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Application of modified Black-Litterman model for active portfolio management

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ABSTRACT

An active policy for portfolio optimization is developed based on repetitive application of modified Black-Litterman (BL) portfolio model and new formal definition of the expert views. New subjective views are defined which are based on the differences between the historical mean asset returns and their implied return values. An algorithm for the implementation of active management with the modified BL model is derived. The active management policy allows using short time series of historical data of assets, providing portfolio optimization with limited set of assets. New market point is evaluated, because the small set of assets does not allow market index to be used as characteristics of the market. The new formalization of the expert views allows to be compared the Mean Variance and BL portfolios on common basis. The experiments and comparisons between the Mean Variance optimization and the modified BL problem give advantages to the last one.

1. Introduction

The active portfolio management relies on the proper forecasting of assets' characteristics: risk and return. In modern portfolio theory such forecasts mainly contain assessments of previous, historical behavior of the assets' returns. These forecasts strongly influence the input parameters of the portfolio problem and its solutions in this case can be far from the real market dynamics (Becker & Gürtler, 2010; Calvo, Ivora, & Liern, 2012; Garcıa, Quintana, Galvan, & Isasi, 2013; Gorgulho, Neves, & Horta, 2011; Jørgensen, 2016; Kolm, Tutuncu, & Fabozzi, 2014; Michaud, Esch, & Michaud, 2013; Sharpe, 1999; Walters, 2014). With the appearance of the Black-Litterman (BL) portfolio model, additional information for future return and risk are used for the portfolio management generated by experts, which target more close forecasts to the market changes. For detailed description of the Black-Litterman model one can refer to (Black & Litterman, 1991; Silva, Da & Pornrojnangkool, 2009; Walters, 2014). Additional developments of the BL model can be found in (Kara, Ulucan & Atici, 2019; Kolm, Tutuncu & Fabozzi, 2014; Palczewski & Palczewski, 2018; Pang & Karan, 2018; Xu, Chen & Tsui, 2008). Thus, the assessment of the assets' characteristics formally is evaluated according to more complex model. It takes into consideration both the historical data of assets' returns used by the classical Mean Variance (MV) Model and additional information, generated by the experts' views for the assets returns. The expert views can be defined not only in absolute values, but in relative way as well for increase or decrease the values of the returns. Thus, the Black-Letterman portfolio model extends the opportunities for precise estimation and forecast of the assets characteristics.

Nevertheless that the Black-Litterman model is complicated, due to the integration of data about the historical and current behavior of the asset returns and information, generated by the subjective experts' views, its formal description is well analytically justified. However, for its practical application and potential for utilization, there are controversial conclusions. In (Cheung, 2011) is concluded that this model does not have wide applications for active portfolio management yet. The reasons for that are practical due to the requirements for assessment and estimation of a set of parameters which influence the portfolio problem. For example, very important and needed parameters, which have to be evaluated for the BL model, are:

- the confidence parameter which participates in the optimization problem and which quantitatively defines the level of risk which the investor is willing to undertake;
- the market equilibrium point, which gives the levels of market return and market volatility. The market point in general is used as a

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benchmark point for the BL model and according to it the forecasts of the future returns are evaluated;
- the need to justify the parameter τ, which defines the level of usage of the historically based covariance matrix in BL model.

Because these parameters are very important for the accuracy of the BL model, its practical usage is limited for active and short term optimization. The criticism towards the BL model is also available. In (Michaud & Michaud, 2013) the doubts about the efficiency of the BL model are explicitly stated: "…The BL portfolio is often uninvestable in applications due to large leveraged or short allocations."

Nevertheless the aforementioned constraints which restrict the application of the BL model for active short term usage in a set of financial software suits, the portfolio constructions apply the BL model in Morningstar Direct (URL1). Additionally, one can find the top 20 investment management software for 2019 in (URL2). Following (Bertsimas, Gupta & Paschalidis, 2012) "…..The U.S. investment bank Goldman Sachs regularly publishes recommendations for investor allocations based on the BL model and has issued reports describing the firm's experience using the model. A host of other firms (Zephyr Analytics, BlackRock, Neuberger Berman, etc.) also use the BL model at the core of many of their investment analytics….".

Despite that the Black-Litterman is widely assessed in academic and research works, the Black Litterman model does not have wide applications for active and short term portfolio management. An overview trough the references, claiming theoretical proof of the BL model, looking to the real and experimental tests, they give evidences that the BL model is checked and applied in global market environment and for long period of time for the asset returns. The examples, which can be found, are based on global and/or long periods for analysis and parameter estimation for the BL model.

Example for active management of portfolio for short terms with limited number of assets is not met in the reference materials. This lack of such short term applications is the main reason for this paper to try to develop an algorithm allowing the BL model to be applied for short periods of investment and limited number of assets for the portfolio.

From user's point of view it is worth to follow practical algorithm and utilization rules, which can provide successful implementation of the Black-Litterman model for the cases of active short term investments. This paper tries to develop a practical and easy for implementation algorithm which is based on the Black-Litterman model for portfolio management. The algorithm targets implementation of active policy of portfolio optimization, which can be successfully applied in practical cases. The paper illustrates such portfolio optimization with up to date values of returns and portfolio parameters by example. Comparisons with the classical Mean Variance Model are given. The paper illustrates also a new form of formalization of the expert views. These views are based on additional assessment of the difference between the implied and historical data about the asset returns.

2. Overview of examples with BL model usage

This part of the research makes an overview of numerical examples, which are published for illustration of the BL model for portfolio optimization. The BL model application requires the estimation of a set of parameters which are evaluated from a time series of returns' data, named here as historical data. Here attention is paid to the duration of the time series and the scale of markets which were considered in different references, applying and assessing the BL model for portfolio optimization. Several papers lack of evidences with numerical data, which make difficult implementation of their results.

In (Seimertz, 2015) the example, which is developed in that research, uses data for a long time horizon, 2007–2014. In (Harris, Stoja & Tan, 2017) the time horizon for historical data is 1994–2015. In (Kolm & Ritter, 2017) has been studied the US equity market over period 1992–2015.

A conclusion can be done that the BL model for portfolio optimization cannot be applied for active management for short time period. The same case is in (Kierkegaard, Lejon & Persson, 2006) where the time series of data, used for portfolio management belongs for a long period of time, 1996–2006. In another wide cited paper (Fabozzi, Focardi & Kolm, 2006), the numerical data for the evaluations in BL model addresses returns from long and past period from 1980 to 2004. In (Allaj, 2017) data for the BL model are taken for a wide period from 1988 till 2016. The master thesis (Dove & Norell, 2016) also deals with data from 2006 to 2015 for the BL model.

The distinguish paper (Satchell & Scowcroft, 2000) takes data for explanations of BL model from the global markets, consisting of 11 countries. In (Stotz, 2005) the sample period covers monthly data from December 1989 up to December 2000.

In (Allaj, 2013) quantitative evaluations in global markets considering 13 countries constitute the numerical proofs and illustrations. Again the application of BL model is not performed for active and short term investment and portfolio optimization. The same case for data from global markets, considering many countries, is met in (Bevan & Winkelmann, 1998; Litterman, The Quantitative Resources Group, 2003). In (Cheung, 2011) comparisons in global markets, taking data from 7 countries are done, considering annualized volatilities, market capitalization, weights. In (Kocuk & Cornuéjols, 2018) it has been collected returns and market capitalizations for 30 years period from 1987 to 2016 and the portfolio optimization has been considered for 11 sectors, which define a global scale of the market.

Apparently, it is seen that such an approach for active management on local basis is not feasible. In (Idzorek, 2002) the numerical evaluations apply data from 60 months' period. The same duration with 5 years monthly sales (data of 60 months) is applied in (Walters, 2014). In (Michaud, Esch & Michaud, 2013), 18 years period of monthly returns are used for quantitative assessment of BL model. In (Becker & Gürtler, 2010) the Black-Litterman model is applied with real capital market data with monthly data from 12/01/1993 to 01/01/2008 of all stocks of HDAX and DAX100. The wide cited paper (Walters, 2014) does not make proofs with numerical experiments and data, despite that it provides valuable explanations about the BL model.

This analysis proves that the BL model is intensively assessed and used for long time periods for global scale of the market. But the model has not been reported for active management in short term duration and practically this is a constraint for small set of assets. Table1 summarizes the long time application of the BL history periods and the global scale of the markets.

The small set of assets needs the estimation of new market point, which can differ from this one, given by market indices. Nevertheless the strong theoretical background of the BL model, algorithms for its implementations in active investment policies are not popular neither discussed.

This paper targets development of an algorithm for practical application of BL model in active portfolio management. The active management is considering for cases with short time horizon and with limited number of assets for the portfolio. The derived algorithm is based on the usage of BL modified portfolio model and it provides additional calculations which implement rules for active portfolio management. The modifications of the BL model concern new formal descriptions of the expert views. The expert views are made according to the differences between the implied asset returns and their mean values. Such modification of the BL model allows its solutions to be compared with the portfolio solutions of the classical MV problem. The present research uses small number of securities, because no more than a handful of "views" can be accommodated at one time. If there were a large number of securities in the asset universe, the investor would get swamped not to accommodate more than a handful of "views" at one time. The application of the rolling investment policy allows being implemented active portfolio management, with the modified BL model.

Table 1
BL model applications giving duration of history periods and global scale markets.

Source	History period	Scale of assets
Seimertz (2015)	2007–2014	OMXSBPI Stockholm indices (Oil&Gas; Basic materials; Industrials; Consumer goods; Health care; Consumer services; Telecommunications; Utilities; Financial ; Technology)
Harris, Stoja and Tan (2017)	1994–2015	10 FTSE industry sectors in USA, UK and Japan
Kolm & Ritter (2017)	1992 2015	CRISP(Center for Research In Security Prices) indices; IBES(Institutional Brokers Estimate System) indices.
Kierkegaard, Lejon and Persson (2006)	1996–2006	10 indices OMX Stockholm Group
Fabozzi, Focardi and Kolm (2006)	1980–2004	23 market country indices
Allaj (2017)	1988–2016	10 S&P sector indices; 5 FTSE Europe sector indices; 2 FTSE Canada sector indices; 3 FTSE Japan sector indices
Dove and Norell (2016)	2006–2015	10 FTSE100 sector indices(Oil & Gas; Consumer goods; Financial; Consumer services; Health care; Basic Materials; Industrials; Telecommunications; Utilities; Technology)
Satchell and Scowcroft (2000)	not defined	Universe of 11 European equity markets
Stotz (2005)	1989–2000	DJSTOXX50 sector indices (indices from 17 European countries)
Allaj (2013)	2000–2006	Indices from 13 countries(Austria, Denmark, Finland, France, Germany, Italy, Nederland, Norway, Portugal, Spain, Sweden, Switzerland)
Bevan and Winkelmann (1998)	1988–1997	Indices from 14 countries; Goldman-Sachs 13 government bound indices
Litterman (2003)	1980–2002	13 country indices; 7 government bounds; 6 currencies
Cheung (2011)	not given	Global market of 7 countries
Kocuk and Cornuéjols (2018)	30 years 1987–2016	11 sector indices
Idzorek (2002)	60 months	8 sector indices

3. Algorithm for active management of BL - based portfolio

The developed here algorithm is based on a modified BL model, targeting active management of the portfolio investments. The content of the algorithm consists of sequence of calculations, which provide estimation of future average returns of the portfolio assets. Following the BL model, the sequence of calculations are graphically presented in Fig. 1. The manners of evaluations are described below.

3.1. *Evaluation of the implied returns: Π_i, $i = 1,N$- number of assets in the portfolio.*

The algorithm makes difference between the two parameters "implied returns Π_i" and "implied excess returns Π_i*, $i = 1,N$ ". One can meet in the references a mixture of usage of these two terms but they provide different values for the parameters of the BL model. For illustration purposes the following references interchange the usage of these two parameters:

- in (Idzorek, 2002) with the notation **Π** relation (1) concerns the vector of excess return, but in relation (3) the same notation represents the vector of implied equilibrium return;
- in (Jørgensen, 2016; Kooli & Selam, 2010; Mishra, Pisipati & Vyas, 2011) the same mixture of usage of implied and excess returns are met.

In this paper "implied equilibrium" is used and appropriate calculations are undertaken to render into account the risk-free value for the portfolio management.

The value for the "implied excess return" **Π*** is undertaken from the Security Market Line (SML), derived by the Capital Asset Pricing Model (CAPML). Following (Sharpe, 1999), the SML has analytical description

$$E_i - r_f = \beta_i(E_M - r_f) \tag{1}$$

where E_i is expected return of asset i ;
E_M - expected return of the market;
r_f - risk free rate;
β_i - the well-known beta parameter, defined by $\beta_i = \frac{cov(R_i, R_M)}{\sigma_M^2}$;
σ_M^2 - the variance of the market portfolio;
R_i, R_M are the current returns of asset i and the market.

The difference $E_i - r_f = \prod_i^*$ is the excess implied return of asset i. The paper follows (Fabozzi, Focardi & Kolm, 2006) for the evaluation of $\prod_i^*$. Hence, from (1) the following equation holds

$$\prod_i^* = \beta_i(E_M - r_f) = \frac{cov(R_i, R_M)}{\sigma_M^2}(E_M - r_f)$$

or

$$\prod_i^* = \frac{E_M - r_f}{\sigma_M^2} cov(R_i, R_M). \tag{2}$$

The covariance value $cov(R_i, R_M)$ can be substituted for the analytical relations, derived according to the sequence of substitutions, described as follows.

At the market point, the current market return is equal to

$$R_M = \sum_{j=1}^{N} R_j w_j^*, \tag{3}$$

where R_j are the current returns of assets $j = 1,N$,
w_j^* are the market capitalization weights of asset j.

Respectively, for the expected values E_i and E_M at the market point, it follows the same relation like (3) or

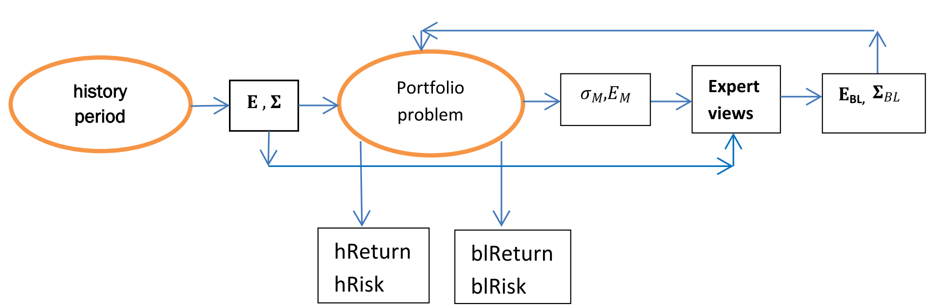

Fig. 1. Graphical presentation of the BL sequence of evaluations.

$$E_M = \sum_{j=1}^{N} E_j w_j^*. \tag{4}$$

The content of the notation $cov(R_i, R_M)$ means

$$cov(R_i, R_M) = \frac{1}{n-1} \sum_{k=1}^{n} (R_i^{(k)} - E_i)(R_M^{(k)} - E_M) \tag{5}$$

where n are the number of data in the historical period.

Substituting (3) and (4) in (5) it follows

$$cov(R_i, R_M) = \frac{1}{n-1} \sum_{k=1}^{n} (R_i^{(k)} - E_i) \sum_{j=1}^{N} (R_j^{(k)} - E_j) w_j^*$$

or

$$cov(R_i, R_M) = \sum_{j=1}^{N} \frac{1}{n-1} \sum_{k=1}^{n} \left(R_i^{(k)} - E_i\right)\left(R_j^{(k)} - E_j\right) w_j^*.$$

The internal sum gives the co variation between assets i and j or

$$cov(R_i, R_M) = \sum_{j=1}^{N} cov(R_i, R_j) w_j^*. \tag{6}$$

Substituting (6) in (2) and applying the notation λ for risk aversion parameter (or market price of risk)

$$\lambda = \frac{E_M - r_f}{\sigma_M^2}, \tag{7}$$

it follows the well-known relation for the implied excess return

$$\prod_i^* = \lambda \sum_{j=1}^{N} cov(R_i, R_j) w_j^*. \tag{8}$$

The value of the parameter λ is the market price of risk. The expected implied return in matrix form is

$$\prod = \prod^* + r_f. \tag{9}$$

For the evaluations, needed for the implementation of BL model, relation (9) is applied in the paper.

The peculiarity of this research is that it uses small amount of securities. This is a constraint to use given values for the market expected return E_M and market risk σ_M^2. These values have to be evaluated for the particular market of the investor, which contains small number of assets.

3.2. Evaluation of the market price of risk λ

This parameter is needed for the evaluation of the excess implied returns according to (2). For that case the values of the market expect return E_M and the market risk (volatility) σ_M^2 have to be evaluated. But the particular case of small number of assets, the market point as reference one is not known. For the case of active portfolio management it is needed for this particular market to be evaluated new market return and volatility. The new market parameters are evaluated according to the next algorithmic steps.

3.2.1. Initial choice of assets.

Investor makes choice about the limited set of assets N, which will participate in the portfolio and will define the particular investor's market. The number of N is determined according to the investor preferences. The experiments in this paper were done with $N = 5$ types of shares of the technological companies Apple, Google, Amazon, Microsoft and Facebook. Because the target of this research is to derive an active management algorithm, this was the reason to have short list of assets, which prevents the usage of global market characteristics. The average returns of these companies have been estimated for a short time period of 6 months. It has been chosen the first 6 months from the beginning of 2018.

3.2.2. Definition of initial portfolio data

As initial data it has been used the daily returns of the company shares. These data are freely available from (URL3) and (URL4). Later these data have been used for the evaluation of the monthly average returns because the experiments have been provided on monthly basis. From these information sources it has been collected also the weights of capitalization for each company. Thus, the initial data for experiments were the average monthly returns and average monthly capitalization weights.

The average monthly data for the first 6 months of 2018 were used for the evaluation of the expected returns E_i, volatilities σ_i, covariance matrix $cov(i,j) i,j = 1,N$, $N = 5$ and the definition of the average capitalization weights $\mathbf{w}^* = \begin{vmatrix} w_1^* \\ \cdots \\ w_N^* \end{vmatrix}$.

3.2.3. Evaluation of the "efficient frontier" of our market.

Using the estimated values E_i, σ_i, $cov(i,j)$ the efficient frontier of our particular market is calculated, according to the constraint optimization problem

$$\min_{w} \quad \left[\delta \mathbf{E}^T \mathrm{w} - (1-\delta) \mathbf{w}^T cov() w\right] \tag{10}$$

$$\mathbf{w}^T 1 = 1, \mathbf{w}^T = (w_1, \cdots, w_N) \geq 0, 1 = \begin{vmatrix} 1 \\ . \\ 1 \end{vmatrix}_{Nx1}.$$

Problem (10) is solved for different values of δ. For values of δ from the set $0 \leq \delta \leq 1$, the repetitive solution of (10) gives points from the "efficient frontier" of the portfolio. As a result the "efficient frontier" is estimated as a set of numerical values.

3.2.4. Evaluation of risk aversion parameter λ

The risk aversion parameter (or market price of risk) is given with relation (7), which insists the market portfolio characteristics to be evaluated. But the limited number of assets makes our particular market to have its own market characteristics, which differ from the available global market indices. For the active portfolio management here it has been used relations, which origin from the CAPM theory. According to CAPM, the market portfolio is the access point of the tangent line towards the "efficient frontier", which passes through the risk free point with coordinates$(0, r_f) \equiv (Risk, Return)$. Using the evaluations from the "efficient frontier" it is found as a set of points in the space$(Risk, Return)$. This research applies an analytical approximation of the "efficient frontier", given as a set of numerical points. Having such analytical description, it is possible additionally to define in analytical form the tangent line, which passes through the risk-free point. This tangent line is the Capital Market Line and the coordinates of the tangent point can be evaluated analytically. The coordinates of the tangent point will give the values of the new market return and volatility of our particular market.

3.2.5. Analytical approximation of the "efficient frontier"

This line can be analytically derived in the space $(Risk, Return)$ for the case of unconstrained optimization problem (10), (Merton, 1972). But for the case of constraint optimization, which takes place in (10), the "efficient frontier" is not a smooth line. In (Calvo, Ivora & Liern, 2012) it is assessed numerically the lack of continuity of the "efficient frontier" for constraint portfolio optimization. It is proved that the "efficient frontier" in that case is a combination of parabolic quadratic curves. Hence, an approximation of the "efficient frontier", given as a set of points, with only one unique quadratic curve will not be precise and accurate. Such an approximation is needed to find analytically the tangent point between the efficient frontier" and the Capital Market Line for our particular market.

In this research the approximation is applied only for a subset of points of the "efficient frontier", which belongs to the prospective area, where the new market point could be situated on the "efficient frontier". This reduction of the size of the "efficient frontier" is made mainly for the initial and final points of the "efficient frontier". This reduction of the set of points of the "efficient frontier" increases the accuracy of the approximation for the area, where the market point belongs. A formal condition for decrease of the points could be the requirement analytically defined as:

if (abs(Return(i + 1)-Return(i)) < ε_R . Δ_{return}), n = n-1; where
n is a number of points, evaluated for the "efficient frontier",

$\Delta_{return} = Return_{max}$ - $Return_{min}$ is the difference between the maximal and minimal values of the returns on the "efficient frontier",

ε_R is a part of the value of Δ_{return} (in this research it has been chosen as 3 %).

The decrease of the number of points of the "efficient frontier" increases the accuracy of the approximation for the part of the "efficient frontier" where the tangent point is expected.

The analytical approximation for this short line of the "efficient frontier" is chosen to be approximated with a quadratic curve in analytical form

$$y = a_2x^2 + a_1x + a_0, \tag{11}$$

where y is the portfolio *Return* and x is the portfolio *Risk* for points of *the* "efficient frontier". The approximation is illustrated in Fig. 2.

The parameters a_0, a_1, a_2 of the approximation curve $y = y(x)$ are evaluated by the usage of the least square method (Kantar, 2015). The unknown coefficients a_0, a_1, a_2 are the solutions of the linear equation system

$$\begin{vmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{vmatrix} \begin{vmatrix} a_o \\ a_1 \\ a_2 \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} y_i x_i \\ \sum_{i=1}^{n} y_i x_i^2 \end{vmatrix}, \tag{12}$$

where the points $(y_i, x_i), i = 1, n$ concern the chosen points from the "efficient frontier" as the couples $(Return, Risk)$; n is the decreased number of points for which analytical approximation is calculated.

Relations (12) allow having a more manual approach for home investors to make their calculations without usage of complex software suit.

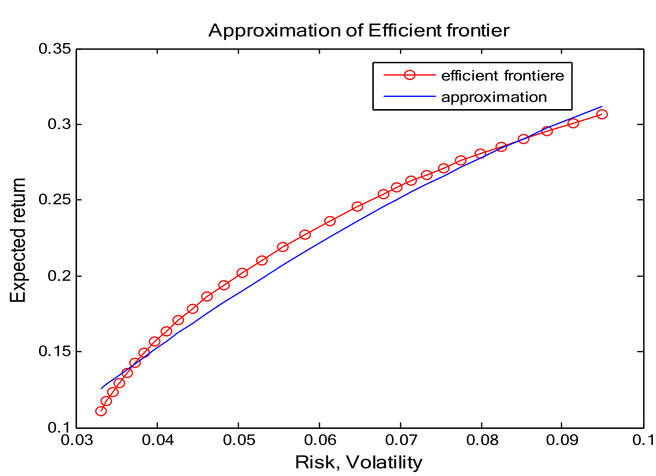

Fig. 2. Approximation of the "efficient frontier".

3.2.6. Evaluation of the new market characteristics(E_M, σ_M^2).

Following the CAPM theory according to (Sharpe 2000), the Capital Market Line (CML) is tangential to the curve $y = y(x)$ and passes through the point $(0, r_f)$ in the space $(Risk, Return)$, Fig. 3.

Because the "efficient frontier" is derived analytically according to relation (11), applying the rules of the analytical geometry, the tangent line towards the unknown new market point (y_M, x_M) has analytical form as

$$t \equiv y - y_M = \frac{dy}{dx}(y_M, x_M)(x - x_M), \tag{13}$$

where $\frac{dy}{dx}(y_M, x_M) = 2a_2x_M + a_1$.

Because the tangent line t has to pass through the point of the risk free asset$(0, r_f)$, the following equation must hold

$$t(0, r_f) \equiv r_f - y_M = 2a_2x_M + a_1(0 - x_M). \tag{14}$$

Using relation (11) for the market point ($y = y_M$, $x = x_M$) and relation (14), both they make a linear system of equations with unknown values of the new market return $y_M = E_M$ and the new market risk (volatility) $x_M = \sigma_M^2$ or

$$t(0, r_f) \equiv r_f - y_M = 2a_2x_M + a_1(0 - x_M) \tag{15}$$

$$y(x) \equiv y_M = a_2{x_M}^2 + a_1x_M + a_0.$$

By adding these two equations, an explicit analytical relation for the new market risk is found

$$x_M = \sqrt{\frac{a_0 - r_f}{a_2}}. \tag{16}$$

The new market return y_M is calculated from relation (11) when x_M is given.

Finally, the risk aversion parameter λ, defined by (7) is evaluated with the values of $(y_M = E_M, x_M = \sigma_M^2)$, which gives $\lambda = \frac{E_M - r_f}{\sigma_M^2}$.

3.2.7. Evaluation of the excess implied returns $\prod^*$. Following (8) the implied excess returns in matrix form are

$$\prod^* = \lambda \Sigma \mathbf{w}^*$$

where the covariance matrix is evaluated from p.3, $\mathbf{cov}(.) = \Sigma = \begin{vmatrix} cov_{11} & \cdots & cov_{1N} \\ \cdots & \cdots & \cdots \\ cov_{N1} & \cdots & cov_{NN} \end{vmatrix}$, N is number of portfolio assets,

$\mathbf{w}^* = \begin{vmatrix} w_1^* \\ \cdots \\ w_N^* \end{vmatrix}$ are the capitalization weights of the assets, available from

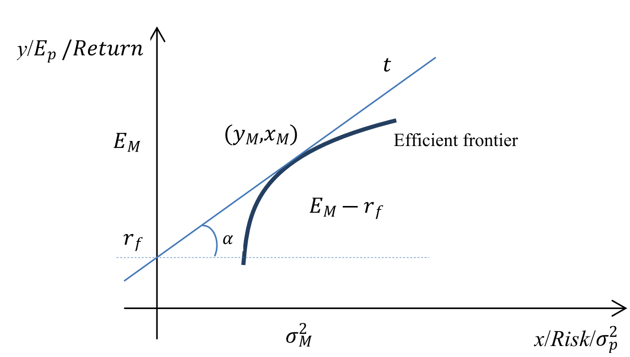

Fig. 3. The market point as the tangent point between CML and the "efficient frontier".

p.3.

Respectively, the implied returns are $\prod = \prod^* + r_f$.

3.3. Application of BL relations

For the sake of complete presentation of the Black-Litterman model, here in brief the main BL relations are derived. The starting point is the assumption that the implied returns $\prod$ must be equal to the BL returns, noted E_{BL}. Because a noise ε influences the real values of the implied returns, the relation between the implied returns $\mathbf{\Pi}$ and the BL returns E_{BL} is

$$\prod = \mathbf{I}\mathbf{E}_{BL} + \varepsilon$$

$\mathbf{I}$ is a diagonal identity matrix with dimensions $N \times N$, ε is a noise with normal distribution and zero mean and volatility, proportional to the covariance matrix. It is accepted the notation for the covariance to be the symbol $\mathbf{\Sigma}$. The proportional coefficient $\tau < 1$ is one of the parameter in BL model, which is intensively discussed about its consistent estimation, (Allaj, 2013). Value of τ less than 1 means that the volatility of the noise ε has less power than the volatility of the asset returns, $\varepsilon\ N(0, \tau\mathbf{\Sigma})$.

The main feature of the BL model is the integration of data about the historical estimated mean returns E_i of the assets with a set of the experts, subjective views about the return values for future time. Analytically, the subjective views are formally described as

$$\mathbf{Q} = \mathbf{P}\boldsymbol{E}_{BL} + \eta,$$

where $\boldsymbol{Q}$ is a vector of the subjective view about the change of the values of returns, $\mathbf{P}$ is a matrix, defining which assets are under subjective corrections, η is the noise for the expert views, assumed to be with normal distribution, zero mean and Ω volatility, $\eta\ N(0, \Omega)$.The matrix $\mathbf{\Omega}$ has meaning of confidence about the subjective views. The problem for integration of historical data about the asset returns with these expert views is analytically presented by a linear stochastic system

$$\prod = \mathbf{I}\mathbf{E}_{BL} + \varepsilon$$

$$\mathbf{Q} = \mathbf{P}\mathbf{E}_{BL} + \eta \tag{17}$$

The subjective experts' views are defined analytically with the matrices $\mathbf{P}$and $\mathbf{Q}$. This research derives particular relations for the definitions of these matrices. It is based on comparisons between the evaluated implied asset returns and the historically defined mean asset returns. The analytical relations for the definition of the new expert views are given in p.3D.

The solution of (17) gives the unknown returns $\mathbf{E}_{\mathrm{BL}}$, which will give the closest approximation of (17) by means to minimize the influence of the volatilities of the stochastic variables ε and η. To simplify the solution of (17) matrix notations are applied

$$\mathbf{Y} = \mathbf{X}\mathbf{E}_{BL} + \psi \tag{18}$$

where $\mathbf{Y} = \begin{vmatrix} \prod \\ \mathbf{Q} \end{vmatrix}, \mathbf{X} = \begin{vmatrix} \mathbf{I} \\ \mathbf{P} \end{vmatrix}, \psi = \begin{vmatrix} \varepsilon \\ \eta \end{vmatrix}, \overline{\psi} = \begin{vmatrix} \tau\Sigma & 0 \\ 0 & \Omega \end{vmatrix}$

The linear regression relation is solved by means to estimate the values of $\mathbf{E}_{BL}$, which will minimize the influence of the noise $\boldsymbol{\psi}$. The solution of this linear regression equation is found, applying the Mahalonobis distance (used in general least square method). The problem for the application of the general least square method to (18) is described analytically by the unconstrained optimization problem

$$\mathbf{E}_{BL}^{min} \equiv arg\left\{ \begin{matrix} min \\ \mathbf{E}_{BL} \end{matrix} \quad (\mathbf{Y} - \mathbf{X}\mathbf{E}_{BL})^T \overline{\psi}^{-1} (\mathbf{Y} - \mathbf{X}\mathbf{E}_{BL}) \right\} \tag{19}$$

Due to the unconstrained form of (19) the analytical solution can be derived

$$\mathbf{E}_{BL}^{min} = (\mathbf{X}^T\overline{\psi}^{-1}\mathbf{X})^{-1}\mathbf{X}^T\overline{\psi^{-1}}\mathbf{Y}.$$

Substituting $\boldsymbol{Y}, \boldsymbol{X}, \overline{\psi}$ with their initial content from (18) it follows the main relation in BL model.

$$\mathbf{E}_{BL} = \left[(\tau\Sigma)^{-1} + \mathbf{P}^T\Omega^{-1}\mathbf{P}\right]^{-1}\left[(\tau\Sigma)^{-1}\prod + \mathbf{P}^T\Omega^{-1}\mathbf{Q}\right] \tag{20}$$

The amendment to the covariance matrix, due to $\mathbf{E}_{BL}$ is defined as

$$Vol(\mathbf{E}_{BL}) = \Delta_{BL} = \left[(\tau\Sigma)^{-1} + \mathbf{P}^T\Omega^{-1}\mathbf{P}\right]^{-1}. \tag{21}$$

The final BL modified covariance matrix, due to relation (18) is

$$\Sigma_{BL} = \Sigma + \Delta_{BL}. \tag{22}$$

3.4. Derivation of formal relations of modified expert views

The problem how to generate expert views is explored in many researches. We are not going to illustrate the well-known absolute and relative ways of definition of the parameters of the expert views according to analysts' forecasts. But this problem is tightly connected with the need to forecast the behavior, respectively the dynamics of the assets returns in the future.

According to the Dow theory, the analyst can follow the overall market and can identify the directions of the individual stocks. Additionally, the Elliott Theory claims that the trades in the market have repetitive patterns. This behavior of the market gives ground for the attempts to forecast future level of asset returns (Batyrbekova, 2015; Mikaelian, 2013). In (Didenko & Demicheva, 2013) it has been proposed a procedure for group decision-making by application of Ensemble Learning algorithm, which clams that it can be achieved accurate of asset returns. This procedure in explicit way makes usage of subjective assessments about the future level of asset returns. In (Geyer & Lucivjanská, 2016; Geyer, Lucivjanská & Atlas, 2017) a new approach for the definition of the views and their uncertainty are derived. It is used a predictive regressions estimated in a Bayesian framework. The subjective element concerns the investors' belief about the degree of predictability of the regression. The uncertainty of the views is derived from the Bayesian regression, rather than by using the covariance of returns. In (Becker & Gürtler, 2010) the views are quantitatively defined based on dividend discount model. They claim that such forecasts can be made using other valuation models (discounted cash flow model, residual income model). In (Beach & Orlov, 2007) it has been used GARCH-derived views as an input into the Black–Litterman model. In (Kara, Ulucan & Atici, 2019) the GARCH modeling is used to forecast technical indicators of the assets, which then are used for the estimation of the values of the future assets returns. In (Becker & Gürtler, 2010) analysts' dividend forecasts are used for determination of a-priori-estimation of the expected returns. Additionally, confidences of the investors' views are determined from the number of analysts' forecasts as well as from a Monte- Carlo simulation.

Attempts for usage of Neural networks for the approximation of the asset returns trends and for the case of expert forecasts are made in (Vena, 2018). Nevertheless of the wide range of models, applied for the definition of expert views for the Black-Litterman portfolio model, these models insist considerable amount of data and large time for calculations. This restricts their applications for real time active portfolio management.

This research derives a simple procedure for the evaluation of the expert views, which allows fast application with no many input data for active portfolio management. The idea behind this new form of views' formalization is based on a comparison between the evaluated implied asset returns and their mean values defined from the historical trend of the asset returns, (Vladimirov, Stoilov & Stoilova, 2017). The values of the expert views are based formally on the estimation of the difference between the average historical returns hE_i and the corresponding implied returns $\prod_i, i = 1, N$. The assessment about the future increase or decrease of the asset returns are based on the relations:

- $if(\nabla_i = \frac{\prod_i - hE_i}{\sigma_i^2} > 0)$, security i is underestimated and it is expecting its return to increase.
- $if(\nabla_j = \frac{\prod_j - hE_j}{\sigma_j^2} < 0)$, security j is overestimated and it is expecting its return to decrease.

Finding this index i^*, which corresponds to the maximal value of∇_i

$$i^* \equiv \max_i \left(\nabla_i = \frac{\prod_i - hE_i}{\sigma_i^2} \right), i = 1, N$$

this security i^* will participate in the view matrix **P** for increase.

Respectively, security j^*, which corresponds to the minimum value of∇_j

$$j^* \equiv \min_j \left(\nabla_j = \frac{\prod_j - hE_j}{\sigma_j^2} \right), j = 1, N$$

This security j^* will participate in the view matrix **P** for decrease.

The components in the matrix **P** are calculated in normalization way as

$$P(i^*) = \frac{\max(\nabla_{i^*})}{|\max(\nabla_{i^*})| + |\min(\nabla_{j^*})|} P(j^*) = \frac{\min(\nabla_{j^*})}{|\max(\nabla_{i^*})| + |\min(\nabla_{j^*})|}. \quad (23)$$

These new weighted forms of relative views are based on additional assessment of the historical data and the evaluations of the implied asset returns, used for the portfolio management. Relations (25) give no binary values for the components of matrix **P**, which does not correspond to the classical case, (Satchell & Scowcroft, 2000; He & Litterman, 2002). The two no binary values of the components of P have meaning of asset risk of a virtual portfolio where the assets have equal weights (Vladimirov, Stoilov & Stoilova 2017). For the numerical experiments in paragraph 5, this form of expert views is noted as P(α). To be compared this weighed expert policy with a classical BL one, additional policy, noted as P(1) is applied. In P(1) matrix the corresponding elements i^* and j^* of matrix **P** has values 1 and −1 according to the relations:

$$if(P(i^*) > 0) then P(i^*) = 1, if(P(\alpha j^*) < 0) then P(j^*) = -1.$$

The next parameters of the expert views **Q** and **Ω** are chosen in this research as

$$q_i = \max_i \left(\prod_i - hE_i\right), Q = (q_1, \cdots, q_N)$$

$$\Omega = \tau diag\left(P\Sigma P^T\right).$$

As a result, these new forms of expert views allow to be compared on common bases the results of MV portfolio problem and these ones, based on the modified BL problem. Both problems use common historical data and the comparison of their solutions will give objective assessment about the usefulness of their results.

The value of τ is recommended to be linked with the number of available data from the historical period, used for the average calculations of the mean asset returns, $\tau = \frac{1}{n_t}$, where the notation n_t is equal to the number of points in the historical period.

3.5. Solving a portfolio problem with BL data

Having the new estimates about the returns $\mathbf{E}_{BL}$ and covariance matrix Σ_{BL} the Mean Variance Portfolio problem with the BL data is defined and solved. It will give the weights of the assets in the new BL portfolio

$$\min_{\mathbf{w}} \left(\mathbf{w}^T \mathbf{E}_{BL} - \lambda^* \mathbf{w}^T \Sigma_{BL} \mathbf{w}\right)$$

$$\mathbf{w}^T 1 = 1, \mathbf{w} \geq 0. \quad (24)$$

The portfolio optimization, problem (24) is solved by changing accordingly the risk aversion parameter λ^* from the set $0 \leq \lambda^* \leq \infty$. The repetitive solution of (24) with different values of λ^* will give points from the "efficient frontier" for the BL model.

4. Application of the algorithm for active short term portfolio management

The active portfolio management has been implemented in sequential, rolling procedure following Fig. 3. The monthly average returns of the securities for the first 11 months of 2018 were used for the chosen five companies' shares. Three types of active managements are illustrated graphically in Fig. 4.

The idea of the experiments is to use the portfolio solutions, evaluated with data of the previous 6 months and the portfolio optimization to apply these solutions for a future time for 1, 2 and 3 months ahead. Then the history period is moving ahead, respectively with 1, 2 or 3 months and the next portfolio optimization is performed for the next future 1, 2, or 3 months. Thus, a rolling procedure of implementation of the active portfolio algorithm is applied. The portfolio weights, which are invested, are always evaluated for the last 6 months, but their implementation is optionally implemented for a time cycle of 1, 2 or 3 months ahead. Fig. 4 graphically explains the procedure of active management with different future time cycles.

The paper illustrates the calculations for one iteration of the active portfolio management with time cycle of one month investment ahead.

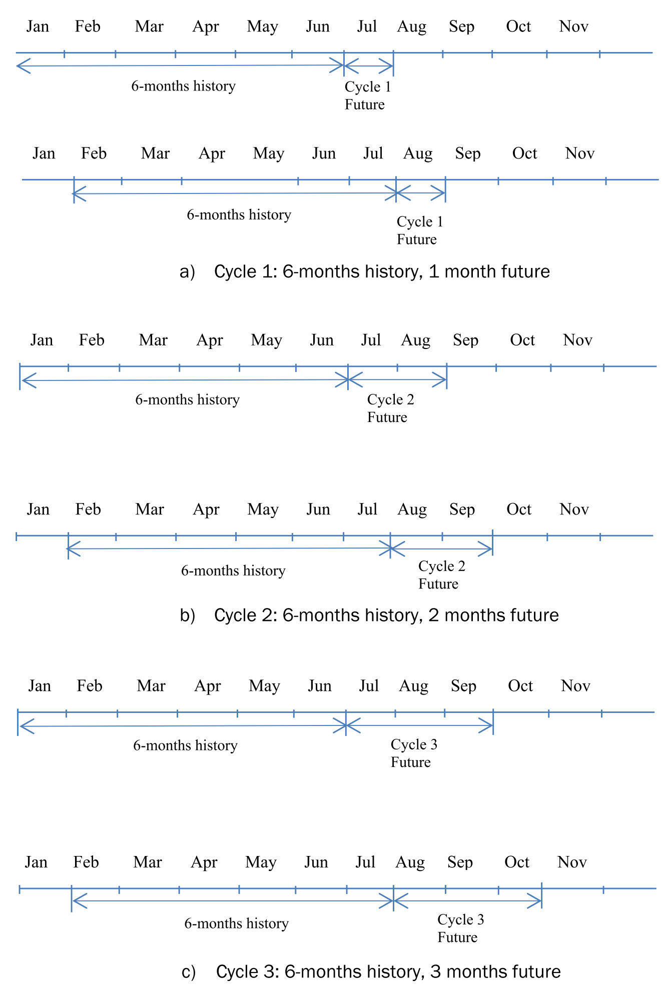

Fig. 4. Active portfolio management with rolling procedures.

For that case the parameter of rolling time horizon is cycle $= 1$. The notation (**h**) means that the evaluations are performed according to MV model and only historical data are applied for the portfolio active management. Respectively, the notation **(bl)** defines the application of BL model and additional views, according to (23) are used to modify the historical data.

4.1. Case h: Evaluation of the weights $\mathbf{hw}^{opt}$ with historical 6 months data.

1 h. Using the monthly average data for the first 6 months of 2018 (from January till June) the average historical returns **hReturn = hE** and **hCOVAR = hΣ** are calculated. The dimension of vectors **hReturn** is 5x1, while **hCOVAR** is a matrix 5x5.

2 h. With the historical data **hE** and **hΣ** a portfolio optimization problem is solved

$$\min_{\mathbf{hw}} \left[\delta \mathbf{hE}^{\mathrm{T}}\mathbf{hw} - (1-\delta)\mathbf{hw}^{T}\mathbf{h\Sigma h}\boldsymbol{w}\right]$$

$$\mathbf{hw}^{T}1 = 1, \mathbf{hw}^{\mathrm{T}} = (\mathrm{hw}_1, \cdots, \mathrm{hw}_{\mathrm{N5}}) \geq 0, 1 = \begin{vmatrix} 1 \\ . \\ 1 \end{vmatrix}_{5x1}$$

The parameter $\delta = 0 : 0.01 : 1$ takes value from the diapason [0,1] with a step of 0.01. Hence, 101 points from the "efficient frontier" are numerically found, which correspond to the different solutions of the portfolio problem. Thus, with historical data (**h**) the "efficient frontier" is numerically defined.

3 h. The portfolio (and the corresponding weights), which is chosen from the "efficient frontier" for comparison with the BL model is this one, which has a maximal value of the modified Sharpe ratio

$$\mathbf{hw}^{opt} \equiv max\left[\frac{hReturn}{hRisk}\right] = max\frac{\mathbf{hw}^{T}\mathbf{hE}}{\mathbf{hw}^{T}\mathbf{h\Sigma hw}}.$$

By calculation of the modified Sharpe ratio for each point of the "efficient frontier", the portfolio with maximal ratio defines the weights $\mathbf{hw}^{opt}$. These weights are applied for active management for the next future period. Particularly, for cycle with 1 month ahead, the weights $\mathbf{hw}^{opt}$ are used for the investment for the future (**f**) month of July 2018. For the future month, this set of assets will have average returns **fE**. Respectively, using these data from July, the future actual values of the covariance matrix fCOVAR = **fΣ** are calculated but using the new 6 months' time period from February till July, skipping January. The active portfolio management will apply in July the evaluated weights $\mathbf{hw}^{opt}$, according to the 6 months historical data. But in the end of July the portfolio Return and Risk will be evaluated with the real data **fE** and **fΣ** of July

$$hfReturn = \mathbf{fE}^{T}\mathbf{hw}^{\mathbf{opt}}, hfRisk = \mathbf{hw}^{optT}\mathbf{f\Sigma hw}^{opt}. \tag{25}$$

These values define the future portfolio return and risk, which the portfolio will have if the weights $\mathbf{hw}^{opt}$, evaluated by classical mean–variance portfolio model, will be applied in July for investment. By rolling this procedure of cycle = 1 month till the end of available data of month of November 2018 this active management policy will provide 5 couples of data for portfolio Risk and Return for the months from July till November.

4.2. Case bl: Evaluation of the weights $\mathbf{blw}^{opt}$.

1bl. From p.2 h, the "efficient frontier" based on historical data is available in numerical form as a set of 101 points in the space Risk/Return.
2bl. Analytical approximation of the "efficient frontier"

An approximation of the "efficient frontier" is derived as a quadratic curve, applying least-square method. Respectively, the coefficients a_0, a_1, a_2 are evaluated solving the linear equation system (12).

3bl. Evaluation of the new market point on the "efficient frontier". The system (15) is solved. Relation (14) gives the risk (volatility)σ_M^2 of the new market. The new market return E_M is calculated from the first equation of (15).
4bl. Calculation of the risk aversion coefficient λ using relation (7).
5bl. Evaluation of the implied excess returns $\prod^{*}$ using relation (8). Respectively, the implied return **Π** is found according to (9).
6bl. Evaluation of the modified expert views

Evaluation of the differences ∇_i between the average historical returns hE_i and the corresponding implied returns $\prod_i$, $i = 1, N$and identification of the most underestimated $i*$ and overestimated j^{*}securities according to the relations

$$i^{*} \equiv \max_{i}\left(\nabla_i = \frac{\prod_i - hE_i}{\sigma_i^2}\right), i = 1, N \text{and} j^{*} \equiv \min_{j}\left(\nabla_j = \frac{\prod_j - hE_j}{\sigma_j^2}\right), j = 1, N$$

The components in the matrix **P** are calculated in normalization way from (23)

$$P(i^{*}) = \frac{\max(\nabla_{i^{*}})}{|\max(\nabla_{i^{*}})| + |\min(\nabla_{j^{*}})|}, P(j^{*}) = \frac{\min(\nabla_{j^{*}})}{|\max(\nabla_{i^{*}})| + |\min(\nabla_{j^{*}})|}$$

The second parameters of the expert views **Q** and **Ω** are chosen as

$$q_i = \max_{i}\left(\prod_i - hE_i\right), Q = (q_1, \cdots, q_N)$$

$$\Omega = \tau diag\left(P - \Sigma P^{T}\right)$$

The value of τ in this research is assumed to be 0.5.

7bl. Having the BL model parameters **Π**, **P**, **Q**, τ the corresponding BL returns $\mathbf{E}_{BL}$ and covariance matrix $\Sigma_{BL}$are calculated from (20)-(21).
8bl. The BL parameters for returns $\mathbf{E}_{BL}$ and covariance matrix Σ_{BL} define a new "BL efficient frontier". Numerically points of the "BL efficient frontier" are calculated by solving the constrained optimization problem

$$\min_{blw} \left[\delta \mathbf{E}_{BL}^{T}\mathbf{blw} - (1-\delta)\mathbf{blw}^{T}\Sigma_{BL}\mathbf{blw}\right]$$

$$\mathbf{blw}^{T}1 = 1, \mathbf{blw}^{T} = (blw_1, \cdots, blw_5) \geq 0, 1 = \begin{vmatrix} 1 \\ . \\ 1 \end{vmatrix}_{5x1}$$

9bl. The portfolio (and the corresponding weights), which is chosen from the "efficient frontier" for comparison with the classical mean variance optimization is this one, which has maximal value of the modified Sharpe ratio

$$\mathbf{blw}^{opt} \equiv max\left[\frac{blReturn}{blRisk}\right] = max\frac{\mathbf{blw}^{T}\mathbf{E}_{BL}}{\mathbf{blw}^{T}\Sigma_{BL}\mathbf{blw}}$$

10bl. These weights$\mathbf{blw}^{opt}$ are used to make an investment for the beginning of July. Following the rules of calculation like part 3 h and relations (24) the future portfolio return and risk are calculated.

$$blfReturn = \mathbf{fE}^{T}\mathbf{blw}^{opt}, blfRisk = \mathbf{blw}^{optT}\mathbf{f\Sigma blw}^{opt}. \tag{26}$$

The values from (25) and (26) are compared for assessment the efficiency of the classical mean–variance and BL models both used for active investment management.

Applying rolling evaluations with a cycle of one month, 5 couples of data for Risk and Return are found. The comparison of this data for the both portfolio models, MV and BL, for this rolling investment procedure is done by averaging the data for the Risk and Returns. When increasing the cycle to 2 and 3 months ahead, the rolling procedure gives less data, respectively 4 and 3 couples for portfolio Risks and Returns. The algorithm for active portfolio management is presented graphically in Fig. 5.

5. Numerical example

The average monthly returns of the securities of 5 technological companies are the initial data, used for the experiments. To have the opportunity for checks and understanding the evaluation steps of the algorithm, it is given a set of intermediate calculations, which will illustrate the sequence of decision making and assessment of the active portfolio management. Thus, the reader can follow and check the derived algorithm and evaluation steps. The numerical example illustrates the calculations only for one iteration of the rolling procedure in the repetitive application of the active management algorithm. The case of cycle = 1 month is considered. The calculations are performed in MATLAB environment.

1. The average monthly returns for January-November 2018 are summarized in matrix [Apple; Google; Amazon; Microsoft; Facebook].
APL = [−0.137829911; 0.358641418; −0.197236016; −0.029712067; 0.485694503; −0.133681429; **0.08684116**; 0.559735222; −0.05384593; −0.076580026; −0.755593892];
GOOG = [0.473629103; −0.274784288; −0.156162942; 0.067457906; 0.221241692; −0.01144791; **0.394652755**; −0.001962958; −0.011595778; −0.448829741; 0.198489214];
AMAZ = [1.00478627; 0.493180968; −0.135007142; 0.687531385; 0.1428834; 0.18257778; **0.191306493**; 0.521574799; −0.088704957; −0.734027637; 0.124664934];
MICR = [0.507669081; −0.000292723; −0.057113674; 0.287911686; 0.192016186; −0.10494842; **0.303632964**; 0.255103499; 0.136785325; −0.347533108; 0.251277339];
FACE = [0.160460385; −0.413620219; −0.447634391; 0.541461231; 0.473397345; 0.016162931; **−0.557679309**; 0.117834282; −0.209550961; −0.266373758; −0.328331338];

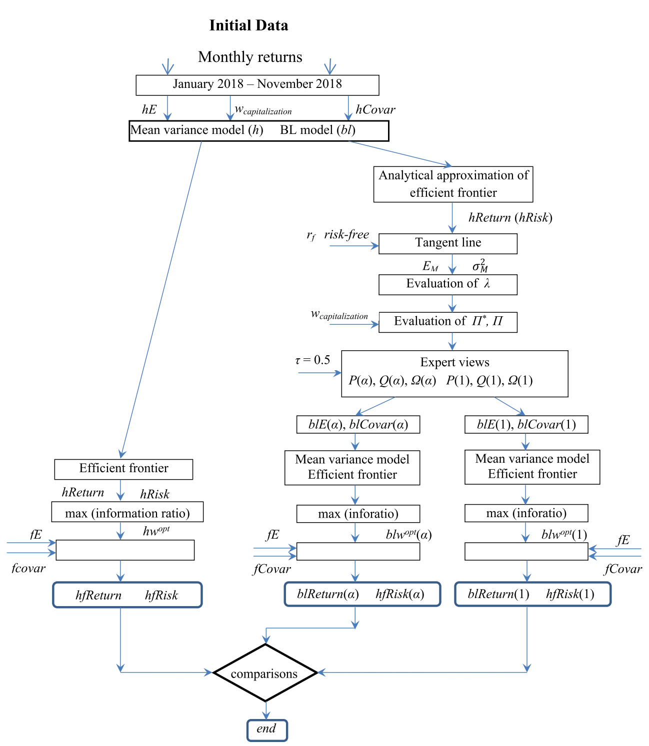

Fig. 5. Algorithm for active portfolio management and comparison of MV and BL models.

For a cycle = 1 month future period of investment, the first 6 months are considered as historical data. The month 7 is assumed as future time, when the solutions of the algorithm will provide active portfolio management. With the historical data, the average returns **E** and the covariance matrix Σ are evaluated as

$\mathbf{E}^{\mathbf{T}}$ = [0.0576 0.0533 0.3960 0.1375 0.0550],
Σ = [0.0842–0.0089–0.0095–0.0015 0.0221
−0.0089 0.0722 0.0621 0.0537 0.0766
−0.0095 0.0621 0.1713 0.0794 0.0660
−0.0015 0.0537 0.0794 0.0557 0.0614
0.0221 0.0766 0.0660 0.0614 0.1793].

2. Evaluation of 101 points from the "efficient frontier" by changing $\delta = 0 : 0.01 : 1$ for problem (10). The graphical presentation of the "efficient frontier" is given on Fig. 6.
3. The maximum value of the modified Sharpe ratio is 4.0196 and the corresponding portfolio characteristics are: *max_hReturn* = 0.1861, *max_hRisk* = 0,0463. The weights of this portfolio are $\mathbf{hw}^{optT}$ = [0.3991 0 0.3111 0.2898 0].
4. Analytical approximation of the "efficient frontier". The analytical approximation is precise if fewer points are used in evaluation of the parameters a_0, a_1, a_2 from (12). To increase the accuracy of the approximation it is worth to drop points of the "efficient frontier" from its initial and ending set of values. The initial values, which are lower from the risk free level, are not in importance for the approximation. Same - for the last points for which the increase in portfolio Risk provides very few increase of the portfolio Return.

a. decrease the initial point from the "efficient frontier". The values of the portfolio return, which are lower to the risk free level are dropped from the approximation set, (*hReturn* < r_f). It was used $r_f = 0.1$, according the data of the international markets. According to this decrease, the initial point of the "efficient frontier", which is used for the approximation, is the 7th one.

b. decrease a set of points from the end of the "efficient frontier". This decrease is done because the points of the efficient frontier with high risk do not change considerably the portfolio return. The decrease of points from the "efficient frontier" is performed according to the comparison *hReturn*(*j*)-*hReturn*(*j*-1)≤ ϵ, ϵ = 4*10^{-5}- is a small value. For the current case the last point here is $j = 53$.

Hence, for this initial set of calculations, the "efficient frontier" is approximated not for the full range of 101 points but for less points – 46

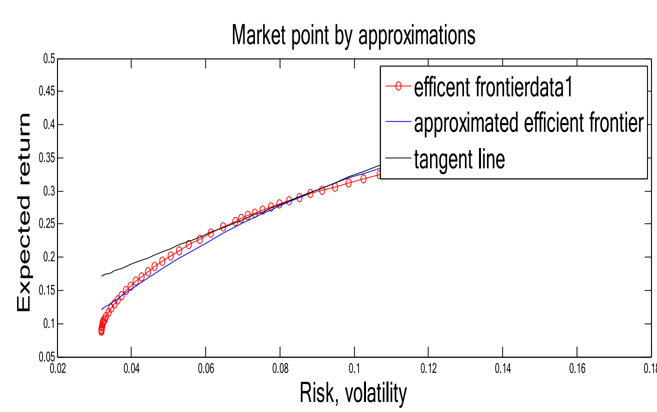

Fig. 6. Analytical approximation of the efficient frontier.

with indices from 7 to 53. This reduction provides close approximation of the "efficient frontier" for the part where the portfolio with maximal information ratio belongs.

5. Analytical approximation of the "efficient frontier" according to (11)-(12). The coefficients of the quadratic approximation are: $a_0 =$ -15.68, $a_1 = 5.02$, $a_2 =$ -0.02. On Fig. 6 it is illustrated the "efficient frontier" (in red), its approximation (in blue) and the tangent line - Capital Market Line (CML) (in black).

The tangent point of CML and the approximated "efficient frontier" give the new market characteristics *hReturn_market* $= 0.2988$, *hRisk_market* $= 0.0888$. These values define the risk aversion coefficient, according to (7) as $\lambda = 2.2385$.

6. Evaluation of the implied excess returns $\mathbf{\Pi}^*$ and the implied returns $\mathbf{\Pi}$ with $r_f = 0.1$, according to (8) and (9)

$\mathbf{\Pi}^{*T} = [0.0746\ 0.0898\ 0.1129\ 0.0882\ 0.1846]$
$\mathbf{\Pi}^{T} = [0.1746\ 0.1898\ 0.2129\ 0.1882\ 0.2846]$.

7. Definition of the expert views:
a. evaluation of the weighted matrix $\mathbf{P}(\alpha)$. For the current case $\mathbf{P}$ is a row vector with dimension 1x5
b. evaluation of $\nabla_i = \frac{\prod_i - hE_i}{\sigma_i^2}$, $i = 1{,}5$ which gives

$\nabla^{T} = [0.4028\ 0.5081 – 0.4423\ 0.2149\ 0.5421]$,

a. the maximal value $\max(\nabla_i) = 0.5421$, $i = 5$,
b. the minimal value $\max(\nabla_j) = -0.4423$, $j = 3$,
c. following (22), the vector $\mathbf{P}(\alpha)^{T} = [0\ 0 – 0.4493\ 0\ 0.5507]$,
d. for comparison reasons, the evaluation has been done also with the classical definition of matrix $\mathbf{P}$. The notation used was defined as P (1) . For the current numerical case $\mathbf{P}(1) = [0\ 0 – 1\ 0\ 1]$.
e. the corresponding value of $Q = 0.2296$. With $\tau = 0.5$ the value of $\Omega = 0.0282$.

8. Evaluation of the BL returns, and risk following (17) and (19)

$\mathbf{E}_{\mathbf{BL}}{}^{T} = [\ 0.1991\ 0.2112\ 0.1521\ 0.1854\ 0.3880\ ]$
$\Delta_{BL} = [\ 0.1252 – 0.0144 – 0.0113 – 0.0022\ 0.0281;$
$-0.0144\ 0.1073\ 0.0958\ 0.0807\ 0.1105;$
$-0.0113\ 0.0958\ 0.2497\ 0.1188\ 0.1114;$
$-0.0022\ 0.0807\ 0.1188\ 0.0835\ 0.0926;$
$0.0281\ 0.1105\ 0.1114\ 0.0925\ 0.2478]$.

9. Evaluation the corresponding new "BL efficient frontier" with the BL data $\mathbf{E}_{\mathrm{BL}}$ and Δ_{BL}. Applying the computational rules, defined in **p.8bl** the new "BL efficient frontier" is numerically defined as a set of 101 points, Fig. 7.

From this set of points this one with maximal modified Sharpe ratio is chosen. For the current state of calculations the maximal information ratio is 4.1726 and the characteristics of this portfolio are: *max_blReturns* $= 0.1998$, *max_blRisk* $= 0.0479$ and the weights of this portfolio are

$\boldsymbol{blw}^{optT} = [0.4244..0.3334..0\ 0.2422..0]$

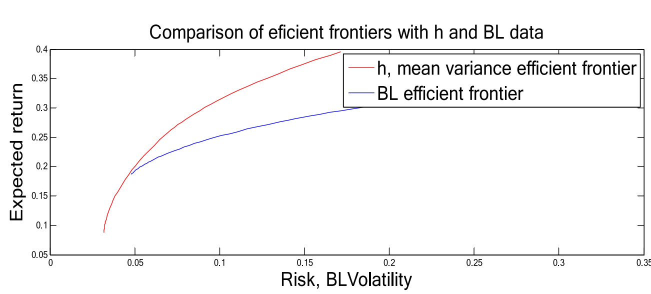

Fig. 7. Efficient frontiers with *h* and *bl* data.

The graphical comparison of the both "efficient frontiers", with historical data (in red) and with the evaluated BL data (in blue) is given in Fig. 7.

10. Application of the evaluated weights $\mathbf{hw}^{opt}$ and $\mathbf{blw}^{opt}$ for the active portfolio management. The weights $\mathbf{hw}^{opt}$ and $\mathbf{blw}^{opt}$ are evaluated with data available for the first 6 months of 2018. The active portfolio management concerns the application of these weights for the future month of July 2018. In the end of July, when its average return and risk will be estimated the comparison between the two portfolios will provide results about the efficiency of each investment policy: the classical Mean Variance model (*h*) or the BL model (*bl*).
a. estimation of the average asset returns for July. Using the data from p.1 (marked bold in the set of initial historical data), these values are

$\mathbf{fE}^{T} = [\ 0.0868\ 0.3947\ 0.1913\ 0.3036 – 0.5577]$

b. evaluation of the new covariance matrix with data from February to July. The covariance matrix is calculated with historical data for 6 months. But these months are from February to July. Thus, a rolling procedure for the calculations is implemented. The new values of the covariance matrix are:
$\mathrm{f}\Sigma = [\ 0.0703 – 0.0060 – 0.0088 – 0.0006\ 0.0159;$
$-0.0060\ 0.0768\ 0.0418\ 0.0529\ 0.0340;$
$-0.0088\ 0.0418\ 0.1488\ 0.0613\ 0.0729;$
$-0.0006\ 0.0529\ 0.0613\ 0.0503\ 0.0366;$
$0.0159\ 0.0340\ 0.0729\ 0.0366\ 0.2031]$.

Using the evaluated weights $\mathbf{hw}^{opt}$ and $\mathbf{blw}^{opt}$ and the portfolio conditions for July according to $\mathbf{fE}$ and $\mathbf{f\Sigma}$, the generated characteristics of the portfolios are:

hfReturn $= \mathbf{fE}^{T}\,\mathbf{hw}^{opt} = 0.1822$ *hfRisk* $= \mathbf{hw}^{optT}\,\mathbf{f\Sigma}\,\mathbf{hw}^{opt} = 0.0386$
blfReturn $= \mathbf{fE}^{T}\,\mathbf{blw}^{opt} = 0.2420$ *blfRisk* $= \mathbf{blw}^{optT}\,\mathbf{f\Sigma}\,\mathbf{blw}^{opt} = 0.0309$.

Applying rolling application of this algorithm following Fig. 3, it has been found the characteristics of 5 portfolios for the months from July till November 2018. For the comparison of the results of these 5 portfolios it has been applied mean values for their Return and Risk results, which gives:

mean (*hfReturn*) = -0.0514 **mean**(*hfRisk*) = 0.0503
mean(bl*fReturn*) = 0.0359 **mean**(bl*fRisk*) = 0.0354 .

An extended illustration of the results with different cycles of 1, 2 and 3 months are given in Table 2. The BL model has been evaluated with both forms of views $P(\alpha)$ and $P(1)$.

The results for cycle = 1 month are presented graphically in Fig. 8.

The five portfolios, resulting from the Mean-Variance model (*h*) are in red and the portfolio from BL model (*bl*) is in blue. The BL based portfolios are situated in Nord-West direction in the Risk/Return plane. This Nord-West rule is generally applied for assessment the efficiency of the portfolios, because in this direction the Return is higher and the Risk is lower of the assessed portfolios.

The average assessment inside Fig. 8 gives evidences that the BL model, applying weighted view vector $P(\alpha)$ gives better result in comparison with the classical Mean-Variance model, because the mean Return is higher and the mean Risk is lower. The same advantage for the expert views of type $P(\alpha)$ gives more benefit for the portfolio returns in

Table 2
Comparison between Mean-Variance (*h*), BL portfolios (*bl*) with P(α) and P(1) views.

cycles	hfReturn	blReturn		hfRisk	blRisk	
		P(α)	P(1)		P(α)	P(1)
1 month	0.1822	0.2420	0.2384	0.0386	0.0309	0.0302
	0.3594	0.2963	0.2914	0.0318	0.0309	0.0306
	0.0003	0.0125	0.0001	0.0295	0.0262	0.0264
	−0.4504	−0.3763	−0.3623	0.0667	0.0499	0.0471
	−0.3484	0.0055	−0.0310	0.0850	0.0392	0.0412
mean	**−0.0514**	**0.0359**	**0.0273**	**0.0503**	**0.0354**	**0.0351**
2 months	0.3209	0.2703	0.2749	0.0425	0.0314	0.0313
	0.1712	0.1469	0.1472	0.0323	0.0285	0.0279
	−0.2585	−0.1754	−0.1826	0.0791	0.0479	0.0477
	−0.1386	−0.1538	−0.1505	0.0584	0.0428	0.0404
mean	**0.0237**	**0.0220**	**0.0222**	**0.0531**	**0.0377**	**0.0368**
3 months	0.2108	0.1824	0.1888	0.0430	0.0290	0.0288
	−0.0492	−0.0224	−0.0176	0.0751	0.0495	0.0473
	−0.1112	−0.0832	−0.0944	0.0710	0.0425	0.0421
mean	**0.0168**	**0.0256**	**0.0256**	**0.0630**	**0.0403**	**0.0394**

comparison with the classical form of views of $P(1)$.

By increasing the cycle to 2 and 3 months, the comparisons of these cases also give advantages of the derived algorithm. But the differences between the cases $P(\alpha)$ and $P(1)$ about the portfolio returns are not so significant. An explanation about this case is because the portfolio weights are evaluated according to the past 6 months, but they are applied for 2 or 3 months ahead, without change. This results in usage of less actual data for the portfolio management. If the weights are evaluated and applied for the near future, as the case cycle = 1 month, the suggested algorithm gives better results for the portfolio management with the case of expert views $P(\alpha)$.

The presented algorithm has been applied on additional examples by means empirically to check its behavior. Comparisons are done with the classical MV portfolio solutions and these ones, resulting from the modified BL application. The experiments were performed for three types of portfolios with data of the Bulgarian Stock Exchange: with shares of four industrial companies (N = 4); with seven mutual funds and with two types of real estates. The market data used for the definition of the portfolio problems were from a period of severe stagnation of the stock market. The practical problem was to find a sector to recommend investments there. The obtained results are numerically presented on Table 3. It is evident that investments in industrial shares and mutual funds are not profitable and the portfolio returns are negative. But the case of the modified BL modeling gives advantages in comparison with the MV problem, because the losses (negative values of returns) are lower. Respectively, the portfolio risks with MV problems are higher. The example with the real estate gave positive returns and that was a recommendation for the investors to invest on this market. In this example the application of the modified BL model is superior for the MV portfolio by higher return and less portfolio risk. These results have to be assumed as empirical ones but not as theoretical proves for the power of the algorithm, based on the modified BL modeling.

6. Conclusions

The paper develops and tests an algorithm, based on modified BL model for implementation for active management of portfolios. The modification concerns new form of the expert views. The expert views are defined by comparison of implied and mean asset returns. Thus, only historical data about the asset returns are used. Such new formalization of the expert views allows to be compared on a common basis the results of the portfolio problems based on MV classical model and the modified BL one. The active management is regarded in this case, as implementation of portfolio decision for 1, 2 or 3 months ahead based on the historical 6 months data about the asset returns. In this case the active management is considered for not long periods of time. The portfolio is constructed with limited number securities. The limitation of the universe of securities allows the investor to define its own market with new market point. The research presents a way for the evaluation of the market characteristics on this particular new limited market. For the experiments in this research, the assets have been chosen in arbitrary way, which gives a hint to the investor not to consider big amount of security data neither to extend his evaluations to other countries or markets. The algorithm, based on BL model applies new form of the expert views $P(\alpha)$, which is based on additional analysis of the historical security returns. The modification of the BL model applies weighted matrix $P(\alpha)$, which in comparison with the classical case of $P(1)$ provides benefits for the investment process.

Table 3
Comparison among three types portfolio problems with MV and BL modified algorithm.

Type of assets	Period duration	MV Return [%]	MV Risk [%]	BL Return [%]	BL Risk [%]
N = 4 shares Sofarma; Himimport; Evrohold; HoldVarna	January 2018 – December 2018, monthly	−1.3864	5.5290	−1.3551	3.9860
N = 7 mutual funds Concord; Elana; Profit; Texim; Lider; Patrim; Growth	January 2018 – September 2018, monthly	−0.3667	0.0812	−0.0241	0.0307
N = 2 real estates	January 2016 – December 2018 quarterly, (3 months)	1.0601	0.5335	1.0374	0.5185

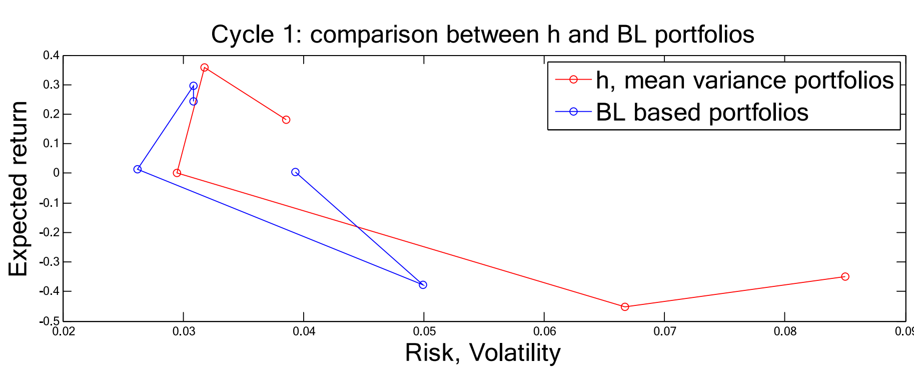

Fig. 8. Comparison between mean variance and BL based portfolios.

The algorithm can be implemented continuously by using rolling horizon for analysis of historical data and applying the decisions about the portfolio weights in next future period. The algorithm has been applied to real market conditions despite the no friendly behavior of the market. On an average, the algorithm provides better results in comparison with the application of the classical Mean-Variance optimization. The results obtained are promising for the implementation of the modified BL model for active management of investor portfolio with limited number of assets.

The algorithm from this research can be complicated taking into account additional constraints, for example with taxes for the implementation of the portfolios.

CRediT authorship contribution statement

Todor Stoilov: Conceptualization, Formal analysis, Methodology, Validation, Writing - original draft. **Krasimira Stoilova:** Formal analysis, Validation, Investigation, Writing - original draft, Writing - review & editing. **Miroslav Vladimirov:** Formal analysis, Validation, Resources, Data curation, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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