budget realization $\overline{w}$ (which is a zero probability event) can afford $\overline{w}$. Let $\overline{b} < \overline{w}$, and bidder 1 have a mass point at $\overline{b} < min\{v_1, v_2\}$. Then, bidder 2 with $w > \overline{b}$ has a profitable deviation: bid beyond his supremum bid, $\overline{b} + \epsilon$ for some $\epsilon > 0$ sufficiently small. This yields a jump in winning probability for an infinitesimally lower payment. □

The next result is a no gap observation for bidders with asymmetric budget constraints.

Lemma A.2. *For any pair $x, y \in [\underline{b}, \overline{b}]$ with $x < y$, it holds that $G_i(x) < G_i(y)$ for any i.*

Proof. Assume by contradiction that there exist $x < y$ in $[\underline{b}, \overline{b}]$, such that $G_i' := G_i(x) = G_i(y)$. Let $\alpha := \inf\{w : G_i(w) = G_i'\}$ and $\beta := \sup\{w : G_i(w) = G_i'\}$. Bidder $j \neq i$ also places zero mass on (α, β), as lowering j's bid in this range yields a strictly lower payment for the same winning probability.

Without loss of generality, let bidder 1 place no atom at β. Then, the usual no gap argument applies: it is suboptimal for bidder 2 to bid just above the gap; bidding closer to α yields the same winning probability for a strictly lower payment.

If both bidders place a mass point at β, both have a strictly profitable (and feasible) deviation to slightly outbid the mass point. This yields a jump in winning probability of a strictly positive event (because $\beta \leq \overline{b} < \min\{v_1, v_2\}$) for an arbitrarily small increase in expected payment. □

Bidder i's payoff from bidding b is

$$u_i(b) = (v_i - b) \lim_{b' \uparrow b} G_j(b) + \frac{1}{2}(v_i - b) \left(G_j(b) - \lim_{b' \uparrow b} G_j(b) \right).$$

The next lemma shows that the payoff is non-decreasing in the bid in the bidding support.

Lemma A.3. *In any equilibrium, for any $a, b \in [\underline{b}, \overline{b}]$ with $a < b$, $u_i(a) \leq u_i(b)$.*

Proof. Fix some candidate equilibrium. Without loss of generality, let bidder 2 have bidding distribution G_2 and let there exist a, b with $a < b$ such that $u_1(a) > u_1(b)$. For some $\epsilon > 0$ sufficiency small, $u_1(b - \epsilon) < u_1(a)$ (bidder 1 strictly prefers to bid a), irrespective of whether G_2 has an atom at b.[22] Therefore, bidder 1 has a gap in his bidding support in some ϵ-neighborhood of b, contradicting Lemma A.2. □

[22] This is because $u_1(b - \epsilon) \leq (v_1 - (b - \epsilon))G_2(b - \epsilon) \leq (v_1 - (b - \epsilon)) \lim_{b' \uparrow b} G_2(b)$.

A.2. Omitted proofs

Proof of Lemma 1. Let $m_i \in \arg\max(v_i - b_i)F_j(b_i)$ be a best reply bid of an unconstrained bidder i who faces a naive bidder. A bid at or below $\underline{w}$ never wins and yields zero payoff. A bid $b_i \in (\underline{w}, v_i)$ yields a strictly positive expected payoff. Hence, $m_i > \underline{w}$.

The derivative of the expected payoff with respect to b is $(v_i - b_i)f_j(b_i) - F_j(b_i)$. This is positive if $\frac{f_j(b_i)}{F_j(b_i)} \geq \frac{1}{v_i - b_i}$ (Inequality (4) in the main text). $\frac{f_j(b_i)}{F_j(b_i)}$ decreases in b_i by Assumption 1. The right-hand side strictly increases in b_i for $b_i < v_i$. There exists a unique m_i such that $\frac{f_j(m_i)}{F_j(m_i)} = \frac{1}{v_i - m_i}$. For all $w < m_i$, Inequality (4) is strict and the lower bound is strictly increasing. For $w > m_i$, Inequality (4) does not hold and the lower bound is constant at $(v_i - m_i)F_j(m_i)$. Any bid $b_i > m_i$ yields a strictly lower payoff than bidding m_i.

Inequality (4) might also hold strictly for all w. In this case, $m_i = \overline{w}$ as the other bidder never bids above $\overline{w}$. The lower bound strictly increases over the entire domain. □

Proof of Lemma 2. By contradiction, let bidder i with a budget $\tilde{w} \in (w', w'')$ bid $\tilde{b} < \tilde{w}$. Choose bidder i with a lower budget $(\tilde{w} - \epsilon) \in (w', w'')$ for sufficiently small $\epsilon > 0$ such that $\tilde{b} \leq \tilde{w} - \epsilon$. The $(\tilde{w} - \epsilon)$-budget bidder can mimic the $\tilde{w}$-budget type: bid $\tilde{b}$ and obtain the same utility as type $\tilde{w}$. This contradicts $U_i(\tilde{w}) > U_i(\tilde{w} - \epsilon)$.

A bidder i with a budget $w \leq w'$ cannot afford a bid above w'. A bidder i with a budget above w'' bids at least w'' as any lower bid yields lower payoff as U_i is strictly increasing on (w', w''). Hence, $G_i(w) = F_i(w)$ for $w \in (w', w'')$. □

Proof of Lemma 3. By contradiction, let U_i strictly increase on (w', w'') and assume there exists a budget level $\hat{w} \in (w', w'')$ such that $U_i(\hat{w}) > \underline{U}_i(\hat{w})$. As U_i is strictly increasing, by Lemma 2, for all $w \in (w', w'')$, $b_i(w) = w$ and $G_i(w) = F_i(w)$.

Case 1: $m_j < w''$. A bid $\tilde{b} \in (\max\{m_j, w'\}, w'')$ yields utility $(v_j - \tilde{b})F_i(\tilde{b})$ to bidder j, as $G_i(\tilde{b}) = F_i(\tilde{b})$. Bidding m_j instead yields a strictly higher utility,

$$(v_j - \tilde{b})F_i(\tilde{b}) < (v_j - m_j)F_i(m_j) = \underline{U}_j(w \geq m_j).$$

Hence, bidder j never bids in the interval $(\max\{m_j, w'\}, w'')$. But then, U_i cannot be strictly increasing on $w \in (\max\{m_j, w'\}, w'')$, leading to a contradiction.

Case 2: $m_j \geq w''$. The payoff of bidder j is strictly increasing in a bid on (w', w''), as it equals $(v_j - w)F_i(w) = \underline{U}_j(w)$, which is strictly increasing below m_j. Any bid below w' yields an even lower payoff (Lemma A.3). Hence, bidder j with budget $w \in (w', w'')$ chooses the highest feasible bid, $b_j(w) = w$ and $G_j(w) = F_j(w)$ for all $w \in (w', w'')$. But then, $U_i(\hat{w}) = (v_i - \hat{w})F_j(\hat{w}) = \underline{U}_i(\hat{w})$, which contradicts $U_i(\hat{w}) > \underline{U}_i(\hat{w})$. □

The next result shows that there is at most one atom in each bidder's bidding distribution.

Lemma A.4. *In any equilibrium, $G_i(.)$ has at most one atom. If an atom occurs, it is at m_i and $G_j(m_i) = F_j(m_i)$.*

Proof. Without loss, let bidder 1 place an atom at $a \in [\underline{b}, \overline{b})$.[23] As $a < \overline{b} < v_2$, bidder 2's payoff of a bid at a is strictly lower than bidding slightly above the atom:

[23] An atom at a is feasible only if $a < \overline{w}$. An atom at $\overline{b}$ is impossible by Lemma A.1.

$$u_2(a) = (v_2 - a)Pr(b_1 < a) + \frac{1}{2}(v_2 - a)Pr(b_1 = a) < (v_2 - a)G_1(a) = \lim_{b_2 \searrow a} u_2(b_2).$$

By Lemma A.3 (payoff weakly increasing in the bid), $u_2(b_2 < a) \leq u_2(a)$. Bidder 2 with any budget above a can afford to slightly outbid the mass point, and hence will never bid at or below a. Thus, $G_2(a) = F_2(a)$. What does this imply for bidder 1 with an atom at a?

Case 1: $a > m_1$. Bidder 1's payoff from bidding a is strictly dominated by bidding m_1 (thus, an atom at a is suboptimal), as $u_1(a) = (v_1 - a)F_2(a) < (v_1 - m_1)F_2(m_1) \leq u_1(m_1)$.

Case 2: $a < m_1$. Bidder 1's payoff from bidding a is $(v_1 - a)F_2(a) = \underline{U}_1(a)$. But then, a positive mass of bidder 1 with budget above a is bidding a and getting a payoff strictly below the lower bound (as $\underline{U}_1(w > a) > \underline{U}_1(a)$). This yields a contradiction. □

Proof of Lemma 4. Without loss of generality, consider bidder 1. By contradiction, let there exist $a \in [\underline{w}, m_1)$ such that $U_1(a) > \underline{U}_1(a)$. By Lemma A.4, G_2 contains no atoms below m_1. Thus, $(v_1 - w)G_2(w)$ and $U_1(w)$ is continuous at every $w < m_1$.

If $a = \underline{w}$, then $U_1(\underline{w}) > 0$. This is possible only if any bid of bidder 1 wins with a strictly positive probability. This is possible only if bidder 1 is not bidding in some neighborhood of $\underline{b}_2$, which contradicts $\underline{b}_1 = \underline{b}_2$ (see Lemma A.1). Therefore, $U_1(\underline{w}) = \underline{U}_1(\underline{w}) = 0$.

Let $a > \underline{w}$. Define $z = \sup\{w < a : \underline{U}_1(w) = U_1(w)\}$ be the supremum budget level below a at which the equilibrium utility equals the lower bound utility. Note that $z \geq \underline{w}$ exists (because $\underline{U}_1(\underline{w}) = U_1(\underline{w}) = 0$), and $z < a$ because U_1 is continuous at a. As $U_1(a) > \underline{U}_1(a) > \underline{U}_1(z)$ and U_1 continuous below m_1, there needs to exist an open interval in (z, a) on which U_1 is strictly increasing and strictly above the lower bound. This contradicts Lemma 3 that shows that if equilibrium utility is strictly increasing, the lower bound binds.

Finally, as $U_i(w) = \underline{U}_i(w)$ is strictly increasing below m_1, all bidders bid their entire budget on $(\underline{w}, m_1)$ by Lemma 2. □

Proof of Lemma 5. *Proof of 1.* Let U_i have a jump discontinuity[24] at $x > \underline{w}$. The discontinuity can only occur if G_j contains an atom at x, at which a bidder i with budget x is bidding.[25] This follows as there are no gaps in the bidding support (Lemma A.2) and without an atom at x the payoff $(v_i - x)G_j(x)$ is continuous at x. By Lemma A.4, atoms can only occur at m_1 for G_1 and m_2 for G_2. Hence, U_1 has at most one jump discontinuity at $x = m_2$, and U_2 has at most one jump discontinuity at $x = m_1$.

If bidder j places an atom at m_j, by Lemma A.4, $F_i(m_j) = G_i(m_j)$. Hence, bidder j's utility from bidding at the atom is $U_j(m_j) = (v_j - m_j)F_i(m_j) = \underline{U}_j(m_j)$.[26]

Proof of 2. Assume by contradiction that there exist a, b with $a < b$ such that $U_i(a) < U_i(b)$ in one of the three intervals in Statement 2. in Lemma 5. U_i is a monotonic function. It is only possible to have $U_i(a) < U_i(b)$ if at least one of the following statements holds:

(i) U_i has a jump discontinuity in $[a, b]$.
(ii) U_i is strictly increasing on some open interval $(c, d) \subseteq [a, b]$.

24 All discontinuities of U_i are jump discontinuities as U_i is monotonic.

25 If the mass point were below, bidder i with a lower budget could afford it and obtain the same utility as budget type x, which contradicts the existence of a jump discontinuity in U_i at budget realization x.

26 A bid m_j is the highest bid bidder j can afford, and payoffs are non-decreasing in bids by Lemma A.3.

Case (i) is not possible in equilibrium as jumps can only occur at m_1 or m_2 (see the first paragraph of this proof). Case (ii) also leads to a contradiction: by Lemma 3, $U_i(w) = \underline{U}_i(w)$ for all $w \in (c, d)$. $\underline{U}_i(w)$ is constant for $w > m_i$, which yields a contradiction if $[a, b] \in (m_i, \overline{w}]$. For $[a, b] \subseteq (m_1, m_2)$, let Case (ii) hold for U_2. By Lemma 2, $G_2(w) = F_2(w)$ for $w \in (c, d)$. But then, bidder 1's payoff from any bid $w \in (c, d)$ yields a payoff of $(v_1 - w)F_2(w)$ which is strictly lower than the payoff from bidding m_1, yielding a contradiction.

Proof of 3. There is no mass point at $\overline{b}$ (Lemma A.1). Therefore, a bid $\overline{b}$ yields a payoff of $v_i - \overline{b}$. A bidder with budget $\overline{w}$ can afford to bid $\overline{b}$, and any lower bid in the bidding support yields weakly lower payoff by Lemma A.3. Thus, $U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2$. □

Proof of Theorem 1. Case $m_1 = m_2 = \overline{w}$. Lemma 4 pins down the utilities for $w < \overline{w}$ and $\overline{b} = \overline{w}$. As there cannot be a mass point at $\overline{b} = \overline{w}$, it also holds that $U_i(\overline{w}) = \underline{U}_i(\overline{w})$.

Case $m_1 < m_2 \leq \overline{w}$. By Lemma 5, $U_2(w)$ is constant for budget realizations $w \in (m_1, \overline{w}]$. Note that $U_2(w) \geq \underline{U}_2(m_2)$ for $w \in (m_1, \overline{w}]$, with $\underline{U}_2(m_2)$ being the highest value for the lower bound, which the equilibrium utility cannot undercut. Moreover, $\underline{U}_2(m_2) > \underline{U}_2(m_1)$, as the lower bound is strictly increasing for $w < m_2$. By Theorem 4, the lower bound binds, $\underline{U}_2(w) = U_2(w)$ for $w < m_1$. In sum, $U_2(w > m_1) - U_2(w < m_1) \geq \underline{U}_2(m_2) - \underline{U}_2(m_1) > 0$. Therefore, if $m_1 < m_2$, G_1 has an atom at m_1 so that U_2 has a jump discontinuity at m_1.

By Lemma 5, U_1 is constant on (m_1, m_2) and $(m_2, \overline{w}]$. If G_2 has no atom at m_2, U_1 is continuous at m_2 and therefore constant on $(m_1, \overline{w}]$. It is equal to the lower bound on (m_1, m_2) as bidder 1 places a mass point at m_1 (by Lemma 5, $U_1(m_1) = \underline{U}_1(m_1)$ if U_2 has a discontinuity at m_1). Thus, if G_2 has no atom at m_2, $U_1(w) = \underline{U}_1(w)$ for all w.

By Lemma 5, $U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2$. When does bidder 2 place a mass point at m_2?

Let (C1) hold: $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$. If G_2 has an atom at m_2, then $U_1(\overline{w}) > \underline{U}_1(m_1)$ and $U_2(\overline{w}) = \underline{U}_2(m_2)$. But then $U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2 > \underline{U}_1(m_1) - \underline{U}_2(m_2)$, which contradicts (C1). Thus, if (C1) holds, G_2 cannot have an atom at m_2.

As bidder 2 does not place a mass point, $U_1(\overline{w}) - U_2(\overline{w}) = \underline{U}_1(m_1) - U_2(w > m_1) = v_1 - v_2$. This pins down $U_2(w > m_1) = \underline{U}_1(m_1) - (v_1 - v_2)$ in any equilibrium. Ties are broken randomly. Thus, the utility from bidding at exactly at the mass point (which only bidder 2 with budget m_2 does), $U_2(m_2)$, is the average of the left and right hand side limit of U_2.

Let (C2) hold: $\underline{U}_1(m_1) - \underline{U}_2(m_2) < v_1 - v_2$. If bidder 2 does not place a mass point, then $U_1(\overline{w}) = \underline{U}_1(m_1)$ and $U_2(\overline{w}) \geq \underline{U}_2(m_2)$. In this case, $U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2 \leq \underline{U}_1(m_1) - \underline{U}_2(m_2)$, which yields a contradiction to (C2). Hence, in any equilibrium G_2 has an atom at m_2, U_2 is constant on (m_1, m_2) and $U_2(m_2) = \underline{U}_2(m_2)$. Furthermore, $U_2(\overline{w}) = \underline{U}_2(m_2)$ and U_1 satisfies $U_1(\overline{w}) - \underline{U}_2(m_2) = v_1 - v_2$, which pins down $U_1(w > m_2) = U_2(\overline{w})$. $U_1(m_2)$ is the average of the right and left hand side limit due to the equal tie-breaking rule.

Case $m_1 = m_2 =: m < \overline{w}$. By Footnote 8, bidders are labeled such that $\underline{U}_1(m) - \underline{U}_2(m) \geq v_1 - v_2$. Let $\underline{U}_1(m) - \underline{U}_2(m) > v_1 - v_2$. Then, by the same arguments as above, bidder 1 places a mass point at m which leads to a jump discontinuity in U_2 at m. This pins down the utility of bidder 1 to $U_2 = \underline{U}_2$ for all $w \geq m$, and raises U_2 via a jump discontinuity to satisfy $U_2(w > m) = \underline{U}_1(m) - (v_1 - v_2)$. Let $\underline{U}_1(m) - \underline{U}_2(m) = v_1 - v_2$. Then, there are no mass points as otherwise $U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2$ cannot be satisfied. □

Proof of Theorem 2. There is no mass point at $\overline{b}$ (Lemma A.1), and payoffs are weakly increasing in bids for the entire bidding support (Lemma A.3). Thus, $U_i(\overline{w}) = v_i - \overline{b}$. Equilibrium utilities are given in Theorem 1. In Case (C1), $U_1(\overline{w}) = \underline{U}_1(m_1) = (v_1 - m_1)F_2(m_1) = v_1 - \overline{b}$.

In Case (C2), $U_2(\overline{w}) = \underline{U}_2(m_2) = (v_2 - m_2)F_1(m_2) = v_2 - \overline{b}$. Solving for $\overline{b}$ in both cases yields the expressions for the supremum bids.

By Lemma 4, $G_i = F_i$ and $b_i(w) = w$ for $w < m_1$. Next, I derive G_1 and G_2 for $w \geq m_1$.

Bidder 1: By Theorem 1, $U_2(w > m_1) = U_2(\overline{w})$. As there are no gaps in the bidding support (Lemma A.2), any bid on $(m_1, \overline{b}]$ yields the same expected payoff

$$(v_2 - b)G_1(b) = v_2 - \overline{b}.$$

Solving for $G_1(b)$ yields the equilibrium bid distribution for $w > m_1$. Finally, as G_1 is right-continuous, $G_1(m_1) = \lim_{w \searrow m_1} G_1(w)$ completes the result for G_1.

Bidder 2: Let (C1) hold. By Theorem 1, $U_1(w \geq m_1) = U_1(\overline{w}) = \underline{U}_1(m_1)$. For all $b \in [m_1, \overline{b}]$, it holds that

$$(v_1 - b)G_2(b) = v_1 - \overline{b} = (v_1 - m_1)F_2(m_1).$$

Solving for $G_2(b)$ yields the equilibrium bid distribution for $w \geq m_1$ in Theorem 2.

Let (C2) hold. By Theorem 1, $U_1(w < m_2) < U_1(m_2)$. Then, $b_1(m_2) = m_2$ (no bidder 1 with a budget $w < m_2$ can mimic m_2's bid and get the same utility), and $\overline{b} \geq m_2$. There are no gaps in the bidding support, and only bidder 1 with a budget $w \in [m_1, m_2)$ bids in $[m_1, m_2)$. Solving for G_2 yields the equilibrium bid distribution in this interval:

$$(v_1 - b)G_2(b) = \underline{U}_1(m_1) = (v_1 - m_1)F_2(m_1).$$

By Theorem 1, $U_1(w > m_2) = U_1(\overline{w}) = v_1 - \overline{b}$. Then, for all $b \in (m_2, \overline{w}]$,

$$(v_1 - b)G_2(b) = v_1 - \overline{b}.$$

Solving for $G_2(b)$ yields the equilibrium bid distribution for $w > m_2$ in Theorem 2. Finally, $G_2(m_2) = \lim_{w \searrow m_2} G_2(w)$ completes the proof. □

Proof of Theorem 3. The proof is by construction. If (C1) holds, let the bidding functions be as in Equations (11) and (12). If (C2) holds, let the bidding functions of bidder 1 be Equation (11) and the bidding function of bidder 2 be

$$b_2(w) = \begin{cases} w & \text{if} \quad w \in [\underline{w}, m_1), \\ v_1 - \frac{(v_1 - m_1)F_2(m_1)}{F_2(w)} & \text{if} \quad w \in [m_1, F_2^{-1}(\frac{(v_1 - m_1)F_2(m_1)}{v_1 - m_2})), \\ m_2 & \text{if} \quad w \in [F_2^{-1}(\frac{(v_1 - m_1)F_2(m_1)}{v_1 - m_2}), F_2^{-1}(\frac{v_1 - \overline{b}}{v_1 - m_2})], \\ v_1 - \frac{v_1 - \overline{b}}{F_2(w)} & \text{otherwise.} \end{cases} \tag{17}$$

First, I show that these bidding functions aggregate into the *bid distributions* $\{G_i\}_{i=1,2}$ in Theorem 2. For $w < m_1$, $b_i(w) = w$ and hence $G_i(b) = F_i(b)$ is satisfied. Let $w \geq m_1$. For a bid in $[m_1, \overline{b}]$, let $w_i(b) = \sup\{w : b_i(w) = b\}$ be the highest budget realization of a bidder i who bids b. Note that $F_i(w_i(b)) = G_i(b)$. Consider bidder 2 in Equation (12) for (C1) with a budget $w \geq m_1$. His bidding function $b_2(w)$ can be rewritten using the inverse bidding function $w_2(b)$: $b = v_1 - \frac{v_1 - \overline{b}}{G_2(b)}$. Solving this expression for $G_2(b)$ yields the bid distribution in Theorem 2. The same approach applied to bidder 1 in Equation (11) and to bidder 2 in Equation (17) for (C2) yields the required bid distributions.

Next, I show the *feasibility* ($b_i(w) \leq w$ for all w and i). For any bid equal to or below m_1, feasibility is trivially satisfied. It is left to show that (1) $v_1 - \frac{(v_1 - m_1)F_2(m_1)}{F_2(w)} \leq w$; (2) $v_i - \frac{v_i - \overline{b}}{F_j(w)} \leq w$; (3) $F_2^{-1}(\frac{(v_1 - m_1)F_2(m_1)}{v_1 - m_2}) \geq m_2$. Rewrite (1) as $(v_1 - w)F_2(w) \leq (v_1 - m_1)F_2(m_1)$, which holds by Lemma 1. Rewrite (2) as $(v_i - w)F_j(w) \le v_i - \overline{b}$. This is true since for $w \ge m_i$, it holds that $(v_i - w)F_j(w) \le \underline{U}_i(\overline{w}) \le U_i(\overline{w}) = v_i - \overline{b}$. For (3), applying F_2 to both sides yields $(v_1 - m_1)F_2(m_1) \ge (v_1 - m_2)F_2(m_2)$. This holds by Lemma 1.

Finally, I show *optimality*. Let $w < m_1$. Any bid $b < w$ yields strictly lower utility than $b_i(w) = w$.[27] Any higher bid $b > w$ is not feasible. Let $w \ge m_1$ and (C1) hold. Any bid of bidder 1 in the interval $[m_1, \overline{b}]$ yields constant utility $v_1 - \overline{b}$. Any bid above $\overline{b}$ or below m_1 yields a strictly lower utility. Bidder 2 with budget m_1 has a higher utility from bidding exactly at the potential atom at m_1 than from any lower bid, and cannot afford to bid higher. Bidder 2 with $w > m_1$ is indifferent between any bid on $(m_1, \overline{b}]$. Any bid outside of this interval is strictly worse. Optimality in (C2) is established by the same technique. □

Proof of Proposition 3. Proof of 2. ⇒ 1. For $w < m_1$, by Lemma 3, it holds that $U_i(w) = \underline{U}_i(w) \ge \underline{U}_j(w) = U_j(w)$. For $w \ge m_1$ the following case distinction establishes the result, utilizing the equilibrium utilities as in Theorem 1.

Case (C1). If $\underline{U}_1(w) \ge \underline{U}_2(w)$, and $v_1 \ge v_2$, then for $w \in [m_1, \overline{w}]$

$$U_2(w) \le U_2(\overline{w}) = (v_2 - v_1) + \underline{U}_1(m_1) \le \underline{U}_1(m_1) = U_1(m_1) \le U_1(w).$$

If $\underline{U}_2(w) \ge \underline{U}_1(w)$ and $v_2 \ge v_1$, then for $w \in [m_1, \overline{w}]$

$$U_1(w) \le U_1(\overline{w}) = \underline{U}_1(\overline{w}) \le \frac{1}{2}\left[\underline{U}_1(m_1) + \underline{U}_2(m_1) + (v_2 - v_1)\right] = U_2(m_1) \le U_2(w).$$

Case (C2). If $\underline{U}_1(w) \ge \underline{U}_2(w)$, and $v_1 \ge v_2$, then for $w \in [m_1, \overline{w}]$

$$U_1(w) \ge U_1(m_1) = \underline{U}_1(m_2) \ge \underline{U}_2(m_2) = U_2(\overline{w}) \ge U_2(w).$$

Finally, let $\underline{U}_2(w) \ge \underline{U}_1(w)$, and $v_2 \ge v_1$. For $w > m_1$, it holds that

$$U_2(w) = \underline{U}_2(m_2) \ge \underline{U}_2(m_2) + (v_1 - v_2) = U_1(\overline{w}) \ge U_1(w).$$

At $w = m_1$, $U_1(m_1) = \underline{U}_1(m_1) \le \underline{U}_2(m_1) \le U_2(m_1)$. This establishes 2. ⇒ 1.

Proof of 1. ⇒ 2. If $v_i < v_j$, then $U_i(\overline{w}) = v_i - \overline{b} < v_j - \overline{b} = U_j(\overline{w})$. Therefore, $v_i \ge v_j$ is a necessary condition for 1.

Let $v_i \ge v_j$. It is left to show that $U_i \ge U_j$ implies $\underline{U}_i \ge \underline{U}_j$ at every w. Assume by contradiction that there exists $\tilde{w}$ such that $\underline{U}_i(\tilde{w}) < \underline{U}_j(\tilde{w})$, and $U_i \ge U_j$ for all w. If $\tilde{w} < m_1$, there is an immediate contradiction by Lemma 3. Let $\tilde{w} \ge m_1$.

First, let $i = 2$. $\underline{U}_1$ is constant above m_1, and hence $\underline{U}_1(m_1) = \underline{U}_1(\tilde{w})$. As $U_2 \ge U_1$ at all w, by Lemma 3, $\underline{U}_2(w) \ge \underline{U}_1(w)$ for $w < m_1$. By continuity of $\underline{U}_1$ and $\underline{U}_2$, it also holds that $\underline{U}_2(m_1) \ge \underline{U}_1(m_1)$. But then, this contradicts the existence of $\tilde{w}$, as

$$\forall w \ge m_1, \qquad \underline{U}_2(w) \ge \underline{U}_2(m_1) \ge \underline{U}_1(m_1) = \underline{U}_1(w).$$

Second, let $i = 1$ such that for all w, $U_1 \ge U_2$. It holds that $\underline{U}_1(m_1) = \underline{U}_1(\tilde{w})$ and $\underline{U}_2(m_2) \ge \underline{U}_2(\tilde{w})$. As established above, $v_1 \ge v_2$. Thus, Case (C2) in Theorem 1 holds. In this case, take any $w \in (m_1, m_2)$.[28] By Theorem 1, this yields a contradiction to $U_1 \ge U_2$, as $U_1(w) = \underline{U}_1(m_1) = \underline{U}_1(\tilde{w}) < \underline{U}_2(\tilde{w}) \le \underline{U}_2(m_2) = U_2(w)$. □

[27] This is because for $w < m_1$, $G_i = F_i$ and $(v_i - w)F_j(w)$ increases in w by Lemma 1.

[28] The interval (m_1, m_2) is non-empty if such a $\tilde{w}$ exists and it holds that $\underline{U}_1 \ge \underline{U}_2$ for $w > m_1$.

Proof of Proposition 4. Let $k = 2$, such that $\hat{F}_2 \geq_{RHR} F_2$. The proof for $k = 1$ with $\hat{F}_1 \geq_{RHR} F_1$ works accordingly and is therefore omitted.

For bidder 2, m_2 and $\underline{U}_2$ are not affected by F_2. For bidder 1, let $m_1 = \arg\max_b (v_1 - b)F_2(b)$ and $\hat{m}_1 = \arg\max_b (v_1 - b)\hat{F}_2(b)$. Due to RHR-dominance (and thus, FOSD), $m_1 \leq \hat{m}_1$ and $\underline{U}_1(w) \geq \underline{\hat{U}}_1(w)$ for all w. As $\underline{U}_1$ is non-decreasing, $\underline{U}_1(m_1) \geq \underline{\hat{U}}_1(\hat{m}_1)$.

If $\hat{m}_1 > m_2$, the labels of the bidders as bidder 1 and 2 are reversed with F_2 versus $\hat{F}_2$. Let $i \in \{1, 2\}$ refer to the bidder identities with F_2, and $\hat{i} \in \{\hat{1}, \hat{2}\}$ with $\hat{F}_2$.

Proof of $\hat{\overline{b}} \geq \overline{b}$: First, let $i = \hat{i}$. There are three possibilities: 1. (C1) holds with both budget distributions, 2. (C2) holds with both budget distributions, 3. (C1) holds under F_2 and (C2) holds under $\hat{F}_2$.[29] Consider the equation for $\overline{b}$ and $\hat{\overline{b}}$ in Theorem 2. If 1., then $\hat{\overline{b}} \geq \overline{b}$ by FOSD and $\hat{m}_1 \geq m_1$. If 2., then $\hat{\overline{b}} = \overline{b}$. If 3., then $\hat{\overline{b}} - \overline{b} = \underline{U}_1(m_1) - \underline{U}_2(m_2) - (v_1 - v_2) \geq 0$, where the last inequality follows from Case (C1) with F_2.

Second, let $i \neq \hat{i}$ ($\hat{1} = 2$ and $\hat{2} = 1$) because $m_1 \leq m_2 < \hat{m}_1$. If with both budget distributions (C1) holds, then $v_2 = v_{\hat{1}}$ and $F_1 = F_{\hat{2}}$. Hence, $\hat{\overline{b}} - \overline{b} = v_1 - (v_1 - m_1)F_2(m_1) - v_2 - (v_2 - m_2)F_1(m_2) \geq 0$. If (C2) holds in both cases, then $\hat{\overline{b}} - \overline{b} = v_1 - v_2 - \left[\underline{\hat{U}}_1(\hat{m}_1) - \underline{U}_2(m_2)\right] \geq v_1 - v_2 - \left[\underline{U}_1(m_1) - \underline{U}_2(m_2)\right] \geq 0$. If (C1) holds with F_2, and $(C2)$ holds with $\hat{F}_2$,[30] then $\hat{\overline{b}} - \overline{b} = v_1 - \underline{\hat{U}}_1(m_1) - v_1 + \underline{U}_1(m_1) \geq 0$.

Proof of $\hat{G}_i$ *being FOSD over* G_i: First, let $\hat{1} = 1$ and $\hat{2} = 2$. The bid distribution of bidder 1 with F_2 is in Equation (10) in Theorem 2. With $\hat{F}_2$, it becomes

$$\hat{G}_1(b) = \begin{cases} F_1(b) \text{ if } b < \hat{m}_1, \\ \frac{v_2 - \hat{\overline{b}}}{v_2 - b} \text{ if } b \in [\hat{m}_1, \hat{\overline{b}}]. \end{cases}$$

Below m_1, $\hat{G}_1 = G_1$. Above $\hat{m}_1$, it is immediate that $\hat{G}_1 \leq G_1$ as $\hat{\overline{b}} \geq \overline{b}$. For $b \in [m_1, \hat{m}_1)$, it holds that $v_2 - \overline{b} = U_2(\overline{w}) \geq \underline{U}_2(m_1) \geq (v_2 - b)F_1(b)$. Thus, $\frac{v_2 - \overline{b}}{v_2 - b} \geq F_1(b)$ and hence, $\hat{G}_1 \leq G_1$ for all b.

The bid distribution of bidder 2 with $\hat{F}_2$ is given in Equation (10) in Theorem 4. Substituting $\hat{F}_2$ for F_2, $\hat{m}_1$ for m_1, and $\hat{\overline{b}}$ for $\overline{b}$, yields the new bid distribution under $\hat{F}_2$. As before, it is apparent that $\hat{G}_2 \leq G_2$ for bids below m_1 or above m_2. For $b \in [m_1, \hat{m}_1)$ and $b \in [\hat{m}_1, m_2)$, the following inequality establishes that $\hat{G}_2 \leq G_2$,

$$(v_1 - m_1)F_2(m_1) \geq (v_1 - \hat{m}_1)\hat{F}_2(\hat{m}_1) \geq (v_1 - b)\hat{F}_2(b).$$

Finally, let the labels of the bidders change such that $\tilde{1} = 2$ and $\tilde{2} = 1$. Bidder 1's bid distribution with F_2 and $\hat{F}_2$ follows from Equation (10). With $\hat{F}_2$, his bid distribution is

$$G_{\hat{2}}(b) = \hat{G}_1(b) = \begin{cases} F_{\hat{2}}(b) = F_1(b) \text{ if } b < m_{\hat{1}} = m_2, \\ \frac{v_{\hat{1}} - m_{\hat{1}} F_{\hat{2}}(m_{\hat{1}})}{v_{\hat{1}} - b} = \frac{(v_2 - m_2)F_1(m_2)}{v_2 - b} \text{ if } b \in [m_{\hat{1}}, m_{\hat{2}}) = [m_2, \hat{m}_1), \\ \frac{v_{\hat{1}} - \hat{\overline{b}}}{v_{\hat{1}} - b} = \frac{v_1 - \hat{\overline{b}}}{v_2 - b} \text{ if } b \in [m_{\hat{2}}, \hat{\overline{b}}] = [\hat{m}_1, \hat{\overline{b}}]. \end{cases} \tag{18}$$

[29] It cannot be the other way since $\underline{U}_1(m_1) \geq \underline{U}_1(\hat{m}_1)$.

[30] Note that it is impossible to have (C2) with F_2 and (C1) with $\hat{F}_2$.

Below m_1 and above $\hat{m}_1$, it immediately holds that $\hat{G}_1 \leq G_1$ using $\hat{\overline{b}} \geq \overline{b}$. For $b \in [m_1, \hat{m}_1)$, it holds that $v_2 - \overline{b} = U_2(\overline{w}) \geq (v_2 - m_2)F_1(m_2) \geq (v_2 - b)F_1(b)$. This establishes the inequality for the entire interval, and $\hat{G}_1 \leq G_1$. For bidder 2, the same approach establishes that $\hat{G}_2 \leq G_2$ when identities change.

The utility at every budget level is also lower with $\hat{F}_2$ than with F_2. Without loss, consider the utility of bidder 1 with budget w. Any bid yields a weakly lower winning probability with $\hat{G}_2$ than with G_2, while the surplus of a win $(v_1 - b)$ remains the same.

Both strictly worse off: I show this by example. Let $v_1 = v_2 = 1$, $F_1(w) = w^2$, $F_2(w) = w$, and $\hat{F}_2(w) = w^2$ for $w \in [0, 1]$. Then, $m_2 = \hat{m}_1 = \frac{2}{3}$ and $m_1 = \frac{1}{2}$. Using Theorem 1,

$$\hat{EU}_1 = \hat{EU}_2 = \int_0^1 U_1(w) f_1(w) dw = \int_0^{\frac{2}{3}} (1-w)w^2 2w dw + (1 - \frac{2}{3})\left(\frac{2}{3}\right)^2 \frac{5}{9} \approx 0.13, \tag{19}$$

$$EU_1 = \int_0^1 U_1(w) f_1(w) dw = \int_0^{\frac{1}{2}} (1-w)w 2w dw + (1 - \frac{1}{2})\frac{1}{2}\frac{1}{2} \approx 0.24, \tag{20}$$

$$EU_2 = \int_0^1 U_2(w) f_2(w) dw = \int_0^{\frac{1}{2}} (1-w)w^2 dw + (1 - \frac{1}{2})\frac{1}{2}\frac{1}{2} \approx 0.15. \quad \square \tag{21}$$

Proof of Proposition 7. Let $s_1 \in S$ and $s_2 \in S$ be the budget type realization of bidders 1 and 2. First, consider the disclosure regime in which s_1 and s_2 are public. Let $U_i^D(w; s_i, s_j)$ be bidder i's equilibrium utility with budget w if the two budget types are s_i and s_j. By Theorem 1, equilibrium utility is above the lower bound,

$$U_i^D(w; s_i, s_j) \geq \underline{U}_i^D(w; s_i, s_j) := \max_{b \leq w}(v - b)F_{s_j}(b).$$

Second, consider the no-disclosure regime in which s_1 and s_2 are private information. Let $\underline{U}_i^{ND}(w; s_i)$ be bidder i's lower bound utility with budget w who knows only his own budget type s_i, where the other bidder with unknown budget type always bids his entire budget,

$$\underline{U}_i^{ND}(w; s_i) = \max_{b \leq w}(v - b)F(b),$$

where the expected budget distribution of the other bidder is $F(b) = \sum_{s_j \in S} p_{s_j} F_{s_j}(b)$. The lower bound at a budget w does not depend on a bidder's own budget type s_i. This is because his own budget distribution F_{s_i} is not payoff relevant after learning w.

Both bidders share the same lower bound for any w, irrespective of the own budget type, $\underline{U}_i^{ND}(w; s_i) = \underline{U}_j^{ND}(w; s_j)$ for any $s_i, s_j \in S$. Thus, by Corollary 2, in any equilibrium, the lower bound and equilibrium utility at all w coincide in the no-disclosure regime,

$$U_i^{ND}(w; s_i) = \underline{U}_i^{ND}(w; s_i), \text{ for all } w, i, s_i.$$

Finally, I establish that at every w for any budget type s_i, equilibrium utility in the no-disclosure regime is weakly lower than in the disclosure regime.

$$\begin{aligned} U_i^{ND}(w; s_i) = \underline{U}_i^{ND}(w; s_i) &= \max_{b \le w}(v-b) \sum_{s_j \in S} p_{s_j} F_{s_j}(b) \\ &\le \sum_{s_j \in S} p_{s_j} \max_{b \le w}(v-b) F_{s_j}(b) \\ &= \sum_{s_j \in S} p_{s_j} \underline{U}_i^D(w; s_i, s_j) \\ &\le \sum_{s_j \in S} p_{s_j} U_i^D(w; s_i, s_j). \end{aligned}$$

The revenue of the auctioneer is the generated total surplus, v, minus the expected utilities of both bidders (see Equation (14)). Taking expectations over bidder i's budget types s_i and budget realizations w, the ex ante expected utility of a bidder under no disclosure is lower than under the disclosure regime. Hence, a lower expected utility for the bidders in the no-disclosure regime corresponds to a higher revenue for the seller. □

Proof of Theorem 4. Under Assumption 2 instead of Lemma 1, the auxiliary Lemmas A.1, A.2, A.3, A.4 and the Lemmas 2, 3, 4, 5 also hold for the all-pay auction, after substituting $\underline{U}_i^a$ for $\underline{U}_i$, m_i^a for m_i, and $\overline{b}^a$ for $\overline{b}$ as the supremum bid. Theorem 1 also holds after substituting the notation: equilibrium utilities are unique. This can be checked using exactly the same steps of the proofs as for the FPA (therefore, I omit the proofs), and using the payment rule of the all-pay auction instead of that for the FPA. The assumption $\underline{w} = 0$ guarantees that the utility of bidding the lowest budget is zero. Differentiate between:

$(C1)$ $\underline{U}_1^a(m_1^a) - \underline{U}_2^a(m_2^a) \ge v_1 - v_2$,
$(C2)$ $\underline{U}_1^a(m_1^a) - \underline{U}_2^a(m_2^a) < v_1 - v_2$.

Supremum bids and unique bid distributions: As Lemma A.1 also holds for the all-pay auction, there is no mass point at $\overline{b}^a$. Thus, $U_i(\overline{w}) = v_i - \overline{b}^a$. Furthermore, as Theorem 1 also holds for the all-pay auction,

$$U_1^a(\overline{w}) = \underline{U}_1^a(m_1^a) = v_1 F_2(m_1^a) - m_1^a = v_1 - \overline{b}^a \text{ if } (C1),$$
$$U_2^a(\overline{w}) = \underline{U}_2^a(m_2^a) = v_2 F_1(m_2^a) - m_2^a = v_2 - \overline{b}^a \text{ if } (C2).$$

Solving both equations for $\overline{b}^a$ yields the supremum bids in the theorem.

As Lemma 4 holds for the all-pay auction, $G_i^a = F_i^a$ for $w < m_1^a$. For G_2^a on $[m_1^a, m_2^a)$ and $(m_2^a, \overline{b}^a]$, and for G_1^a on $(m_1^a, \overline{b}^a]$, the same steps as in the proof of Theorem 2 for the all-pay auction can be used: equating the expected payoff $v_i G_j^a(b) - b$ to a fixed utility given by Theorem 1 and solving for $G_j^a(b)$ yields the unique bid distributions in Theorem 4. The bid distributions at possible atoms, $G_1^a(m_1^a)$ and $G_2^a(m_2^a)$, are obtained as in Theorem 2 by taking the right-hand limit.

Equilibrium existence follows via the following weakly monotonic pure strategies,

$$b_1^a(w) = \begin{cases} w & \text{if } \quad w \in [0, m_1^a), \\ m_1^a & \text{if } \quad w \in [m_1^a, F_1^{-1}(\frac{v_2 - \overline{b}^a + m_1^a}{v_2})], \\ \overline{b}^a - v_2(1 - F_1(w)) & \text{otherwise.} \end{cases} \tag{22}$$

$$b_2^a(w) = \begin{cases} w & \text{if} \quad w \in [0, m_1^a), \\ v_1[F_2(w) - F_2(m_1^a)] + m_1^a & \text{if } w \in [m_1^a, F_2^{-1}\left(F_2(m_1^a) + \frac{m_2^a - m_1^a}{v_1}\right)), \\ m_2^a & \text{if } w \in [F_2^{-1}\left(F_2(m_1^a) + \frac{m_2^a - m_1^a}{v_1}\right), F_2^{-1}(\frac{v_1 - \overline{b}^a + m_2^a}{v_1})), \\ \overline{b}^a - v_1(1 - F_2(w)) & \text{otherwise.} \end{cases} \tag{23}$$

As for the FPA, it can be easily checked that these bidding functions satisfy the bid distributions in Theorem 4, and yield the utilities as in Theorem 1 for the all-pay auction.

Optimality is satisfied: bidders with budget strictly below m_1^a (and bidder 2 with budget m_1^a) prefer to bid higher, but cannot afford it. If (C1) holds, all the remaining budget types have no strictly profitable deviation, as any bid yields the same or lower utility. If (C2) holds, all the remaining bidder 2 types have the same equilibrium utility and cannot improve their payoff strictly by bidding differently. The remaining bidder 1 types have no profitable deviation: note that $b_1^a(m_2^a) = m_2^a$. Thus, bidders with a budget below or at m_2^a cannot afford the strictly profitable deviation to bid above m_2^a, while bidders with a budget above m_2^a achieve the highest possible utility with bids above m_2^a and have no strictly profitable deviation.

Finally, I establish feasibility. For bidder 1, $b_1^a(w) \leq w$ follows immediately for $w \leq F_1^{-1}(\frac{v_2 - \overline{b}^a + m_1^a}{v_2})$. For higher budgets,

$$\begin{aligned} w - b_1^a(w) &= w - \left(\overline{b}^a - v_2(1 - F_1(w))\right) \\ &= (v_2 - \overline{b}^a) - (v_2 F_1(w) - w) \geq U_2^a(\overline{w}) - \underline{U}_2^a(\overline{w}) \geq 0. \end{aligned}$$

For bidder 2, the same argument holds for $w \leq m_1^a$ and $w \geq F_2^{-1}(\frac{v_1 - \overline{b}^a + m_2^a}{v_1})$. Feasibility also holds for the remaining budget realizations, as

$$w - \left(v_1\left[F_2(w) - F_2(m_1^a)\right] + m_1^a\right) = \left(v_1 F_2(m_1^a) - m_1^a\right) - (v_1 F_2(w) - w) \geq 0, \tag{24}$$

where the last inequality holds by Assumption 2. □

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