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The interaction of debt financing, cash grants and the optimal investment policy under uncertainty

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ABSTRACT

Significant progress has been made toward understanding a levered firm's optimal investment policy under uncertainty. To date, however, not much work has concentrated on the time scale trade-off and collectively held real options. We employ a game-theoretic real option model between a firm and a government to analyze the effect of uncertainty and investment stimulus in the form of cash grants on optimal investment timing, financing and investment scaling. We find that the jointly held real option between the firm and the government leads to underinvestment, regardless of whether the firm has the possibility to issue debt. Subsidies, however, reduce the level of underinvestment. Notably, the results indicate that even though levered firms receive less support, they invest more than unlevered firms. This challenges recent findings that a firm's optimal investment level is not affected by the way it finances a project. Similarly, we find that for realistic parameter constellations the levered firm's optimal investment threshold can be higher than that of its unlevered counterpart. This indicates that the availability of tax shield benefits does not necessarily serve as an incentive to invest earlier. Finally, we show that the effect of cash flow uncertainty on the equilibrium level of grants is ambiguous and triggers the switch from a subsidy to non-subsidy regime.

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1. Introduction

One of the key strategic decisions in firms concerns the timing and financing of an investment. According to corporate finance theory, firms have to consider two trade-offs. First, the optimal investment timing decision is determined by the trade-off between early commitment to cash flows and late commitment to maintaining managerial flexibility. Second, the optimal leverage decision is determined by the trade-off between interest tax shield benefits and bankruptcy costs of debt in the event of default.

Option-based valuation of investments has been proposed as an efficacious analytical tool for addressing these trade-offs and the literature provides guidance on how to determine the optimal investment timing of an unlevered firm under uncertainty (see, e.g., Dixit & Pindyck, 1994; Mauer & Triantis, 1994; McDonald & Siegel, 1986; Trigeorgis, 1999; Trigeorgis & Tsekrekos, 2018). These approaches have been extended to take the financing decision into account, especially debt financing by means of corporate bonds (see, e.g., Mauer & Sarkar, 2005; Shibata & Nishihara, 2012).

However, bonds are not the only source of financing key strategic investments. Different kinds of public support are also important sources of financing. According to a recent report by the European Commission, EU Member States granted a total amount of roughly EUR 10.9 billion to promote corporate R&D investments in 2010, which corresponds to circa 18 percent of total aid for industry and services.[1] Besides tax credits, the most common types of governmental support are cash grants and infrastructure assistance (e.g., site procurement and preparation). One program that offers such cash grants is the Texas Enterprise Fund (TEF). Since its creation in 2004, it has awarded over 140 cash grants to levered multinationals such as Apple Inc., eBay, Lockheed Martin, Samsung and T-Mobile totaling nearly $600 million for investment projects (see Table 1).

Even though such grants have become increasingly important to stimulate investment in R&D, infrastructure and other strategic investments, their impact on firms' investment options and optimal investment policies in particular has not yet generated much attention. This paper contributes to the corporate finance literature by

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[1] See European Commission Competition DG Staff Working Paper on *Revision of the state aid rules for research and development and innovation*, December 2012. Available online at: http://ec.europa.eu/competition/state_aid/legislation/rdi_issues_paper.pdf.

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Table 1
Examples of cash grants awarded to private companies by the Texas Enterprise Fund (TEF).[2]

Company	Capital investment	TEF award	Return	Date of announcement
Sematech	$190 million	$40 million	194%	03/2004
eBay	$5.18 million	$1.4 million	612%	04/2011
Apple	$304 million	$21 million	281%	03/2012

identifying the optimal investment policy (scale and timing) when partially financed by internal equity and external debt, and promoted by a government investment grant. We derive the optimal mix of external debt and equity provided by the firm, the optimal investment intensity, and the optimal investment timing for a firm that can profit from such a subsidy support scheme. The latter is the outcome of a non-cooperative game in continuous time between the firm and government. To the best of our knowledge, this problem has not been solved in the literature to date. We find that regardless of the way the firm finances its investment, the jointly held real option between the firm and government generates underinvestment. Our model yields optimal investment, financing and stimulus policies for levered and unlevered firms, which are clearly different from those found in the previous literature. In particular, we find that levered firms do not always invest (in expectancy) earlier, their investment level does not always equate with the one of the unlevered firm, and high uncertainty does not always justify higher investment grants.

The rest of the paper is organized as follows. Section 2 gives a brief overview of related literature, while Section 3 presents the model and characterizes the optimal investment threshold and scale for the unlevered and levered firm. Section 4 numerically illustrates the impact of uncertainty and subsidy on timing, investment level, financing and the size of the first-mover advantage. Section 5 concludes and suggests several directions for future research.

2. Literature review

This paper's chief contribution is the integration of two seminal streams of literature, which have been considered in isolation up to now: investment (stimulus) under uncertainty and joint real options contracting. Historically, the real option framework has been applied to study the investment timing of firms financed with equity. The findings reveal that uncertainty does not always deter investment and may even accelerate it. The positive effect of uncertainty is especially pronounced when discounting the project's cash flows with an adequate risk premium and/or the project life becomes finite (Gryglewicz, Huisman, & Kort, 2008; Mauer & Ott, 1995; Metcalf & Hassett, 1995; Sarkar, 2000; Wong, 2007; among others).

However, recent literature has shown that external financing possibilities can have a positive impact on investment timing.[3] First, allowing the firm to partially finance its investment by debt leads to a trade-off between profiting from tax shield benefits and incurring sunk cost in the case of bankruptcy. Thus, how financing decisions affect investment policy, and investment timing in particular, has been of great interest in recent years (see, e.g., Mauer & Sarkar, 2005; Shibata & Nishihara, 2012; Shibata & Nishihara, 2015; Sundaresan & Wang, 2007). These papers have already shown that the investment threshold for levered (debt-equity financing) firms is smaller than that of unlevered firms (all-equity financing). However, they have largely neglected the scaling decision with respect to investment size.

Second, instead of relying on external debt, the firm can make use of a government support scheme (e.g., reduced taxes, subsidies or concessions) in the context of public–private partnerships (PPPs).[4] Within this domain of literature, it is acknowledged that an optimal combination of taxation and subsidy can promote early investment. As revealed in the findings, higher taxes promote investment if the subsidies are optimally set so as to reduce the total cost of a subsidy support scheme to zero or reducing taxes becomes a more efficient means of promoting investment than subsidies if the government's discount rate is greater than that of the firm's (see, e.g., Pennings, 2000; Sarkar, 2012). Just recently, Armada, Pereira, and Rodrigues (2012) investigated the effect of a broader set of incentive policy instruments in the context of PPPs (i.e., an investment subsidy, a revenue subsidy, a minimum demand guarantee, and a rescue option) on a firm's optimal investment policy. In contrast to the previous literature, the authors analyze the effectiveness of a certain incentive scheme in promoting immediate investment, i.e., the way it compensates the firm for losing the option to defer. According to their findings, a revenue subsidy is the most effective instrument and a demand guarantee is the least. Similarly, Scandizzo and Ventura (2010) model an optimal concession design in a PPP arrangement between a public and private party under two types of uncertainty, i.e., cash flow uncertainty and uncertainty arising from the strategic behavior of the parties involved. An equilibrium concession price exists for reasonable concession periods, and the results indicate that the higher the cash flow volatility, the less attractive the contract becomes for both parties, i.e., the concession price increases, which decreases the propensity to engage in the PPP. Moreover, if the concession design allows one party (i.e., the firm) to renegotiate the price periodically, the concession price is also increasing for the duration of the contract.

Interestingly, a few papers have established a link between the two types of external financing. Sarkar (2008) investigates how a convex tax structure that taxes profits higher than losses affects the financing decision. His findings reveal that tax convexity increases the likelihood of default and also reduces the optimal leverage ratio compared to linear taxation. Similarly, Danielova and Sarkar (2011) investigate how tax cuts and investment subsidies affect the timing and financing decision of the firm. Their results indicate that higher taxation leads to increased use of debt, thereby increasing the likelihood of bankruptcy. Furthermore, higher uncertainty leads to a higher subsidy in order to induce earlier investment.

[2] For more information, see https://businessintexas.com/texas-enterprise-fund. The figures are taken from the TEF award listing, https://businessintexas.com/sites/default/files/tef_listing_2-28-18.pdf, accessed April 10, 2018.

[3] Besides bank loans and corporate bonds, the firm might negotiate with other financial intermediaries, e.g., venture capitalists (see Lukas, Mölls, & Welling, 2016; Luo et al., 2016; Wang, Yang, & Zhang, 2015).

[4] In general, PPP arrangements are contractual agreements between the government and firm for a finite period of time. They are designed to motivate investment in areas private firms would not usually invest in and are implemented immediately, thereby resulting in the firm losing the option to defer. Usually, a set of flexibility options is embedded in PPPs, which makes the application of real option modeling appealing (see, e.g., Alonso-Conde, Brown, & Rojo-Suarez, 2007; Takashima, Yagi, & Takamori, 2010).

What all of these analytical models have in common is that they assume a fixed and verifiable investment cost. Consequently, the optimal investment level at the time of investment cannot be determined from these models. Although this research lacuna was noted by Dixit (1993) and Hubbard (1994), it has received little attention until now. Recent literature aiming at endogenizing the investment level highlights that once the investment level becomes endogenous, the aforementioned timing trade-off becomes a *time scale trade-off*.[5] The findings by Della Seta, Gryglewicz, and Kort (2012) reveal that the optimal investment policy of an unlevered firm depends on the speed of learning: early (late) entry on a smaller (larger) scale is appropriate if the learning process is fast (slow). However, two recent papers have explicitly linked the *time scale trade-off* with the *debt financing trade-off*. In particular, Wong (2010) investigates how debt financing affects both the investment timing and investment scaling decision. Similar to previous results, his findings reveal that debt is not neutral to investment timing and the levered firm invests earlier than the unlevered firm. At the same time, however, debt financing is neutral to the scaling decision, i.e., the optimal investment level for the levered firm is the same as for the unlevered firm. Similarly, Sarkar (2011) investigates the link between investment timing and investment scale when debt financing is possible. His findings contradict those from previous research. In particular, he finds that the optimally levered firm will invest later and at a larger amount than the unlevered firm.

In this context, there is another research gap that we want to address with respect to the second stream of literature dealt with in this paper, i.e., *joint real options contracting*.[6] Within this literature domain, the firm's optimal investment policy is viewed as the optimal outcome of a bargaining process (e.g., Hackbarth & Morellec, 2008; Lambrecht, 2004; Morellec & Zhdanov, 2005). Lukas and Welling (2014) examine the effect of uncertainty on investment timing in a game-theoretic real option model. They extend the timing literature by adding the assumption that the investment is also influenced by the actions of a second player. In contrast to the findings of Gryglewicz et al. (2008), Lukas and Welling (2014) show that a U-shaped investment-uncertainty relationship generally holds. Consequently, the bargaining effect promotes early investment in situations where a game against nature suggests deferring investment. However, due to the non-cooperative game, the investment occurs inefficiently late. Moreover, the authors show that uncertainty has an ambiguous influence on the first-mover advantage. In a more general setting, Banerjee, Güçbilmez, and Pawlina (2014) study how the sequence of timing and bargaining affects the optimal exercise policy. In particular, they develop a two-stage decision model in which the parties bargain over the surplus either before or after the timing decision, which can only be made by one firm. Their findings reveal that if the timing decision is made first, the outcome is socially efficient. However, if the parties negotiate over how to share the surplus first, timing inefficiencies arise.

While these game-theoretic models neglect the investment sizing decision, Pennings (2017) analyzes the timing and sizing decision of an unlevered firm when a real option right is shared. The results indicate that when only the firm responsible for the investment timing incurs the investment-specific cost, it does not lead to underinvestment (measured as the difference between the first best outcome and the bargaining outcome), but to inefficiently late investment.

This paper bridges both literature streams, i.e., investment (stimulus) under uncertainty and joint real options contracting. In particular, we assume that the firm can make three choices. It decides on (1) the optimal timing, (2) the optimal investment scale, and (3) the financing of the investment, thereby taking a possible subsidization by the government in the form of a cash grant into account. Hence, our model becomes an extension of Wong (2010) and Shibata and Nishihara (2012) by applying a non-cooperative real options game between the government and levered/unlevered firm in continuous time. Hence, the subsidy is no longer exogenous to the model as in the previous models.

We find that bargaining over the investment stimulus generally generates underinvestment, regardless of how the firm finances its investment. In equilibrium, however, subsidies reduce the level of underinvestment and the strength of this stimulus is strongly affected by the availability of the firm's financing choices. We find that although levered firms receive less subsidies than unlevered firms, they invest more. This result contradicts recent findings in the literature that a firm's investment level is neutral to its financing decision (see, e.g., Wong, 2010). Moreover, situations may occur in which the levered firm's tax shield benefits do not serve as a general incentive to invest earlier as indicated in the literature (see, e.g., Leland, 1994; Mauer & Sarkar, 2005; Shibata & Nishihara, 2012; Sundaresan & Wang, 2007). Rather, we find that for some reasonable parameter constellations, the levered firm's optimal investment threshold is higher than that of its unlevered counterpart. Therefore, the possibility to issue debt does not have an unambiguous impact on investment timing as commonly believed in the literature. A second important result is that the effect of cash flow uncertainty on optimal subsidies is no longer monotonic as postulated by Danielova and Sarkar (2011). On the contrary, it can become ambiguous, indicating that highly uncertain projects do not necessarily receive high subsidies. In addition, we show under which circumstances a complete switch from a subsidy to non-subsidy regime occurs. The results indicate that high-risk projects of levered firms, in particular, are exposed to such a threat.

3. The model

We consider a firm that has the opportunity to invest in a scalable project. The firm's cost is investment specific and the corresponding investment level $I \in \mathbb{R}_+$ cannot be verified by a third party.[7] We explicitly allow the firm to finance the project using a mix of equity, debt and a cash grant subsidy $S \in \mathbb{R}_+$, which the government has provided to stimulate investment. Neither the firm nor the government has private information. Both observe the uncertain cash flow x_t, which is modeled as a geometric Brownian motion (GBM):

$$dx_t = \mu x_t dt + \sigma x_t dW_t, \ \ x_0 > 0, \tag{1}$$

where dW_t is an increment of a Wiener process, $\mu \in \mathbb{R}$ is the drift rate and $\sigma \in \mathbb{R}_+$ is the level of uncertainty.

If the firm invests I, it receives a cash flow $(1-\tau)x_t\pi(I)$[8], where $\tau \in [0,1]$ is the corporate tax rate and $0 \le \pi(I) < 1$ can be interpreted as a scaling factor and accounts for the decreasing returns to scale that exist in different industries (see, e.g.,

[5] See also, e.g., Hagspiel, Huisman, Kort, and Nunes (2016), Lukas, Spengler, Kupfer, and Kieckhäfer (2017), Huberts, Huisman, Kort, and Lavrutich (2015), and Welling (2016).

[6] For a general overview of a game-theoretic real option application, see, e.g., Azevedo and Paxson (2014) and Chevalier-Roignant, Flath, Huchzermeier, and Trigeorgis (2011).

[7] We restrict our analysis to a single investment decision. For more comprehensive modeling as it might apply to product development, see Koussis, Martzoukos, and Trigeorgis (2013).

[8] For the sake of simplicity, we restrict our analysis by considering taxation on cash flow only and assuming similar riskless debt and equity rates. Thus, the standard real option valuation framework as found in Dixit and Pindyck (1994) can be applied. For a more rigorous study on how different tax schemes affect the valuation of real assets, see Gamba et al. (2008).

Basu & Fernald, 1997). Moreover, we assume that the investment option has an infinite lifetime. Let $r \in \mathbb{R}_+,\ r > \mu$ denote the riskless rate of return. The value of the unlevered firm at time t is given by

$$\begin{aligned} V_U(x_t, I) &= \boldsymbol{E}^{x_t}\left[\int_t^\infty (1-\tau)x_z\pi(I)\exp(-r(z-t))dz\right] \\ &= (1-\tau)\pi(I)\frac{x_t}{r-\mu}, \end{aligned} \tag{2}$$

where $\boldsymbol{E}^{x_t}[\ldots]$ denotes the expectation operator conditional on the state of the cash flow x_t.

We follow Pennings (2017) and assume that $\pi(I)$ is given by $\pi(I) = \frac{I}{I+1}$.[9] The intuition behind this scaling factor $\pi(I)$ is as follows. A higher investment level leads to a higher cash flow with decreasing returns to scale. Thus, if I is chosen infinitely high, the scaling factor goes to 1, while for zero investment the firm receives nothing. In contrast to other literature, however, we assume that the subsidization of investment in the project is the outcome of a non-cooperative game between two participants, i.e., the firm F and the government G. We assume that time is continuous, i.e., $t \in [0, \infty)$, and that the players (i.e., the firm and the government) move sequentially and individually maximize their payoff functions, i.e., $F(\cdot),\ G(\cdot)$, thereby taking the best response of the other player into account.

Specifically, the government moves first and decides in $t = 0$ on the amount of subsidy $S \geq 0$ offered to the firm. The government's strategy only depends on the current state of x at $t = 0$, i.e., on x_0. The government's strategy space $\mathcal{G}$ is given by the set of functions $\{g(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+,\ x_0 \mapsto S\}$.

Conditional on the offered subsidy S, the firm will choose an investment time $t_j \geq 0$, investment level $I_j \geq 0$ and a financing strategy $c \geq 0$, with $j \in \{U, L\}$. Here $j = U$ indicates that the firm is unlevered, while $j = L$ indicates that the firm is levered. We explicitly allow the firm to postpone this decision to any point in time. We assume that each player will rely on a non-cooperative Markovian Perfect Nash Equilibrium (MPNE) to determine the equilibrium strategy for both parties, i.e., if one player uses a Markovian strategy then the other has a best response that is Markovian as well. Given that the firm can either invest or wait subject to the current state x_t alone, the exercise strategy for the firm will also be a stationary Markovian strategy. The firm's set of strategies is represented by the set of (vector-valued) functions $\mathcal{F} = \{f(\cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, \infty] \times \mathbb{R}_+ \times \mathbb{R}_+;\ (x_0, S) \mapsto (t_j, I_j,\ c)\}$.

Since the strategies are Markovian and the order by which the players move is fixed and further rounds of negotiation are not possible, the optimization problem for each player is reduced to a nonlinear maximization problem with the strategies of the remaining player fixed at the equilibrium levels (Dockner et al., 2000; Dutta & Rustichini, 1993).

In the following, we present the objective functions of both players in detail and determine their equilibrium strategies. We will first assume that the firm is all-equity financed. Subsequently, we will allow the firm to issue debt and analyze its impact on the players' equilibrium strategies.

3.1. The unlevered firm

3.1.1. The firm's investment strategy

As the reacting party, the firm will maximize the investment's expected payoff by choosing the investment program, i.e., it will choose to invest the optimal amount at the optimal time, conditional on the subsidy granted by the government.

Let $t_U = \inf\{t \geq 0 : x_U \leq x_t\}$ be the (random) first passage time of the state variable to reach the investment threshold x_U from below. Since the option to invest is perpetual, the value of this opportunity only depends on the current state x_0 of the process $\{x_t, t \geq 0\}$ and the cash grant offered by the government. The value of the investment option in $t = 0$ is given by

$$F_U(x_0, S) = \sup_{x_U \geq x_0, I_U \geq 0} \boldsymbol{E}^{x_0}\left[e^{-rt_U}(V_U(x_U, I_U) - I_U + S)\right], \tag{3}$$

where $V_U(x_U, I_U)$ is defined as in Eq. (2). Due to linearity, we can rewrite Eq. (3) as:

$$F_U(x_0, S) = \sup_{x_U \geq x_0, I_U \geq 0} \boldsymbol{E}^{x_0}\left[e^{-rt_U}\right](V_U(x_U, I_U) - I_U + S). \tag{4}$$

As shown in the Appendix, a solution exists for the expectation value $\boldsymbol{E}^{x_0}[e^{-rt_U}]$. It is given by:[10]

$$\boldsymbol{E}^{x_0}\left[e^{-rt_U}\right] = \left(\frac{x_0}{x_U}\right)^{\beta_1} \tag{5}$$

with $\beta_1 = 0.5 - \frac{\mu}{\sigma^2} + \sqrt{(\frac{\mu}{\sigma^2} - 0.5)^2 + \frac{2r}{\sigma^2}} > 1$. Thus, Eq. (4) can be rewritten as follows:[11]

$$F_U(x_0, S) = \sup_{x_U \geq x_0, I_U \geq 0} \left\{(V_U(x_U, I_U) - I_U + S)\left(\frac{x_0}{x_U}\right)^{\beta_1}\right\}. \tag{6}$$

Hence, finding the optimal time to invest is tantamount to finding an optimal threshold value x_U such that the investment option is optimally exercised at the first instant when x_t hits x_U from below (see Wong, 2010).

Proposition 1. *Given a subsidy S, the investment level of the unlevered firm is given by*

$$\hat{I}_U(S) = \begin{cases} \frac{1+\sqrt{1-4\beta_1(\beta_1-1)S}}{2(\beta_1-1)} & , x_0 < \hat{x}_U(S) \\ \sqrt{\frac{(1-\tau)x_0}{(r-\mu)}} - 1 & , x_0 \geq \hat{x}_U(S), \end{cases} \tag{7}$$

where the optimal investment threshold is given by:

$$\hat{x}_U(S) = \begin{cases} \max\{x_0, \mathrm{A}\}, & S \leq \frac{1}{4\beta_1(\beta_1-1)} \\ x_0, & \text{else}, \end{cases} \tag{8}$$

where $\mathrm{A} = \frac{r-\mu}{1-\tau}\left(\frac{2\beta_1-1+\sqrt{1-4\beta_1(\beta_1-1)S}}{2(\beta_1-1)}\right)^2$.

Proof. See the Appendix.

After using Eq. (2) as well as the optimal investment strategy $(\hat{I}_U(S), \hat{x}_U(S))$ the value of the investment opportunity represented by Eq. (6) becomes:

$$\begin{aligned} &F_U(x_0, S) \\ &= \begin{cases} \left[\frac{1-\tau}{r-\mu}\hat{x}_U(S)\pi\left(\hat{I}_U(S)\right) - \hat{I}_U(S) + S\right]\left(\frac{x_0}{\hat{x}_U(S)}\right)^{\beta_1} & , x_0 < \hat{x}_U(S) \\ \frac{1-\tau}{r-\mu}x_0\pi\left(\hat{I}_U(S)\right) - \hat{I}_U(S) + S & , x_0 \geq \hat{x}_U(S). \end{cases} \end{aligned} \tag{9}$$

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[9] The functional form of the scaling factor implies decreasing returns to scale. Instead of multiplying the present value of cash flows by $\pi(I) \in (0, 1)$ and subtracting from this the amount I invested, i.e., $\pi(I)\frac{x_t}{r-\mu} - I$ (with $\pi'(I) > 0$ and $\pi''(I) < 0$), we could have alternatively followed the assumption in other literature and multiplied the present value of cash flows by q and subtract from this a particular cost function $I(q)$, i.e., $q\frac{x_t}{r-\mu} - I(q)$ (with $I'(q) > 0$ and $I''(q) > 0$) (see, e.g., Bar-ilan & Strange, 1999; Sarkar, 2011; Wong, 2010). Obviously, both are closely related since the functional form of the latter also implies that the project exhibits decreasing returns to scale (see, e.g., Wong, 2010, p. 339). To see this, define the inverse function of $\pi(I)$, i.e., $\pi^{-1}(I) = \frac{\pi}{(1-\pi)} \equiv I(\pi)$. Thus, our functional form becomes $\pi V - I(\pi)$ for $\pi \in (0, 1)$.

[10] See also, e.g., Karatzas and Shreve (1998, p. 63) or Dixit and Pindyck (1994, Chapter 9, Appendix).

[11] Please note, that the way we solve the model we are considering first hitting times only. The model could alternatively be solved by means of dynamic programming (see e.g., Bar-Ilan and Strange (1999), Dixit (1993), Sarkar (2011), among others). The results are identical to our approach.

Two intermediate results are noteworthy. First, from Eqs. (7) and (8) it is obvious that both the investment threshold and investment level decrease as the level of subsidy S increases. Hence, the higher the subsidy, the lower the firm's investment threshold, which indicates an acceleration of investment. Second, for $\frac{1}{4\beta_1(\beta_1-1)} < S$ the firm will invest immediately, thereby eroding the firm's value of the option to delay the investment. We will see later that even if our model takes immediate investment as a possible viable solution into account, the government will not choose such an extreme level of subsidy.

3.1.2. The government's subsidization strategy

Up to this point, we have analyzed the firm's investment strategy. In the following, we analyze the government's decision at t_0. Upon the firm's investment, the government receives taxes from the operating project's cash flow. Naturally, the question arises as to how much the government should offer ex-ante such that the firm exercises its option right in the future. We assume that the government will choose the level of subsidy $\hat{S}_U$ so as to maximize its net tax income[12] at the time of investment. Hence, the discounted net tax income is given by:[13]

$$G_U(x_0) = \sup_{S>0}\left\{\left(\tau\frac{\hat{x}_U(S)}{r-\mu}\pi\left(\hat{I}_U(S)\right) - S\right)\left(\frac{x_0}{\hat{x}_U(S)}\right)^{\beta_1}\right\}. \tag{10}$$

Eq. (10) needs a brief explanation. The first part of the RHS represents the present value of all future tax revenues the government will receive from the moment the firm invests. However, this present value is received in exchange for paying a lump sum subsidy S at the investment instant. In order to assess the profitability of an investment stimulus, this net income has to be discounted because the firm controls the timing and will naturally delay the investment decision. The second factor on the RHS discounts the net income received upon investment back to $t = 0$.

Obviously, the government will choose the income-maximizing subsidy from its action set $S \in [0, \infty)$. Since $\hat{x}_U(S)$ is provided by Eq. (8), an analytical solution can be obtained.

Proposition 2. *The government will choose the subsidization strategy $\hat{S}_U$ according to:*

$$\hat{S}_U = \begin{cases} 0 & ,\left(\frac{\beta_1}{\beta_1-1}\right)^2 \le \frac{(1-\tau)}{(r-\mu)}x_0 \\ \min\left\{\hat{S}_{U,1}, \hat{S}_{U,2}\right\} & ,\left(\frac{2\beta_1-1}{\beta_1-1}\right)^2 \le \frac{(1-\tau)}{(r-\mu)}x_0 \le \left(\frac{\beta_1}{\beta_1-1}\right)^2 \\ \hat{S}_{U,2} & ,\text{else} \end{cases} \tag{11}$$

with

$$\hat{S}_{U,1} = \frac{1-\left(2(\beta_1-1)\sqrt{\frac{(1-\tau)x_0}{(r-\mu)}} - 2\beta_1 + 1\right)^2}{4\beta_1(\beta_1-1)} \tag{12}$$

and

$$\hat{S}_{U,2} = \begin{cases} \frac{2\xi^2-\psi^2-2\xi(1-\tau)\beta_1+\psi\sqrt{\psi^2+4\xi(1-\tau)\beta_1}}{8\beta_1(\beta_1-1)\xi^2} & ,\tau \ge \tau_U \\ 0 & ,\text{else}, \end{cases} \tag{13}$$

where $\xi = \tau + \beta_1 - 1$, $\psi = (\tau(1+\beta_1) - 1)(2\beta_1 - 1)$, $\tau_U = \frac{1}{\beta_1+1}$. *Here τ_U denotes the minimum tax required to justify subsidies.*

Proof. See the Appendix.

As discussed above, the firm's investment program is subject to the chosen subsidy (see Eqs. (7) and (8)). Hence, the following corollary summarizes the solutions for the optimal investment threshold and investment level.

Corollary 1. *In equilibrium, the unlevered firm will invest as soon as x_t hits the threshold $\boldsymbol{\hat{x}_U}$ from below, where*

$$\hat{x}_U = \begin{cases} \frac{r-\mu}{(1-\tau)}\left(\frac{2\xi(2\beta_1-1)-\psi+\sqrt{\psi^2+4\xi(1-\tau)\beta_1}}{4(\beta_1-1)\xi}\right)^2 & ,\tau \ge \tau_U \\ \frac{r-\mu}{(1-\tau)}\left(\frac{\beta_1}{\beta_1-1}\right)^2 & ,\text{else} \end{cases}. \tag{14}$$

The unlevered firm will choose an investment level at the time of investment.

$$\hat{I}_U = \begin{cases} \left(\frac{2\xi-\psi+\sqrt{\psi^2+4\xi(1-\tau)\beta_1}}{4(\beta_1-1)\xi}\right) & ,\tau \ge \tau_U \text{ and } x_0 < \hat{x}_U \\ \frac{1}{\beta_1-1} & ,\tau < \tau_U \text{ and } x_0 < \hat{x}_U \\ \sqrt{\frac{(1-\tau)}{(r-\mu)}x_0} - 1 & ,x_0 \ge \hat{x}_U, \end{cases} \tag{15}$$

Proof. This follows directly from Propositions 1 and 2.

From this corollary, it becomes apparent that the optimal investment threshold and the optimal investment level are driven by a critical tax rate τ_U which controls whether a subsidy is paid by the government. Taking a closer look at the investment threshold first, we see that for tax rates below τ_U, i.e. $\tau < \tau_U$, the government is not willing to grant a subsidy to the firm. This is simply driven by the fact that paying subsidies will negatively affect the governments net income. Hence, the problem is reduced to a single firm's investment decision problem under uncertainty and the results replicate the findings of Pennings (2017), i.e., $\hat{x}_U = (r-\mu)/(1-\tau)\left(\frac{\beta_1}{\beta_1-1}\right)^2$. If, however, the corporate tax rate is above the critical tax rate, the government is willing to grant subsidies to the firm and this considerably affects $\hat{x}_U$. By referring to Eq. (8), we can conclude that if $0 < S$ the optimal investment threshold will be reduced.

Since the investment level depends on the investment instant it is obvious, that the critical tax rate also controls for how much is invested. From Eq. (15) it becomes apparent that three cases are possible. First, for $\hat{x}_U \le x_0$ the firm will immediately invest. Hence, there is no need for the government to pay a subsidy to stimulate investment. However, if $x_0 < \hat{x}_U$ then the critical tax rate determines whether the government stimulates investment or not. Hence, for $x_0 < \hat{x}_U$ and $\tau < \tau_U$ the government will not grant a subsidy to the firm because this will reduce its net tax income and as a consequence, the problem is again reduced to a single firm's investment decision problem under uncertainty. If, however, the corporate tax rate is higher than the critical tax rate τ_U the government will stimulate investment by granting a subsidy to the firm. Again, referring to Eq. (7) reveals that if $0 < S$ the firm will invest less compared to a no-subsidy regime.

3.2. The levered firm

3.2.1. The levered firm's investment strategy

In this subsection we will analyze the investment decision problem for the levered firm. At the investment instant, i.e., when the state variable x_t hits the investment threshold $\hat{x}_L$ for the first time from below, we allow the firm to additionally issue debt to finance the investment expenditures. We denote this time by $t_L = \inf\{t \ge 0 : x_t \ge x_L\}$. We follow Leland (1994) and assume that the firm issues debt with infinite maturity at no extra cost. Debtholders will receive a constant coupon payment $0 \le (1-\tau_c)c$ per unit time until the firm defaults. Here, c denotes the coupon payment which represents a choice variable of the firm and $\tau_c < \tau$ is the tax rate, at which debtholders pay taxes on capital income (see

[12] It is common in the literature to assume that the government maximizes its tax income (see, e.g., Pennings, 2005). In addition, it would have been possible to analyze a zero-cost investment stimulus system, where the government sets the subsidy so as to reduce its total cost to zero (see, e.g., Pennings, 2000).

[13] It is certainly arguable that the government can set an optimal tax rate such that it maximizes its income. However, since we focus on a particular firm, this would imply that tax rates are different across industries, which is not the case in reality.

e.g., Goldstein, Ju, & Leland, 2001). Thus, the net income to shareholders will be $(1-\tau)(\pi(I)x_t - c)$ per unit time.

We furthermore assume that shareholders have limited liability. Consequently, the optimal policy for shareholders is to default at the first instant when the equity value vanishes. Let x_b denote the critical default threshold. Hence, at the default instant $t_b = \inf\{t \geq t_L : x_t \leq x_b\}$, i.e., when $x_t = x_b$ for the first time, the firm is immediately liquidated and absolute priority is enforced.[14] As a result, the shareholders get nothing and the debtholders become the new owners of the firm. We follow Mello and Parsons (1992), Morellec (2001) and Wong (2010) and assume that after default the debtholders receive the liquidation value of the firm which is a fraction $(1-\alpha)$ with $0 \leq \alpha \leq 1$ of the going-concern value $V_U(\cdot)$ (see Eq. (2)), i.e., the unlevered firm value as debtholders will continue to operate the firm in its current use.

In the following, we will derive the firm's value of equity E and debt D at the investment instant, i.e., we assume that the investment and financing policy is fixed and given by the triple (x_L, I_L, c). Then, the value of the firm's equity is given by:

$$E(x_L, I_L, c) = \boldsymbol{E}^{x_L}\left[\int_{t_L}^{t_b} e^{-r(z-t_L)}(1-\tau)(\pi(I_L)x_z - c)dz\right], \tag{16}$$

which states that the equity value E is the expected value of the discounted net income accrued to shareholders conditional on x_L. Solving Eq. (16) yields:[15]

$$E(x_L, I_L, c) = V_U(x_L, I_L) - \frac{(1-\tau)c}{r} + \left(\frac{(1-\tau)c}{r} - V_U(x_b, I_L)\right)\left(\frac{x_L}{x_b}\right)^{\beta_2} \tag{17}$$

with $\beta_2 = 0.5 - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - 0.5\right)^2 + \frac{2r}{\sigma^2}} < 0$. Here, the last term highlights the shareholders' option to default.

Hence, they choose the default trigger in such a way that it maximizes equity value (Leland, 1994). Hence, upon differentiating (17) with respect to x_b and solving the first-order condition we get for the optimal bankruptcy threshold:

$$\hat{x}_b(c, I_L) = \frac{r-\mu}{r}\frac{\beta_2}{\beta_2 - 1}\frac{c}{\pi(I_L)}. \tag{18}$$

We define the first time after investment at which x_t hits $\hat{x}_b$ from above as $\hat{t}_b = \inf\{t \geq t_L : x_t \leq \hat{x}_b\}$. Given that the debtholders receive a constant taxable coupon payment $(1-\tau_c)c$ per unit time when the firm is active and are left with the liquidation value upon default, the value of debt at the investment instant equals:

$$D(x_L, I_L, c) = \boldsymbol{E}^{x_L}\left[\int_{t_L}^{\hat{t}_b} e^{-r(z-t_L)}(1-\tau_c)cdz + e^{-r(\hat{t}_b - t_L)}(1-\alpha)V_U\left(\hat{x}_b, I_L\right)\right]. \tag{19}$$

Solving Eq. (19) yields:[16]

$$D(x_L, I_L, c) = \frac{(1-\tau_c)c}{r} + \left((1-\alpha)V_U\left(\hat{x}_b(c, I_L), I_L\right) - \frac{(1-\tau_c)c}{r}\right)\left(\frac{x_L}{\hat{x}_b(c, I_L)}\right)^{\beta_2} \tag{20}$$

where the latter part indicates what debtholders gain/loose should the shareholders exercise their option to default. In the following we will determine the value of the levered firm and deduce the optimal investment and financing policy. As in the previous section, the firm has a perpetual option to invest in the project. In the levered case, however, the firm can also raise debt at the investment instant from the debtholders to finance the investment of size I_L. The difference, i.e. $I_L - D(x_L, I_L, c)$ is then raised from the shareholders. Thus, the ex-ante value of equity prior to debt issuance amounts to $E(x_L, I_L, c) - (I_L - D(x_L, I_L, c)) = E(x_L, I_L, c) + D(x_L, I_L, c) - I_L \equiv V_L(x_L, I_L, c) - I_L$ and reflects the firm value i.e., the net present value of the project. Because $E(x_L, I_L, c) + D(x_L, I_L, c) = V_L(x_L, I_L, c)$ we get:

$$V_L(x_L, I_L, c) = V_U(x_L, I_L) + \frac{(\tau-\tau_c)c}{r} - \left(\frac{(\tau-\tau_c)c}{r} + \alpha V_U\left(\hat{x}_b(c, I_L), I_L\right)\right)\left(\frac{x_L}{\hat{x}_b(c, I_L)}\right)^{\beta_2} \tag{21}$$

As Eq. (21) indicates, the value of the levered firm comprises three parts. The first term of the RHS is the value of the unlevered firm while the second and third term denote the value of the tax-shield benefits of debt and the cost of bankruptcy, respectively.[17] We follow the framework of structural models proposed by Goldstein et al. (2001) and Leland (1994) and assume that managers maximize this firm value rather than ex-post equity.[18] Considering the fact that the investment project is supported by a subsidy, it follows that the value of the levered firm $F_L(x_0, S)$ at $t = 0$ is given by:

$$F_L(x_0, S) = \sup_{x_L \geq x_0, c \geq 0, I_L \geq 0} \boldsymbol{E}^{x_0}\left[e^{-rt_L}(V_L(x_L, I_L, c) - I_L + S)\right], \tag{22}$$

Using Eqs. (5) analogously and (21), Eq. (22) becomes:

$$F_L(x_0, S) = \sup_{x_L \geq x_0, c \geq 0, I_L \geq 0}\left\{\left(V_U(x_L, I_L) + \frac{(\tau-\tau_c)c}{r} - \left(\frac{(\tau-\tau_c)c}{r} + \alpha V_U\left(\hat{x}_b(c, I_L), I_L\right)\right)\left(\frac{x_L}{\hat{x}_b(c, I_L)}\right)^{\beta_2} - I_L + S\right)\left(\frac{x_0}{x_L}\right)^{\beta_1}\right\}. \tag{23}$$

We follow Wong (2010) and solve this maximization problem in two steps. First, we analytically solve Eq. (23) for the optimal coupon payment $\hat{c}$ for a given investment level $I_L > 0$ and investment threshold $x_L \geq x_0$. After substituting the solution, i.e., $\hat{c}(x_L, I_L)$, into Eq. (23), we solve the remaining two first order conditions for x_L and I_L.

Proposition 3. *Given an investment level I_L and investment threshold $\boldsymbol{x_L}$, the financing strategy of the levered firm is characterized by the coupon*

$$\hat{c}(x_L, I_L) = \omega^{-\frac{1}{\beta_2}}\left(\frac{r}{r-\mu}\right)\left(\frac{\beta_2 - 1}{\beta_2}\right)\pi(I_L)x_L, \tag{24}$$

where $\omega = \left(\frac{\tau-\tau_c}{(\tau-\tau_c)(1-\beta_2) - \alpha(1-\tau)\beta_2}\right) \in (0, 1)$.

Proof. See the Appendix.

[14] Please note that the default is only defined after the investment is exercised.

[15] We assume existence and uniqueness of a solution. For a detailed derivation see Appendix A.2.

[16] For a detailed derivation see Appendix A.2.

[17] Obviously, we have that for $\lim_{c\to 0} V_L(x_L, I_L, c)$ the optimal bankruptcy threshold $x_b(c, I_L)$ and the value of debt $D(x_L, I_L, c)$ become zero and thus the value of the levered firm equates the value of the unlevered firm, i.e., $V_L(x_L, I_L, 0) \equiv V_U(x_L, I_L)$.

[18] Obviously, this assumption rules out agency conflicts between debtholders and management which arise if managers maximize ex-post equity value. One could argue that our rigid assumption holds because if managers are solely concerned with the wellbeing of its shareholders, debtholders will take action that leads to second-best solutions (Jensen & Meckling, 1976; Mauer & Sarkar, 2005; Shibata & Nishihara, 2015 among others). By anticipating this, the managers avoid such an outcome.

Substituting Eqs. (24) and (18) into Eq. (23) yields:

$$F_L(x_0, S) = \sup_{x_L \geq x_0, I_L \geq 0} \left\{ (\phi V_U(x_L, I_L) - I_L + S)\left(\frac{x_0}{x_L}\right)^{\beta_1} \right\}, \tag{25}$$

with

$$\phi = 1 + \omega^{-\frac{1}{\beta_2}} \left(\frac{\tau - \tau_c}{1-\tau}\right). \tag{26}$$

Analogous to the derivation of $\hat{I}_U, \hat{x}_U$ in the previous section, we solve Eq. (25) to get the closed form solution for the optimal investment threshold $\hat{x}_L(S)$ and investment level $\hat{I}_L(S)$, which leads to:

Proposition 4. *Given a subsidy S, the investment level $\hat{I}_L(S)$ of the levered firm is given by*

$$\hat{I}_L(S) = \begin{cases} \frac{1+\sqrt{1-4\beta_1(\beta_1-1)S}}{2(\beta_1-1)} & , x_0 < \hat{x}_L(S) \\ \sqrt{\frac{\phi(1-\tau)x_0}{(r-\mu)}} - 1 & , x_0 \geq \hat{x}_L(S), \end{cases} \tag{27}$$

where the optimal investment threshold $\hat{x}_L(S)$ is given by

$$\hat{x}_L(S) = \begin{cases} \max\{x_0, B\} & , S \leq \frac{1}{4\beta_1(\beta_1-1)} \\ x_0 & , \text{else} \end{cases} \tag{28}$$

where $B = \frac{r-\mu}{\phi(1-\tau)}(\frac{2\beta_1-1+\sqrt{1-4\beta_1(\beta_1-1)S}}{2(\beta_1-1)})^2$.

Proof. See the Appendix.

3.2.2. The Government's Subsidization Strategy

Given that the firm reacts to the proposed subsidy of the government, both the optimal investment threshold and optimal investment scale depend on the subsidy offered, i.e., $\hat{x}_L(S)$ and $\hat{I}_L(S)$. Hence, the government's discounted net tax income is given by:

$$G_L(x_0) = \sup_S \left\{ \left(\tau \frac{\hat{x}_L(S)}{r-\mu}\pi\left(\hat{I}_L(S)\right) - \frac{(\tau-\tau_c)c}{r}\left(1 - \left(\frac{\hat{x}_L(S)}{\hat{x}_b(S)}\right)^{\beta_2}\right) - \alpha\tau \frac{\hat{x}_b(S)}{r-\mu}\pi\left(\hat{I}_L\right)\left(\frac{\hat{x}_L(S)}{\hat{x}_b(S)}\right)^{\beta_2} - S \right) \left(\frac{x_0}{\hat{x}_L(S)}\right)^{\beta_1} \right\}. \tag{29}$$

As can be seen from the above equation, the government's objective function comprises two parts. First, the government cares about the taxes raised from the firm. Second, the government also considers the risk of bankruptcy of the firm and the corresponding loss of tax revenue by government when this occurs. Eq. (29) can be rearranged into:

$$G_L(x_0) = \sup_S \left\{ \left(\tau\zeta \frac{\hat{x}_L(S)}{r-\mu}\pi\left(\hat{I}_L(S)\right) - S\right)\left(\frac{x_0}{\hat{x}_L(S)}\right)^{\beta_1} \right\}, \tag{30}$$

where $\zeta = 1 - \frac{\tau-\tau_c}{\tau}\omega^{-\frac{1}{\beta_2}}(\frac{\beta_2-1}{\beta_2})(1-\omega) - \alpha\omega^{(1-\frac{1}{\beta_2})}$.

Analogous to the previous section, the government will choose the income-maximizing subsidy. Following the same analytical steps as in the previous section, we can derive an analytical solution for the optimal subsidy, which is provided by the following proposition:

Proposition 5. *In the case of a levered firm, the government will choose the optimal subsidization strategy $\hat{S}_L$ according to:*

$$\hat{S}_L = \begin{cases} 0 & , \left(\frac{\beta_1}{\beta_1-1}\right)^2 \leq \frac{1-\tau}{r-\mu}\phi x_0 \\ \min\left\{\hat{S}_{L,1}, \hat{S}_{L,2}\right\} & , \left(\frac{2\beta_1-1}{\beta_1-1}\right)^2 \leq \frac{1-\tau}{r-\mu}\phi x_0 \leq \left(\frac{\beta_1}{\beta_1-1}\right)^2 \\ \hat{S}_{L,2} & , \text{else} \end{cases} \tag{31}$$

with

$$\hat{S}_{L,1} = \frac{1 - \left(2(\beta_1-1)\sqrt{\frac{(1-\tau)\phi}{(r-\mu)}x_0} - 2\beta_1 + 1\right)^2}{4\beta(\beta_1-1)}, \tag{32}$$

$$\hat{S}_{L,2} = \begin{cases} \frac{2\eta(\eta-1)+\theta\left(\sqrt{\theta^2+4\eta}-\theta\right)}{8\eta^2\beta_1(\beta_1-1)}, & \tau \geq \tau_L, \\ 0, & \text{else} \end{cases} \tag{33}$$

where $\eta = \frac{\zeta}{\phi}\frac{\tau}{(1-\tau)} + \frac{\beta_1-1}{\beta_1}$, $\theta = (\eta-1)(2\beta_1-1)$ *and* $\tau_L = \frac{\phi}{\zeta}\tau_U$. *Here, τ_L denotes the minimum tax level required to justify subsidization of the investment.*

Proof. See the Appendix.

Finally, inserting Eq. (33) into Eqs. (27) and (28) yields the solution for the investment threshold and investment level, respectively, as summarized by the following corollary.

Corollary 2. *In equilibrium, the levered firm will invest as soon as x_t hits the threshold $\hat{x}_L$ from below, where*

$$\hat{x}_L = \begin{cases} \frac{r-\mu}{\phi(1-\tau)}\left(\frac{2\eta(2\beta_1-1)-\theta+\sqrt{\theta^2+4\eta}}{4(\beta_1-1)\eta}\right)^2 & , \tau \geq \tau_L \\ \frac{r-\mu}{\phi(1-\tau)}\left(\frac{\beta_1}{\beta_1-1}\right)^2 & , \text{else} \end{cases} \tag{34}$$

At the time of investment, the levered firm's investment and financing strategy is given by:

$$\hat{I}_L = \begin{cases} \left(\frac{2\eta-\theta+\sqrt{\theta^2+4\eta}}{4(\beta_1-1)\eta}\right) & , \tau \geq \tau_L \text{ and } x_0 < \hat{x}_L \\ \frac{1}{\beta_1-1} & , \tau < \tau_L \text{ and } x_0 < \hat{x}_L \\ \sqrt{\frac{\phi(1-\tau)}{(r-\mu)}x_0} - 1 & , x_0 \geq \hat{x}_L \end{cases} \tag{35}$$

and

$$\hat{c} = \begin{cases} \hat{c}_1 & , \tau \geq \tau_L \text{ and } x_0 < \hat{x}_L \\ \hat{c}_2 & , \tau < \tau_L \text{ and } x_0 < \hat{x}_L, \\ \hat{c}_0 & , x_0 \geq \hat{x}_L \end{cases} \tag{36}$$

where

$$\hat{c}_1 = \omega^{-\frac{1}{\beta_2}}\left(\frac{r}{r-\mu}\right)\left(\frac{\beta_2-1}{\beta_2}\right) \times \frac{\left(2\eta(2\beta_1-1) - \theta + \sqrt{\theta^2+4\eta}\right)\left(2\eta - \theta + \sqrt{\theta^2+4\eta}\right)}{16(\beta_1-1)^2\eta^2}, \tag{37}$$

$$\hat{c}_2 = \omega^{-\frac{1}{\beta_2}}\left(\frac{r}{r-\mu}\right)\left(\frac{\beta_2-1}{\beta_2}\right)\frac{\beta_1}{(\beta_1-1)^2}, \tag{38}$$

$$\hat{c}_3 = \omega^{-\frac{1}{\beta_2}}\left(\frac{r}{r-\mu}\right)\left(\frac{\beta_2-1}{\beta_2}\right)\sqrt{\frac{\phi(1-\tau)}{(r-\mu)}x_0}\left(\sqrt{\frac{\phi(1-\tau)}{(r-\mu)}x_0} + 1\right). \tag{39}$$

Proof. This follows directly from Propositions 3–5.

3.3. The first-best outcome

Let us assume that both investment participants cooperatively agree on the timing and level of investment. This is similar to the situation where an external central planner decides. Totaling the two parties' payoffs, the central planner's option value results in:

$$F(x_0) = \max_{\bar{x}_1,\bar{x}_2,\bar{I}} \left\{ \left(\frac{\pi(\bar{I})}{r-\mu}\bar{x}_1 - \bar{I} - \alpha \frac{\pi(\bar{I})\bar{x}_2}{r-\mu}\left(\frac{\bar{x}_1}{\bar{x}_2}\right)^{\beta_2}\right)\left(\frac{x_0}{\bar{x}_1}\right)^{\beta_1} \right\}, \tag{40}$$

where $\bar{x}_1, \bar{x}_2$ and $\bar{I}$ denote the investment threshold, liquidation threshold and investment level, respectively. In order to capture both the option to invest as well as the subsequent option to default we assume that $\bar{x}_2 \le x_0 \le \bar{x}_1$. From this maximization problem, it becomes apparent that the central planner will choose $\bar{x}_2^* = 0$ in order to minimize the present value of bankruptcy cost. Hence, the problem reduces to:

$$F(x_0) = \max_{\bar{x}_1,\bar{I}} \left\{ \left(\frac{\pi(\bar{I})}{r-\mu}\bar{x}_1 - \bar{I}\right)\left(\frac{x_0}{\bar{x}_1}\right)^{\beta_1} \right\}. \tag{41}$$

Following Dixit and Pindyck (1994, 140ff.), we can determine the optimal threshold $\bar{x}_1^*$ indicating optimal investment, taking into account that the central planner will choose an optimal investment level of $\bar{I}^*$. This leads to the following proposition:

Proposition 6. *The central planner will invest as soon as x_t hits the threshold $\bar{x}_1^*$ from below, where*

$$\bar{x}_1^* = (r-\mu)\left(\frac{\beta_1}{\beta_1 - 1}\right)^2. \tag{42}$$

At the time of investment, the central planner's investment level is given by:

$$\bar{I}^* = \begin{cases} \sqrt{\frac{\bar{x}_1^*}{(r-\mu)}} - 1 & , x_0 < \bar{x}_1^* \\ \sqrt{\frac{x_0}{(r-\mu)}} - 1 & , x_0 \ge \bar{x}_1^* \end{cases} \tag{43}$$

Proof. See the Appendix.

To what extent does underinvestment occur (i.e., does the investment policy of the unlevered and levered firm deviate from this first-best solution)? Since we rely on both the time dimension and the level of investment, it seems appropriate to compare the discounted investment level because later investment timing implies a larger discount factor. Consequently, we use the following definition to identify underinvestment:

Definition 1. *We define the level of underinvestment Δ as the positive part of the difference between the first-best discounted optimal investment level and the discounted optimal investment level due to non-cooperative bargaining,* i.e.,

$$\Delta := \max\left\{0, \bar{I}^*\left(\frac{x_0}{\bar{x}_1^*}\right)^{\beta_1} - \hat{I}_j(\hat{S})\left(\frac{x_0}{\hat{x}_j(\hat{S}_j)}\right)^{\beta_1}\right\}, \quad j \in \{U, L\}. \tag{44}$$

4. Comparative statics

In the following, our model is analyzed numerically. Unless otherwise stated, we assume the following values: $r = 0.08$, $\mu = 0.02$, $\sigma = 0.3$, $\tau = 0.48$, $\tau_c = 0.33$, $\alpha = 0.3$, $x_0 = 0.06$.[19] We first analyze the impact that corporate tax rate has on the firm's optimal investment and financing policy, and the government's optimal investment stimulus. Subsequently, we analyze how the level of uncertainty affects their optimal choices.

4.1. The impact of corporate tax rate

Let us first neglect the possibility for the firm to issue debt. Hence, the government's equilibrium strategy is to offer a subsidy $\hat{S}_U$ according to Eq. (11), while the firm's investment policy in equilibrium, i.e., $(\hat{x}_U, \hat{I}_U)$, is given by Eq. (14) and Eq. (15). Fig. 1 shows the firm's optimal investment policy, i.e., the optimal investment thresholds (black solid line) and levels (gray solid line) for different tax levels in comparison to the first-best outcome (dotted line). The first result is that for $\tau < \tau_U = 0.38$ no subsidy is granted to the unlevered firm. Thus, for corporate tax rates up to this point, the firm's investment level stays constant at $\hat{I}_U = 1.56$, while its optimal investment threshold increases steadily from 0.39 (no taxation) to 0.63. Obviously, higher taxation increases the firm's propensity to delay investment (see also Eq. (14)). There are two reasons for this. First, the government's tax income is too low in this region, thus paying a subsidy would negatively affect its payoff. Second, Eq. (15) reveals that in equilibrium the investment level is solely driven by risk characteristics, i.e., $\hat{I}_U = 1/(\beta_1 - 1)$. Therefore, in this non-subsidy region, the level of investment is tax neutral, i.e., $\frac{\partial I}{\partial \tau} - (\frac{\partial I}{\partial x})\frac{dx}{d\tau} = 0$. Comparing this solution with the solution of the central planner reveals that in such a non-subsidy region, the firm's investment level equals the investment level of the first-best solution, i.e., $\hat{I}_U = \bar{I}^*$ (see also Eq. (43)). However, commitment to this first-best investment level occurs inefficiently late. As Fig. 1 indicates, the optimal investment threshold of the first-best solution is lower than that of the unlevered firm, which is a direct consequence of taxation because it reduces the investor's payoff.

Obviously, an increase in the corporate tax rate will improve the government's tax income. Consequently, for corporate tax rates above the critical tax rates, i.e., $\tau_U \ge 0.38$, the government has an incentive to pay a subsidy. It follows that $0 \le d\hat{S}_U/d\tau$ holds for the subsidy region. Of course, the subsidy will lower the firm's net investment cost, thus the investment threshold $\hat{x}_U$ will decrease compared to the threshold of a non-subsidized firm. This investment stimulus, however, will also lower the firm's equilibrium investment level. Put differently, the subsidy will accelerate investment, which directly translates into a lower investment level at the time of investment. Hence, in this region the tax neutrality on the investment level does not hold anymore. Nevertheless, it becomes apparent that, despite the subsidy, the investment decision results in underinvestment when compared to the first-best solution. However, Fig. 1 shows that the general effect of the tax level on the firm's investment threshold is ambiguous. Specifically, as the level of tax increases above $\tau = 0.5$, the positive impact of the investment stimulus is eroded by the firm's higher tax burden. At this reflection point, any increase in the tax rate will make the firm postpone the investment further. Notably, an increase in the investment threshold does not result in a higher or more efficient investment level at the time of investment. On the contrary, the firm will invest a lower amount of capital if the corporate tax level increases further. This is due to two effects. First, a change in the tax rate does not directly affect the investment level. However, an increase in the tax rate τ will directly increase the level of subsidy granted to the unlevered firm. This subsidy lowers the net cost of investment, which in turn will lower the firm's investment threshold. We call this the *timing stimuli effect*, and it is due to this effect that the investment level will decrease further if the tax rate increases, i.e., an *investment stimuli effect* occurs. Formally, this can be best seen from Eq. (7).

With respect to the first-best solution, we find that timing inefficiency also exists in the subsidy regime, i.e., $\bar{x} < \hat{x}_U$ for $\tau_U = 0.38 < \tau$. However, the subsidy may reduce the timing inefficiencies. Moreover, Fig. 1 indicates that the firm no longer chooses the first-best investment level. Consequently, for high levels of corporate tax rates, an increase in τ and thus increased subsidy levels

[19] We choose parameters according to common values in the literature (see, e.g., Danielova and Sarkar, 2011; Shibata and Nishihara, 2012; Wong, 2010; Goldstein et al., 2001). For these basecase values, we get: $\beta_1 = 1.64$, $\beta_2 = -1.084$, $\psi = 0.609$, $\xi = 1.12$, $\omega = 0.311$, $\phi = 1.098$, $\zeta = 0.827$, $\eta = 1.0853$, $\theta = 0.2$.

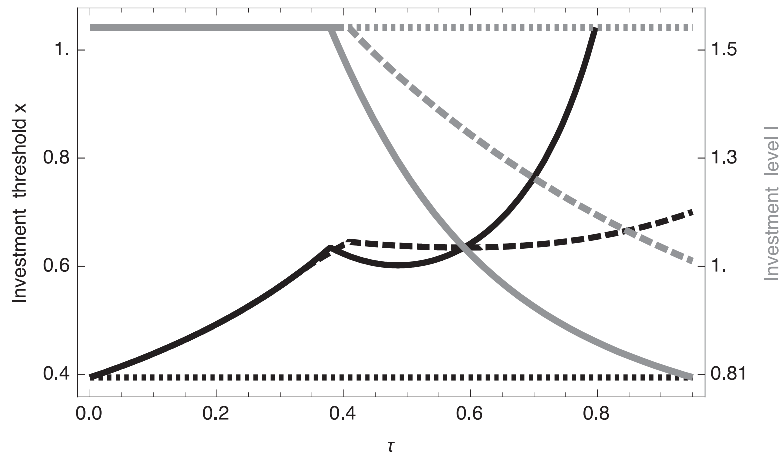


Fig. 1. Optimal investment threshold (black) and optimal investment level (gray) of central planner (dotted), levered (dashed) and unlevered firm (solid) as a function of the tax rate. ($\sigma = 0.3, r = 0.08, \mu = 0.02, \alpha = 0.3, x_0 = 0.06, \tau_c = 0.33$).

lead to lower investment levels, increased timing inefficiencies and thus higher levels of underinvestment.

In analyzing the case of the levered firm, we find that both regimes, i.e. a subsidy and a non-subsidy supported regime, still exist. However, the regime switch occurs at higher tax rates, i.e., at $\tau = \tau_L = 0.41$. In the non-subsidy region, the investment level again equals that of the central planner and remains constant at $\hat{I}_L = 1.56$. Consequently, the investment level is tax neutral and neutral to the possibility of issuing debt until the corporate tax rate reaches $\tau_L = 0.41$. Higher taxation again incentivizes the government to grant a subsidy in order to prevent the firm from further postponing the investment. However, a key finding is that the investment level of the levered firm is substantially higher than that of the unlevered firm for all tax rates if the government favors $S > 0$ (e.g., for a corporate tax rate of $\tau = 0.45$, we have $\hat{I}_L = 1.5$ as compared to $\hat{I}_U = 1.36$). Another key finding is related to the levered firm's propensity to invest. Obviously, the investment threshold equals that of the unlevered firm if $\tau_c \geq \tau$ because the firm does not profit from issuing debt in this region, which is a standard result in the literature. Similarly, for corporate taxes above $\tau_c = 0.33$, the firm profits from tax shield benefits and has an incentive to issue debt, which leads to earlier investment than the unlevered firm. These results, however, only hold for the non-subsidy regime. Thus, for $0.41 < \tau$ the levered firm will be subsidized and we see two effects. First, the investment threshold $\hat{x}_L$ becomes less sensitive to changes in the corporate tax rate. Second, regions exist where the threshold of the levered firm is higher than that of the unlevered firm. For example, for a corporate tax rate of $\tau = 0.45$, the levered firm's investment threshold amounts to $\hat{x}_L = 0.64$, while the unlevered firm's investment amounts to $\hat{x}_U = 0.61$. Notably, for reasonable tax rates, e.g., $\tau \in (0.4,\ 0.6)$, the levered firm will invest later (see Fig. 1).

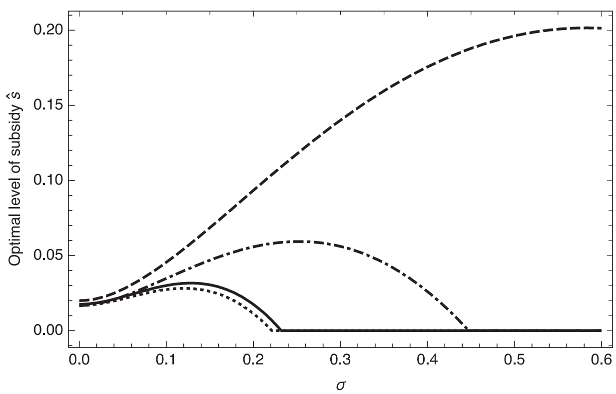
4.2. The impact of uncertainty

To what extent are these results affected by the degree of uncertainty? We again focus on the parties' equilibrium strategies as provided by Eqs. (11)–(15) and Eqs. (31)–(39) for the unlevered and levered firm, respectively. As can be deduced from Fig. 2, a levered firm subject to a corporate tax rate of $\tau = 0.48$ will invest $\hat{I}_L = 1.47$ as soon as x_t hits $\hat{x}_L = 0.63$. The investment will be financed by debt, i.e., the coupon level amounts to $\hat{c}_L = 0.33$ and is supported by a subsidy of size $\hat{S}_L = 0.056$.

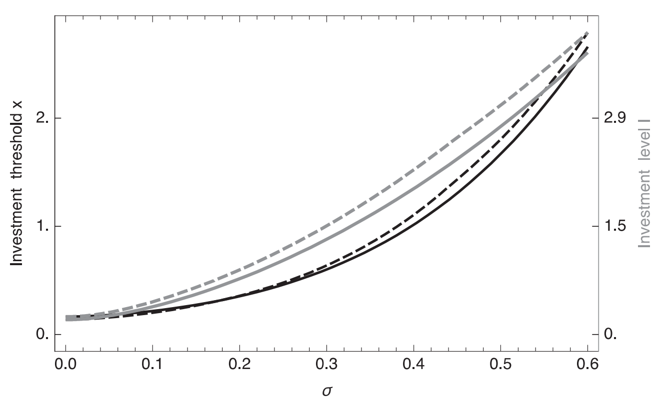
Let us first look at the overall impact of uncertainty on the absolute size of the subsidy granted by the government to promote investment. Since an increase in uncertainty generally discourages investment, we would intuitively expect the equilibrium subsidy to increase as uncertainty increases. Contrary to the findings of Danielova and Sarkar (2011), however, we find that this is not necessarily the case. As Fig. 2(a) indicates, uncertainty will affect the level of subsidy that the government grants to the (un)levered firm, but the effect is ambiguous. For low levels of uncertainty, the results indicate that an increase in uncertainty leads to an increase in the subsidy level. For high levels of uncertainty, however, we observe that an increase in uncertainty might lower the level of subsidy. For example, an increase in the levered firm's cash flow volatility from $\sigma = 0.25$ to $\sigma = 0.3$ lowers the subsidy level granted by the government from $\hat{S}_L = 0.056$ to $\hat{S}_L = 0.059$ (see Fig. 2(a), dash-dotted). The intuition behind this is as follows. In general, the level of subsidy influences the government's objective function in three different ways. First, subsidies represent sunk cost for the government. Second, subsidies influence the size of the cake that the parties negotiate over. Finally, since subsidies affect investment timing, they impact the discounting of the government's payoff. While the latter effect is positive (i.e., higher subsidies mitigate the effect of discounting), the former has a strictly negative influence on the government's payoff. The strength of these effects is heavily determined by the level of uncertainty. For low levels of uncertainty, an increase in the level of subsidy leads to benefits that outweigh the costs. For $0 < \frac{d\hat{S}_j}{d\sigma}$, an increase in the level of uncertainty would lead to more investment later. Anticipating this reaction, the government responds by increasing the subsidy in order to limit the negative effect of discounting, thereby maximizing its payoff.

In contrast, from a certain level of uncertainty, i.e., $\frac{d\hat{S}_j}{d\sigma} < 0$, the benefits of early investment for the government's payoff diminish if uncertainty increases further. In particular, assume that the government further increases the subsidy. Then the overall surplus split between the parties shrinks and the cost of subsidization borne by the government rises. Consequently, the latter two forces erode the benefit of less discounting. Hence, the government is better off reducing the subsidy if uncertainty increases further. However, the strength of the opposing effects depends on the tax rate. From Eqs. (13) and (33), it follows that for sufficiently high tax rates, the government's share of the surplus is such that it will only work against the discounting effect by increasing the subsidy. Consequently, in such a setting we only observe strictly increasing levels of subsidy as a reaction to increased uncertainty, which replicates the findings of Danielova and Sarkar (2011).

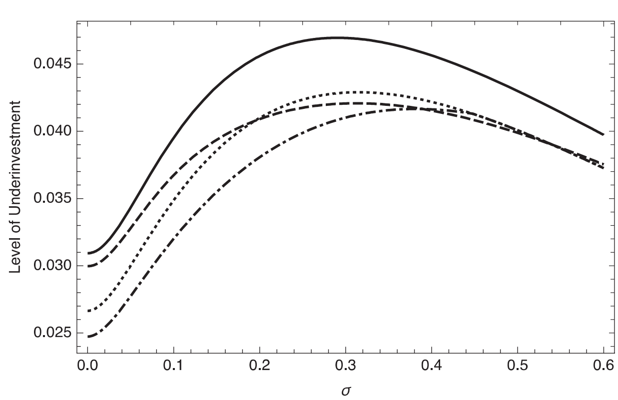
Finally, the results also indicate that a regime switch could occur, i.e., the subsidy regime may become a non-subsidy regime if



(a)



(b)



(c)

Fig. 2. (a) Optimal level of subsidy as a function of uncertainty for the levered firm (dotted, $\tau = 0.35$; dash-dotted, $\tau = 0.48$) and the unlevered firm (solid, $\tau = 0.35$; dashed, $\tau = 0.48$). ($r = 0.08, \mu = 0.02, \alpha = 0.3, x_0 = 0.06,\ \tau_c = 0.33$) (b) optimal investment threshold (black) and optimal investment level (gray) of unlevered firm (solid) and the levered firm (dashed) as a function of uncertainty. ($r = 0.08, \mu = 0.02, \alpha = 0.3, x_0 = 0.08, \tau = 0.48,\ \tau_c = 0.33$) (c) the level of underinvestment as a function of uncertainty for the unlevered firm (non-subsidized, solid; subsidized, dashed) and the levered firm (non-subsidized, dotted; subsidized, dash-dotted). ($r = 0.06, \mu = 0.02, \alpha = 0.3, x_0 = 0.08, \tau = 0.48, \tau_c = 0.33$).

the investment projects are highly uncertain. Furthermore, it can be seen that the level of subsidy granted to the unlevered firm is substantially higher than for the levered firm. This is due to the fact that the levered firm already benefits from the tax shield of debt, which lowers the government's net tax income. Put differently, the tax shield already acts as a kind of investment stimulus, therefore the government is not willing to issue higher lump sum subsidies. As an intermediate result, these findings imply that levered firms that have high cash flow volatility will have a lower chance of getting government subsidies than unlevered firms with the same risk characteristics.

With regard to the firm's investment level and investment threshold, we find that both are increasing in uncertainty for the levered as well as unlevered firm (Fig. 2(b)). However, Fig. 2(b) also shows that the levered firm invests more than the unlevered firm. This means that the firm's decision regarding investment intensity is no longer neutral to debt financing, which contradicts the findings of Wong (2010). As can be deduced from Fig. 2(b), at a corporate tax rate of $\tau = 0.48$ and cash flow uncertainty of $\sigma = 0.3$, the levered firm will invest $\hat{I}_L = 1.47$, while the unlevered firm will choose $\hat{I}_U = 1.28$. This is mainly due to the fact that the levered firm receives a lower amount of subsidy than the unlevered firm (equilibrium subsidy levels are drawn from Fig. 2(a) for the same levels of uncertainty). Since the investment level decreases as subsidies increase, the levered firm invests more. For extremely high values of uncertainty, the levered firm acts like an unlevered firm because neither receives any subsidies (see Fig. 2(a)). Consequently, the amount invested by the unlevered firm equals that of the levered firm and therefore replicates the findings of Wong (2010). Therefore, we can conclude that $\hat{I}_U \leq \hat{I}_L$ holds for all levels of uncertainty.

While this is a distinct effect, it becomes apparent from Fig. 2(b) that the possibility to issue debt might have an ambiguous effect on the investment threshold. For low levels of uncertainty, the levered firm invests sooner than the unlevered firm. This is explained by the trade-off theory of capital alluded to earlier, i.e., investment becomes more attractive due to the benefits arising from the tax shield of debt financing. However, as Fig. 2(b) depicts, this relationship might reverse. Again, for $\tau = 0.48$ and $\sigma = 0.3$, the levered firm will invest as soon as x_t hits $\hat{x}_L = 0.64$, while the unlevered firm will invest when x_t hits $\hat{x}_U = 0.6$. The intuition behind this is as follows. For higher levels of uncertainty, the government grants substantially more subsidies to the unlevered firm than to the levered firm (optimal subsidy levels are drawn from Fig. 2(a)). Since an increase in the level of subsidy will decrease the investment threshold, the unlevered firm will invest earlier than the levered firm. Given that the firm simultaneously decides on timing and scale, the overall effect of uncertainty and the investment stimulus will be discussed by referring to the condensed measure of underinvestment as stated in Eq. (44). As Fig. 2(c) indicates, the first finding is that underinvestment persists, i.e., $\Delta \geq 0$, regardless of whether the firm is levered or subsidized. This is mainly due to the non-cooperative game setting of our model as opposed to the cooperative solution given by the central planner (which represents the first-best outcome).

Second, we find that a subsidy reduces underinvestment. In particular, Fig. 2(c) contrasts the level of underinvestment of the levered and unlevered firm in equilibrium with a situation where both firm types do not receive financial support from the government, i.e., $S = 0$. Consequently, for $\tau > \tau_U(\sigma)$ ($\tau > \tau_L(\sigma)$) we find that the unlevered (levered) firm's level of underinvestment is reduced when government support is received. For $\tau \leq \tau_U(\sigma)$ ($\tau \leq \tau_L(\sigma)$) on the other hand, the same level of underinvestment persists. Taking the levered firm taxed at $\tau = 0.48$ as an example, we see from Fig. 2(c) that at $\sigma = 0.42$ the level of underinvestment of the subsidized firm equates the level of underinvestment of the non-subsidized firm. This marks the transition from a subsidy to a non-subsidy regime, i.e., $\tau = 0.48 = \tau_L$. Consequently, for higher uncertainty levels we have that $\tau = 0.48 < \tau_L(\sigma)$ and thus no subsidy is granted to the firm. As a result, the firm's equilibrium investment and financing policies for the subsidy regime equate those of the non-subsidy regime (see also Section 4.1). For low levels of uncertainty, i.e., $\tau = 0.48 > \tau_L(\sigma)$, the subsidy regime prevails since both the firm and government benefit from a subsidy. From the results alluded to earlier, we know that subsidies decrease the optimal investment threshold and thus incentivize both types of firms to invest earlier, thereby reducing timing inefficiencies when compared to the first-best outcome.

Third, in comparing the unlevered and levered firm, we find that for low levels of uncertainty the underinvestment of the unlevered firm is of greater magnitude than the underinvestment of the levered firm. Consequently, incentivizing the levered firm to invest generates less underinvestment than incentivizing the unlevered firm. However, as the figure depicts, this relation might reverse for higher levels of uncertainty. Hence, the investment policy of the levered firm might suffer from higher levels of underinvestment than that of the unlevered firm. This is mainly driven by the fact that the levered firm will no longer be subsidized, whereas the unlevered firm will still receive subsidies upon investment.

Finally, we see that the level of underinvestment for both types of firms is ambiguous with regard to the level of uncertainty. This is due to two reasons. First, the non-subsidy regime is the preferred government strategy for very high levels of uncertainty, therefore increasing uncertainty will increase the propensity to delay investment. Consequently, the discounting effect dominates in the cooperative and non-cooperative solution and any other effects due to uncertainty vanish as uncertainty goes to infinity. In contrast, for low levels of uncertainty the subsidy regime prevails and the subsidy incentivizes the firm to invest a lower amount at a lower threshold. Since the timing stimuli effect exceeds the investment stimuli effect, underinvestment is reduced as uncertainty decreases further.

Finally, we briefly provide some policy implications. First, our results indicate that levered firms that are subsidized exhibit the lowest level of underinvestment. Hence, this emphasizes the importance of frictionless access to debt financing for firms. Second, when industries suffer from uncertainty, i.e., $\sigma > 0$, it does not always pay to provide firms with subsidies. Hence, for very volatile sectors the government should not encourage earlier investment by means of a subsidy. We show that the transition from a subsidy to a non-subsidy regime is triggered by the government's tax policy and industry risk characteristics. Finally, we have to stress that these policy implications are driven by the government's objective to maximize its net income. Obviously, there may be other objectives for which the firm's reaction function provides valuable information, e.g., if the government wants to maximize the future investment level. In such a situation, it makes sense not to promote investment. However, if the government wishes to increase the firm's propensity to invest earlier, it should subsidize the firm.

5. Conclusion

Significant progress has been made toward understanding optimal investment timing under uncertainty. To date, however, not much work has concentrated on the optimal level of investment and the time scale trade-off. In this paper, we study optimal investment timing under uncertainty, while considering the possibility of investment scaling and financing through debt. In particular, we employ a game-theoretic real option model between a firm and a government to analyze the effect of uncertainty and investment stimulus on optimal investment timing, financing and investment scaling.

Due to the nature of the non-cooperative game, we find that bargaining over the investment stimulus generally generates underinvestment, regardless of how the firm finances its investment. Optimal subsidies, however, reduce the level of underinvestment and the strength of this stimulus is strongly affected by the firm's financing decision. Our results indicate that levered firms are subsidized to a lesser degree. We show that even though levered firms receive fewer subsidies, they invest more than their unlevered counterparts. This result contradicts recent findings in the literature, which postulate that a firm's investment level is neutral to its financing decision (see, e.g., Wong, 2010). Furthermore, situations may occur in which the levered firm's tax shield benefits do not serve as a general incentive to invest earlier. On the contrary, our findings reveal that under some taxation schemes, the levered firm's optimal investment threshold is higher than that of its unlevered counterpart. Finally, an important result is that the effect of cash flow uncertainty on optimal subsidies is no longer monotonic as postulated by Danielova and Sarkar (2011). In fact, we find that it is ambiguous for most reasonable parameters, indicating that highly uncertain projects will most likely receive low or even no subsidies. The paper provides an analytical threshold under which the switch from a subsidy to non-subsidy regime occurs.

Obviously, our model is not without limitations and we leave it to future research to relax some of the assumptions we make (e.g., that governments often have to deal with multiple firms when designing an optimal subsidy regime). Hence, it would be interesting to analyze how a government would deal with heterogeneous firms that differ with regard to size or risk characteristics (see, e.g., Laincz, 2009). Further research might also add financing constraints as suggested by Wong (2010) and Shibata and Nishihara (2012) in order to cap the amount of debt financing and subsidies provided or to analyze how a convex tax schedule deters the optimal investment policy. In addition, a more fine-grained analysis of other incentive structures, such as in Armada et al. (2012), could guide future research directions. Moreover, the price of the bond may be affected by frictions in the secondary markets (e.g., if credit supply becomes uncertain due to distressed bondholders, agency problems or search costs for additional investors). Therefore, future research might explore these effects on the firm's capital structure in more detail (Ericsson & Renault, 2006; Hugonnier, Malamud, & Morellec, 2015; Ni, Chu, & Li, 2017).

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Appendix: Proofs

A.1. Expected present value

We follow Dixit and Pindyck (1994, 315f.) and use a dynamic programming-like recursive expression to estimate the optimal stopping time and thus the corresponding threshold (critical value). Let Y_t follow a geometric Brownian motion, i.e.,

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t, \ \ Y_0 > 0 \tag{45}$$

where dW_t is an increment of a Wiener process, $\mu \in \mathbb{R}$ is the drift rate and $\sigma \in \mathbb{R}_+$ is the level of uncertainty. Further, let T_a be the random first time the process Y_t reaches a fixed level $a < \infty$ starting from Y_0, i.e., $T_a = inf\{t \geq 0 : a \leq Y_t\}$. Define

$$f(Y_0) = \boldsymbol{E}^{Y_0}\left[e^{-rT_a}\right] \tag{46}$$

where $\boldsymbol{E}^{Y_0}[\ldots]$ denotes the expectation operator conditional on the state of Y_0 at time 0 and $r > \mu$ is a discount factor with $r \in \mathbb{R}_+$. It follows that if $Y_0 < a$ we can choose dt sufficiently small, such that it is very unlikely that Y_t hits a in the next short interval dt. Set $Y_0 = y$ as in Karlin and Taylor (1975, p. 364), we have $f(y) = e^{-rdt}\boldsymbol{E}^y[f(y + dy)]$. By means of Itô's Lemma we get:

$$f(y) = (1 - rdt + o(dt))\Big(f(y) + \mu y f'(y)dt + \frac{1}{2}\sigma^2 y^2 f''(y)dt + o(dt)\Big). \tag{47}$$

For $dt \to 0$, $o(dt)$ vanishes and after simplifying and note that $dtdt = 0$ we get the classical differential equation:

$$0 = \frac{1}{2}\sigma^2 y^2 f''(y) + \mu y f'(y) - r f(y) \tag{48}$$

This ordinary differential equation (ODE) has a general solution, i.e.:

$$f(y) = A_1 y^{\beta_1} + A_2 y^{\beta_2} \tag{49}$$

while β_i with $i \in \{1, 2\}$ is the positive and negative root of the standard quadratic equation. The coefficient A_i with $i \in \{1, 2\}$ is determined by a pair of boundary conditions. In particular, as y approaches the boundary a, it is very likely that T_a is very small and hence e^{-rT_a} close to 1. If y is very small, however, T_a is likely to be large and thus the discount factor close to zero. It follows that $f(a) = 1$ and $f(0) = 0$ which results in $A_2 = 0$ and $A_1 a^{\beta_1} = 1$. Upon inserting these results into (49) and back substitution we get:

$$f(Y_0) = \boldsymbol{E}^{Y_0}\left[e^{-rT_a}\right] = (Y_0/a)^{\beta_1} \tag{50}$$

A.2. Equity and debt values

Eq. (16) can be rewritten as:

$$\begin{aligned} E(x_L, I_L,\ c) &= \boldsymbol{E}^{x_L}\left[\int_{t_L}^{\infty} e^{-r(z-t_L)}(1-\tau)(\pi(I_L)x_z - c)dz\right] \\ &\quad -\boldsymbol{E}^{x_L}\left[\int_{t_b}^{\infty} e^{-r(z-t_L)}(1-\tau)(\pi(I_L)x_z - c)dz\right]. \end{aligned} \tag{51}$$

While the first term of the RHS can be easily solved the second term can be solved by means of the strong Markov property, i.e.:

$$\begin{aligned} E(x_L, I_L,\ c) &= (1-\tau)\left(\frac{\pi(I_L)x_L}{r-\mu} - \frac{c}{r}\right) - \boldsymbol{E}^{x_L}\left[e^{-r(t_b-t_L)}\right]\boldsymbol{E}_{x_b} \\ &\quad \times \left[\int_{t_b}^{\infty} e^{-r(z-t_b)}(1-\tau)(\pi(I_L)x_z - c)dz\right] \end{aligned} \tag{52}$$

which can further be simplified to:

$$\begin{aligned} E(x_L, I_L,\ c) &= (1-\tau)\left(\frac{\pi(I_L)x_L}{r-\mu} - \frac{c}{r}\right) \\ &\quad -\boldsymbol{E}^{x_L}\left[e^{-r(t_b-t_L)}\right](1-\tau)\left(\frac{\pi(I_L)x_b}{r-\mu} - \frac{c}{r}\right). \end{aligned} \tag{53}$$

As shown in the previous section of the Appendix a solution exists for the expectation value $\boldsymbol{E}^{x_L}[e^{-r(t_b-t_L)}]$, i.e.:

$$\boldsymbol{E}^{x_L}\left[e^{-r(t_b-t_L)}\right] = \left(\frac{x_L}{x_b}\right)^{\beta_2} \tag{54}$$

with $\beta_2 = 0.5 - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - 0.5\right)^2 + \frac{2r}{\sigma^2}} < 0$

Consequently, we get:

$$E(x_L, I_L, c) = V_U(x_L, I_L) - \frac{(1-\tau)c}{r} + \left(\frac{(1-\tau)c}{r} - V_U(x_b, I_L)\right)\left(\frac{x_L}{x_b}\right)^{\beta_2}, \tag{55}$$

where $V_U(x_L, I_L)$ indicates the value of the unlevered firm.

Similar steps are necessary to solve Eq. (19):

$$D(x_L, I_L, c) = \boldsymbol{E}^{x_L}\left[\int_{t_L}^{t_b} e^{-r(z-t_L)}(1-\tau_c)cdz + e^{-r(t_b-t_L)}(1-\alpha)V_U(x_b, I_L)\right]. \tag{56}$$

Hence, we can rewrite Eq. (56) which yields:

$$\begin{aligned} D(x_L, I_L, c) &= \boldsymbol{E}^{x_L}\left[\int_{t_L}^{\infty} e^{-r(z-t_L)}(1-\tau_c)cdz - \int_{t_b}^{\infty} e^{-r(z-t_L)}(1-\tau_c)cdz\right] \\ &\quad +\boldsymbol{E}^{x_L}\left[e^{-r(t_b-t_L)}\right](1-\alpha)V_U(x_b, I_L). \end{aligned} \tag{57}$$

By means of the strong Markov property, i.e.:

$$\begin{aligned} \boldsymbol{E}^{x_L}\left[\int_{t_b}^{\infty} e^{-r(z-t_L)}(1-\tau_c)cdz\right] &= \boldsymbol{E}^{x_L}\left[e^{-r(t_b-t_L)}\right] \\ &\quad \times \boldsymbol{E}^{x_b}\left[\int_{t_b}^{\infty} e^{-r(z-t_b)}(1-\tau_c)cdz\right] \end{aligned} \tag{58}$$

we can further simplify the equation:

$$D(x_L, I_L, c) = \frac{(1-\tau_c)c}{r} - \boldsymbol{E}^{x_L}\left[e^{-r(t_b-t_L)}\right]\left(\frac{(1-\tau_c)c}{r} - (1-\alpha)V_U(x_b, I_L)\right). \tag{59}$$

Using the closed form solution for $\boldsymbol{E}^{x_L}[e^{-r(t_b-t_L)}]$ as provided by Eq. (54) we get the value of debt at the investment trigger:

$$D(x_L, I_L, c) = \frac{(1-\tau_c)c}{r} + \left((1-\alpha)V_U(x_b, I_L) - \frac{(1-\tau_c)c}{r}\right)\left(\frac{x_L}{x_b}\right)^{\beta_2}. \tag{60}$$

As the firm and its investment project, respectively is financed by debt and equity, the firm value is the sum of equity and debt. Hence, at the investment trigger, we have that $V_L(x_L, I_L, c) = E(x_L, I_L, c) + D(x_L, I_L, c)$ amounts to:

$$\begin{aligned} V_L(x_L, I_L, c) &= V_U(x_L, I_L) - \frac{(1-\tau)c}{r}\left(1 - \left(\frac{x_L}{x_b}\right)^{\beta_2}\right) - V_U(x_b, I_L)\left(\frac{x_L}{x_b}\right)^{\beta_2} \\ &\quad +\frac{(1-\tau_c)c}{r}\left(1 - \left(\frac{x_L}{x_b}\right)^{\beta_2}\right) + (1-\alpha)V_U(x_b, I_L)\left(\frac{x_L}{x_b}\right)^{\beta_2} \end{aligned} \tag{61}$$

which can further be simplified:

$$V_L(x_L, I_L, c) = V_U(x_L, I_L) + \frac{(\tau-\tau_c)c}{r} - \left(\frac{(\tau-\tau_c)c}{r} + \alpha V_U(x_b, I_L)\right)\left(\frac{x_L}{x_b}\right)^{\beta_2}. \tag{62}$$

A.3. Proofs of the propositions

The following lemmas will be helpful in proving the propositions stated throughout the paper.

Lemma 1. *Let*

$$F(x, I) = \left(\frac{x_0}{x}\right)^{\beta_1}[x\lambda(I) - (I-S)],\ 0 < x_0 < x,\ 0 \le I,\ \ 0 \le S < I \tag{63}$$

with $\beta_1 > 1$ and $\lambda : \mathbb{R} \to \mathbb{R}^+,\ I \to \lambda(I)$ a twice continuously differentiable function fulfilling the properties

$$\frac{\partial\lambda}{\partial I}(I) > 0,\ \ \frac{\partial^2\lambda}{\partial I^2}(I) < 0 \quad \forall I > 0,$$

and $\varepsilon(I) := \frac{\partial\lambda}{\partial I}(I)\frac{I}{\lambda(I)}$, $I > 0$ is a strictly decreasing function. Let the point $(\hat{x}, \hat{I})$ satisfy the conditions

$$\hat{x}\frac{\partial\lambda}{\partial I}(\hat{I}) = 1, \tag{64}$$

$$\hat{x}\lambda(\hat{I}) = \frac{\beta_1}{\beta_1 - 1}(\hat{I} - S), \tag{65}$$

$$\frac{\lambda(\hat{I})}{\left(\frac{\partial\lambda}{\partial I}(\hat{I})\right)^2}\frac{\partial^2\lambda}{\partial I^2}(\hat{I}) < \frac{1}{1-\beta_1}. \quad (66)$$

Then $F(\hat{x},\hat{I}) \geq F(x,I)\ \forall (x,I) \in (x_0,\infty) \times (S,\infty)$.

Proof. The first order condition of the optimization problem is given by $\nabla F(\hat{x},\hat{I}) = 0$. The derivatives of $F(x,I)$ regarding x and I are

$$F_x(x,I) = \left(\frac{x_0}{x}\right)^{\beta_1}\left[(1-\beta_1)\lambda(I) + \beta_1(I-S)x^{-1}\right] \quad (67)$$

and

$$F_I(x,I) = \left(\frac{x_0}{x}\right)^{\beta_1}\left[x\frac{\partial\lambda}{\partial I}(I) - 1\right]. \quad (68)$$

Equating to zero and simplifying these equations leads to the desired assertions in Eqs. (64) and (65). For the second order condition, we show that the Hessian matrix

$$H(\hat{x},\hat{I}) = \begin{pmatrix} F_{xx}(\hat{x},\hat{I}) & F_{xI}(\hat{x},\hat{I}) \\ F_{Ix}(\hat{x},\hat{I}) & F_{II}(\hat{x},\hat{I}) \end{pmatrix} \quad (69)$$

is negative definite, i.e., $F_{xx}(\hat{x},\hat{I}) < 0$ and $F_{xx}(\hat{x},\hat{I})F_{II}(\hat{x},\hat{I}) - F_{xI}^2(\hat{x},\hat{I}) > 0$. For the derivatives, we find

$$F_{xx}(x,I) = \left(\frac{x_0}{x}\right)^{\beta_1}\beta_1\left[(\beta_1-1)x^{-1}\lambda(I) - (\beta_1+1)(I-S)x^{-2}\right], \quad (70)$$

$$F_{xI}(x,I) = \left(\frac{x_0}{x}\right)^{\beta_1}\left[(1-\beta_1)\frac{\partial\lambda}{\partial I}(I) + \beta_1 x^{-1}\right], \quad (71)$$

$$F_{II}(x,I) = \left(\frac{x_0}{x}\right)^{\beta_1}\left[x\frac{\partial^2\lambda}{\partial I^2}(I)\right]. \quad (72)$$

Inserting $(\hat{x},\hat{I})$ into Eq. (70), we get

$$F_{xx}(\hat{x},\hat{I}) = \left(\frac{x_0}{\hat{x}}\right)^{\beta_1}x^{-2}\beta_1\left[(\beta_1-1)\hat{x}\lambda(\hat{I}) - (\beta_1+1)(\hat{I}-S)\right]. \quad (73)$$

By using the relation from Eq. (65), this can be simplified to

$$F_{xx}(\hat{x},\hat{I}) = \left(\frac{x_0}{\hat{x}}\right)^{\beta_1}x^{-2}\beta_1\left[\beta_1(\hat{I}-S) - (\beta_1+1)(\hat{I}-S)\right] < 0. \quad (74)$$

Further, we have $F_{xx}(\hat{x},\hat{I})F_{II}(\hat{x},\hat{I}) - F_{xI}^2(\hat{x},\hat{I})$

$$\begin{aligned} &= \left(\tfrac{x_0}{\hat{x}}\right)^{2\beta_1}\hat{x}^{-2}\beta_1\Bigg[\Bigg((\beta_1-1)\underbrace{\hat{x}\lambda(\hat{I})}_{\underset{(64)}{\rightarrow}\frac{\beta_1}{\beta_1-1}(\hat{I}-S)} - (\beta_1+1)(\hat{I}-S)\Bigg)\hat{x}\tfrac{\partial^2\lambda}{\partial I^2}(\hat{I}) \\ &\qquad - \tfrac{1}{\beta_1}\Bigg((1-\beta_1)\underbrace{\frac{\partial\lambda}{\partial I}(\hat{I})\hat{x}}_{\underset{(64)}{\rightarrow}1} + \beta_1\Bigg)^2\Bigg] \\ &= \left(\tfrac{x_0}{\hat{x}}\right)^{2\beta_1}\hat{x}^{-2}\left[-(\hat{I}-S)\hat{x}\tfrac{\partial^2\lambda}{\partial I^2}(\hat{I})\beta_1 - 1\right] \\ &= \left(\tfrac{x_0}{\hat{x}}\right)^{2\beta_1}\hat{x}^{-2}\left[(1-\beta_1)\lambda(\hat{I})\tfrac{\partial^2\lambda}{\partial I^2}(\hat{I})\left(\tfrac{\partial\lambda}{\partial I}(\hat{I})\right)^{-2} - 1\right] > 0 \end{aligned} \quad (75)$$

by the premise.

To see that the solution is unique, write Eqs. (64) and (65) as

$$\frac{\hat{I}}{\hat{I}-S} = \frac{\beta_1}{\beta_1-1}\varepsilon(\hat{I}). \quad (76)$$

The LHS of this equation is a strictly decreasing, convex function of I, with the properties:

$$\lim_{I\to S^+}\frac{I}{I-S} = +\infty, \lim_{I\to\infty}\frac{I}{I-S} = 1.$$

Furthermore, the RHS is a strictly decreasing, continuously differentiable function with

$$\lim_{I\to 0}\frac{\beta_1}{\beta_1-1}\varepsilon(I) = \frac{\beta_1}{\beta_1-1}, \lim_{I\to S}\frac{\beta_1}{\beta_1-1}\varepsilon(I) = \frac{\beta_1}{\beta_1-1}\varepsilon(S) < \frac{\beta_1}{\beta_1-1}.$$

Therefore, Eq. (76) can have at most 2 solutions, i.e., $\hat{I}_1 < \hat{I}_2$, which are characterized by the slope of the LHS and RHS respectively, i.e.

$$\frac{\partial}{\partial I}\left(\frac{I}{I-S}\right)\Bigg|_{I=\hat{I}_1} < \frac{\beta_1}{\beta_1-1}\frac{\partial}{\partial I}(\varepsilon(I))\Bigg|_{I=\hat{I}_1}$$

and

$$\frac{\partial}{\partial I}\left(\frac{I}{I-S}\right)\Bigg|_{I=\hat{I}_2} \geq \frac{\beta_1}{\beta_1-1}\frac{\partial}{\partial I}(\varepsilon(I))\Bigg|_{I=\hat{I}_2}.$$

The latter condition is equivalent to Eq. (66) and therefore, the only point satisfying the second order condition of the maximization problem.
Q.E.D.

Lemma 2. *Given* $(\hat{x},\hat{I})$ *fulfilling Eqs. (64) and (65). Then the derivative* $d\hat{I}/dS$ *exists and is given by*

$$\frac{d\hat{I}}{dS}(S) = \frac{\beta_1}{1 + (\beta_1-1)\lambda(\hat{I}(S))\left(\frac{\partial\lambda}{\partial I}(\hat{I}(S))\right)^{-2}\frac{\partial^2\lambda}{\partial I^2}(\hat{I}(S))} < 0. \quad (77)$$

Proof. Let $\hat{I}(S) := \hat{I}$. By combining Eqs. (64) and (65), we have that

$$f(\hat{I}(S),S) := \frac{\hat{I}-S}{\lambda(\hat{I})}\frac{\partial\lambda}{\partial I}(\hat{I}) = \frac{\beta_1-1}{\beta_1} = const, \quad (78)$$

therefore $d\hat{I}/dS = -(\partial f/\partial S)/(\partial f/\partial \hat{I})$. For the partial derivatives, we get

$$\frac{\partial f}{\partial S} = -\frac{1}{\lambda(\hat{I})}\frac{\partial \lambda}{\partial I}(\hat{I}) \tag{79}$$

and

$$\frac{\partial f}{\partial \hat{I}} = \frac{1}{\lambda(\hat{I})}\left[\frac{1}{\beta_1}\frac{\partial \lambda}{\partial I}(\hat{I}) + (\hat{I} - S)\frac{\partial^2 \lambda}{\partial I^2}(\hat{I})\right]. \tag{80}$$

Inserting Eq. (66) and rearranging, this equation becomes

$$\frac{\partial f}{\partial \hat{I}} = \frac{1}{\beta_1}\frac{\partial \lambda}{\partial I}(\hat{I})\frac{1}{\lambda(\hat{I})}\left[1 + (\beta_1 - 1)\lambda(\hat{I})\left(\frac{\partial \lambda}{\partial I}(\hat{I})\right)^{-2}\frac{\partial^2 \lambda}{\partial I^2}(\hat{I})\right] < 0. \tag{81}$$

Q.E.D.

Corollary 3. *With $\hat{I}$ as determined by Lemma 1, we have*

$$\gamma(S) := \left(\frac{\partial \lambda}{\partial I}(\hat{I}(S))\right)^{-1}\frac{\partial^2 \lambda}{\partial I^2}(\hat{I}(S))\frac{d\hat{I}}{ds}(S) > 0, \ \forall S > 0\ . \tag{82}$$

Proof. Note that $\frac{\partial \lambda}{\partial I}(\hat{I}(S)) > 0$ by the premise of Lemma 1 and from Lemma 2 we know that $\frac{d\hat{I}}{ds}(S) < 0$. Further we have

$$\frac{\partial^2 \lambda}{\partial I^2}(\hat{I}(S)) < 0, \ \forall S > 0 \tag{83}$$

by Eq. (77).
Q.E.D.

Lemma 3. *Let*

$$G(S) = \left(\frac{x_0}{\hat{x}(S)}\right)^{\beta_1}\left[\kappa_G \hat{x}(S)\lambda(\hat{I}(S)) - S\right] \tag{84}$$

and

$$\gamma(S) = \left(\frac{\partial \lambda}{\partial \hat{I}}(\hat{I}(S))\right)^{-1}\frac{\partial^2 \lambda}{\partial \hat{I}^2}(\hat{I}(S))\frac{d\hat{I}}{dS}(S) \tag{85}$$

with $\lambda(I)$ as in Lemma 1 and let the point $(\hat{x}(S), \hat{I}(S))$ satisfy Eqs. (64)–(66). For $\kappa_G\beta_1 > 1$, $\hat{x}(S) \geq x_0$, the optimal level of subsidy $\hat{S} := \operatorname{argmax}_{S \geq 0} G(S)$ is determined by:

$$\hat{S}\gamma(\hat{S}) = \frac{\kappa_G\beta_1 - 1}{\beta_1}, \tag{86}$$

and

$$1 + \frac{\kappa_G\beta_1 - 1}{\beta_1}\frac{1}{\gamma^2(\hat{S})}\frac{d\gamma}{dS}(\hat{S}) > 0. \tag{87}$$

Proof.

$$\frac{dG}{dS}(S) = \left(\frac{\partial G}{\partial \hat{I}}(S) + \frac{\partial G}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial I}(S)\right)\frac{d\hat{I}}{dS}(S) + \frac{\partial G}{\partial S}(S) \tag{88}$$

$$= \left(\frac{x_0}{\hat{x}(S)}\right)^{\beta_1}\Bigg[\Bigg(\kappa_G\Big(\hat{x}(S)\frac{\partial \lambda}{\partial I}(\hat{I}(S)) + (1-\beta_1)\frac{\partial \hat{x}}{\partial I}(S)\lambda(\hat{I}(S))\Big) + \beta_1\frac{S}{\hat{x}(S)}\frac{\partial \hat{x}}{\partial I}(S)\Bigg)\frac{d\hat{I}}{dS}(S) - 1\Bigg]. \tag{89}$$

By using Eq. (64), we have

$$\frac{\partial \hat{x}}{\partial I}(S) = -\left(\frac{\partial \lambda}{\partial I}(\hat{I}(S))\right)^{-2}\frac{\partial^2 \lambda}{\partial I^2}(\hat{I}(S)) \tag{90}$$

hence,

$$\beta_1\frac{S}{\hat{x}(S)}\frac{\partial \hat{x}}{\partial I}(S) = -\beta_1\frac{S}{\hat{x}(S)}\left(\frac{\partial \lambda}{\partial I}(\hat{I}(S))\right)^{-2}\frac{\partial^2 \lambda}{\partial I^2}(\hat{I}(S)) \underset{(64)}{=} -\beta_1 S\left(\frac{\partial \lambda}{\partial I}(\hat{I}(S))\right)^{-1}\frac{\partial^2 \lambda}{\partial I^2}(\hat{I}(S)) \tag{91}$$

and

$$\begin{aligned}&\hat{x}(S)\tfrac{\partial \lambda}{\partial I}(\hat{I}(S)) + (1-\beta_1)\tfrac{\partial \hat{x}}{\partial I}(S)\lambda(\hat{I}(S))\\ &= \underbrace{\hat{x}(S)\frac{\partial \lambda}{\partial I}(\hat{I}(S))}_{\underset{(64)}{\rightarrow} 1} + (\beta_1 - 1)\left(\tfrac{\partial \lambda}{\partial I}(\hat{I}(S))\right)^{-2}\tfrac{\partial^2 \lambda}{\partial I^2}(\hat{I}(S)) \underset{(77)}{=} \beta_1\left(\tfrac{d\hat{I}}{dS}(S)\right)^{-1}.\end{aligned} \tag{92}$$

Using these equations and simplifying leads to

$$\frac{dG}{dS}(S) = \left(\frac{x_0}{\hat{x}(S)}\right)^{\beta_1}\left[\kappa_G\beta_1 - 1 - \beta_1 S\left(\frac{\partial \lambda}{\partial I}(\hat{I}(S))\right)^{-1}\frac{\partial^2 \lambda}{\partial I^2}(\hat{I}(S))\frac{d\hat{I}}{dS}(S)\right]. \tag{93}$$

Equating to zero gives the candidate from Eq. (86). For the second order condition we have

$$\begin{aligned}\frac{d^2G}{dS^2}(\hat{S}) &= -\beta_1\left(\frac{x_0}{\hat{x}(\hat{S})}\right)^{\beta_1}\left[\gamma(\hat{S}) + \hat{S}\frac{d\gamma}{dS}(\hat{S})\right]\\ &= -\beta_1\gamma(\hat{S})\left(\frac{x_0}{\hat{x}(\hat{S})}\right)^{\beta_1}\left[1 + \frac{\kappa_G\beta_1 - 1}{\beta_1}\frac{1}{\gamma^2(\hat{S})}\frac{d\gamma}{dS}(\hat{S})\right] < 0.\end{aligned} \tag{94}$$

Q.E.D

Proof of Proposition 1:

We conduct a case-by-case analysis.

(a) $4\beta_1(\beta_1 - 1)S \leq 1$ and $\hat{x}_U(S) > x_0$:

The value of the unlevered firm's investment option is given by Eq. (6):

$$F_U(x_0, S) = \sup_{x_U \geq x_0, I_U \geq 0}\left\{\left(\frac{1-\tau}{r-\mu}\pi(I_U)x_U - I_U + S\right)\left(\frac{x_0}{x_U}\right)^{\beta_1}\right\}.$$

To solve this optimization problem, we refer to Lemma 1 with $\lambda(I) = \frac{1-\tau}{r-\mu}\pi(I) = \frac{1-\tau}{r-\mu}\frac{I}{I+1}, 0 < \tau < 1, r > \mu > 0\, r > \mu > 0$. We have $\lambda(0) = 0$ and $\lambda'(I) = \frac{1-\tau}{r-\mu}\frac{1}{(I+1)^2} > 0\ \forall I \geq 0$.

The first order optimality conditions become:

$$\frac{1-\tau}{r-\mu}\hat{x}_U\frac{\partial \pi}{\partial I}(\hat{I}_U) = 1 \tag{95}$$

and

$$\hat{x}_U\frac{1-\tau}{r-\mu}\pi(\hat{I}_U) = \frac{\beta_1}{\beta_1 - 1}(\hat{I}_U - S). \tag{96}$$

Inserting the optimal investment level (Eq. (7)), Eq. (95) becomes

$$\frac{1-\tau}{r-\mu}\frac{\hat{x}_U}{(1+\hat{I}_U)^2} = \frac{\left(\frac{2\beta_1 - 1 + \sqrt{1-4\beta_1(\beta_1-1)S}}{2(\beta_1-1)}\right)^2}{\left(1 + \frac{1+\sqrt{1-4\beta_1(\beta_1-1)S}}{2(\beta_1-1)}\right)^2} = 1.$$

We further show that Condition (65) holds true:

$$\hat{x}_U\lambda(\hat{I}_U) = \left(\frac{\partial \lambda}{\partial I}(\hat{I}_U)\right)^{-1}\lambda(\hat{I}_U) = \hat{x}_U\frac{1-\tau}{r-\mu}\pi(\hat{I}_U) = \frac{\beta_1}{\beta_1 - 1}(\hat{I}_U - S),$$

which is equivalent to

$$\hat{I}_U\left(\hat{I}_U+1\right)-\frac{\beta_1}{\beta_1-1}\left(\hat{I}_U-S\right)=\hat{I}_U^2-\frac{1}{\beta_1-1}\hat{I}_U+\frac{\beta_1}{\beta_1-1}S=0,$$

to which Eq. (8) is a solution. Further, we have

$$\frac{\lambda\left(\hat{I}_U\right)}{\left(\frac{\partial\lambda}{\partial I}\left(\hat{I}_U\right)\right)^2}\frac{\partial^2\lambda}{\partial I^2}\left(\hat{I}_U\right)=-2\hat{I}_U=\frac{1+\sqrt{1-4\beta_1(\beta_1-1)S}}{(1-\beta_1)}<\frac{1}{1-\beta_1}.$$

(b) $4\beta_1(\beta_1-1)S\le 1$ and $\hat{x}_U(S)\le x_0$:

In this case, the investor chooses to invest immediately. Thus, $\hat{x}_U=x_0$. The optimal investment level is then optimally given by $F_I(x,I)=0$, that is:

$$x_0\frac{\partial\lambda}{\partial I}\left(\hat{I}_U\right)=1,$$

which is fulfilled by

$$\hat{I}_U=\sqrt{\frac{(1-\tau)x_0}{(r-\mu)}}-1.$$

(c) $4\beta_1(\beta_1-1)S>1$:

In this case, the condition $\hat{I}_U(\hat{I}_U+1)-\frac{\beta_1}{\beta_1-1}(\hat{I}_U-S)=0$ has no solution, thus a local maximum does not exist. The optimal solution is to be found at the border, i.e., $\hat{x}_U\in\{x_0,\infty\}$. Since $\lim_{x\to\infty}F(x,I)=0$, we get $\hat{x}_U=x_0$ and

$$\hat{I}_U=\sqrt{\frac{(1-\tau)x_0}{(r-\mu)}}-1.$$

Q.E.D.

Proof of Proposition 2:

We refer to Lemma 3 with $\lambda(I)=\frac{1-\tau}{r-\mu}\pi(I)=\frac{1-\tau}{r-\mu}\frac{I}{I+1}$, $0\le\tau<1, r>\mu>0$ $\kappa_G=\tau/(1-\tau)$. We conduct a case-by-case analysis. Note that for $0\le S\le\frac{1}{4\beta_1(\beta_1-1)}$

$$\left(\frac{2\beta_1-1}{2(\beta_1-1)}\right)^2\le\frac{(1-\tau)}{r-\mu}\hat{x}_U(S)\le\left(\frac{\beta_1}{\beta_1-1}\right)^2. \tag{97}$$

(a) $x_0<\hat{x}(S)$ $\forall S: 4\beta_1(\beta_1-1)S\ge 1$:

We denote the optimal level of subsidy in this region by $\hat{S}_{U,2}$.

(aa) $\kappa_G\beta_1>1$:

According to Lemma 3, it is given by:

$$\hat{S}_{U,2}\gamma\left(\hat{S}_{U,2}\right)=\frac{\tau}{1-\tau}-\frac{1}{\beta_1}, \tag{98}$$

with

$$\gamma(S)=\frac{4\beta_1(\beta_1-1)}{\left(\sqrt{1-4\beta_1(\beta_1-1)S}+2\beta_1-1\right)\sqrt{1-4\beta_1(\beta_1-1)S}}. \tag{99}$$

Thus, Eq. (98) is equivalent to

$$\frac{4\beta_1(\beta_1-1)\hat{S}_{U,2}}{\left(\sqrt{1-4\beta_1(\beta_1-1)\hat{S}_{U,2}}+2\beta_1-1\right)\sqrt{1-4\beta_1(\beta_1-1)\hat{S}_{U,2}}}=\frac{\tau}{1-\tau}-\frac{1}{\beta_1}.$$

This equation has one solution given by

$$\hat{S}_{U,2}=\frac{2\xi^2-\psi^2-2\xi(1-\tau)\beta_1+\psi\sqrt{\psi^2+4\xi(1-\tau)\beta_1}}{8\beta_1(\beta_1-1)\xi^2}.$$

Note further that $\frac{d\gamma}{dS}>0$ $\forall S: 4\beta_1(\beta_1-1)S<1$. Hence, the second order condition from relation (87) is fulfilled.

(ab) $\kappa_G\beta_1\le 1$:

Since Eq. (86) has no solution, the optimal subsidy is to be found at the border, i.e., $\hat{S}_U^2=0$.

(b) $x_0>\hat{x}(S)$ $\forall S>0$, i.e., $(\frac{\beta_1}{\beta_1-1})^2\le\frac{(1-\tau)}{(r-\mu)}x_0$:

In this region, the investor would always invest immediately. Therefore, the optimal subsidy is $\hat{S}_U=0$.

(c) $(\frac{2\beta_1-1}{2(\beta_1-1)})^2\le\frac{\phi(1-\tau)}{r-\mu}x_0\le(\frac{\beta_1}{\beta_1-1})^2$:

Let $\hat{S}_{1,U}$ denote the subsidy that induces immediate investment, i.e., $\hat{x}_U(\hat{S}_{1,U})=x_0$. Solving the latter condition for $\hat{S}_{1,U}$ leads to

$$\hat{S}_{1,U}=\frac{1-\left(2(\beta_1-1)\sqrt{\frac{(1-\tau)}{(r-\mu)}x_0}+2\beta_1-1\right)^2}{4\beta_1(\beta_1-1)}. \tag{100}$$

The government's action set in this region is given by $\{\hat{S}_{2,U},\hat{S}_{1,U}\}$.

(ca) $\hat{S}_{2,U}\ge\hat{S}_{1,U}$:

Since $\hat{x}_U(\hat{S}_{2,U})=\hat{x}_U(\hat{S}_{1,U})$ and $\hat{I}_U(\hat{S}_{2,U})=\hat{I}(\hat{S}_{1,U})$, the government's payoff is maximized by choosing the smaller subsidy $\hat{S}_{1,U}$, i.e., $\hat{S}=\hat{S}_{1,U}$.

(cb) $\hat{S}_{2,U}<\hat{S}_{1,U}$:

For the government's payoff, we have $G_U(\hat{S}_{2,U})>G_U(\hat{S}_{1,U})$ since $\hat{S}_{2,U}$ is the solution to the system defined by (86) and (87). Hence, the optimal subsidy in this case is given by $\hat{S}=\hat{S}_{2,U}$.

Therefore, we have $\hat{S}=\min\{\hat{S}_{2,U},\hat{S}_{1,U}\}$ in this region.

Q.E.D.

Proof of Proposition 3:

Given an investment threshold x_L and an investment level of I_L, the value of the firm is given by

$$F_L(x_L,I_L,c)=\left(V_U(x_L,I_L)+\frac{(\tau-\tau_c)c}{r}\left(1-\left(\frac{x_L}{x_b(I_L,c)}\right)^{\beta_2}\right)\right.$$
$$\left.-\alpha V_U(x_b(I_L,c),I_L)\left(\frac{x_L}{x_b(c,I_L)}\right)^{\beta_2}-I_L+S\right)\left(\frac{x_0}{x_L}\right)^{\beta_1}. \tag{101}$$

For $\tau_c>\tau$, we have

$$\hat{c}=\underset{c}{\operatorname{argmax}}\{F(x_L,I_L,c)\},\hat{c}:=\frac{\partial F}{\partial c}\left(\hat{c}\right)=0.$$

Further, we have

$$x_b(c,I_L)=\frac{r-\mu}{r}\frac{\beta_2}{\beta_2-1}\frac{c}{\pi(I_L)},$$

Therefore, we have

$$\frac{\partial F}{\partial c}(c)=\frac{\tau-\tau_c}{r}$$
$$+\left(\frac{x_L}{x_b(I_L,c)}\right)^{\beta_2}\left[-\frac{\tau-\tau_c}{r}+\beta_2\frac{\tau-\tau_c}{r}\frac{c}{x_b(I_L,c)}\frac{\partial x_b(I_L,c)}{\partial c}\right.$$
$$\left.-\frac{\alpha(1-\tau)}{r}\frac{\beta_2}{\beta_2-1}\left(1-\beta_2\frac{c}{x_b(I_L,c)}\frac{\partial x_b(I_L,c)}{\partial c}\right)\right]$$
$$=\frac{\tau-\tau_c}{r}+\left(\frac{x_L}{x_b(I_L,c)}\right)^{\beta_2}\left[-\frac{\tau-\tau_c}{r}+\beta_2\frac{\tau-\tau_c}{r}+\frac{\alpha(1-\tau)}{r}\beta_2\right].$$

Simplifying the condition $F_c(\hat{c})=0$, we get:

$$0=\tau-\tau_c-[\tau-\tau_c-(\tau-\tau_c)\beta_2-\alpha(1-\tau)\beta_2]$$
$$\left(\frac{r}{r-\mu}\frac{\beta_2-1}{\beta_2}\frac{\pi(I)}{\hat{c}(x_L,I_L)}\right)^{-\beta_2}x_L^{\beta_2}. \tag{102}$$

Solving the latter equation leads to the desired result:

$$\hat{c}(x_L,I)=\omega^{-\frac{1}{\beta_2}}\left(\frac{r}{r-\mu}\right)\left(\frac{\beta_2-1}{\beta_2}\right)\pi(I)x_L. \tag{103}$$

Q.E.D.

Proof of Proposition 4: Please refer to the Proof of Proposition 1, but with $\lambda(I) = \phi\frac{1-\tau}{r-\mu}\pi(I) = \phi\frac{1-\tau}{r-\mu}\frac{I}{I+1}$.
Q.E.D.

Proof of Proposition 5: Please refer to the Proof of Proposition 2, but with $\lambda(I) = \phi\frac{1-\tau}{r-\mu}\pi(I) = \phi\frac{1-\tau}{r-\mu}\frac{I}{I+1}$ and $\kappa_G = \frac{\zeta}{\phi}\frac{\tau}{1-\tau}$.
Q.E.D.

Proof of Proposition 6: Please refer to the Proof of Proposition 1, but with $\lambda(I) = \pi(I) = \frac{I}{I+1}$.
Q.E.D.

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