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Chebyshev spectral method for solving fuzzy fractional Fredholm–Volterra integro-differential equation

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Abstract

The fuzzy integral equation is used to model many physical phenomena which arise in many fields like chemistry, physics, and biology, etc. In this article, we emphasize on mathematical modeling of the fuzzy fractional Fredholm–Volterra integral equation. The numerical solution of the fuzzy fractional Fredholm–Volterra equation is determined in which model contains fuzzy coefficients and fuzzy initial condition. First, an operational matrix of Chebyshev polynomial of Caputo type fractional fuzzy derivative is derived in fuzzy environment. The integral term is approximated by the Chebyshev spectral method and the differential term is approximated by the operational matrix. This method converted the given fuzzy fractional integral equation into algebraic equations which are fuzzy in nature. The desired numerical solution is to find out by solving these algebraic equations. The different particular cases of our model have been solved which depict the feasibility of our method. The error tables show the accuracy of the method. We also can see the accuracy of our method by 3D figures of exact and obtained numerical solutions. Hence, our method is suitable to deal with the fuzzy fractional Fredholm–Volterra equation.

Keywords: Fuzzy calculus; Chebyshev polynomial; Operational matrix; Mathematical modeling

1. Introduction

Fractional calculus has emerged as an important area of investigation because of its widespread applications in various fields like medical science, physics, chemistry, bio-mathematics, and groundwater problems. Many developments have taken place in fractional calculus in the last few decades. The concept of fractional calculus originates from the idea of the extension of integer order derivative or integration to the real order derivative or integration. The concept of fractional integration/differentiation is extended with the help of the gamma function. The related theory and concepts of fractional derivative/integration can be found in [19,21,26]. The fractional derivatives can be used in describing those physical phenomena which are complex in nature and cannot be easily

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described by integer-order derivatives. Some examples of those phenomena are anomalous diffusion, groundwater contamination problems, epidemic modeling, and simulation of real data in medical science.

In literature, many types of fractional operators can be found such as Riemann–Liouville, Hadamard, Grunwald–Letnikov, and the Caputo derivative. In the early stage, all fractional operators are defined with a singular kernel. These derivatives follow the power-kernel law. In the last decade, researchers have found out the other types of fractional derivatives which have a non-singular kernel. Some of them are popular and using these days very extensively like as Caputo–Fabrizio derivative which has an exponential kernel and the other one is the Atangana–Baleanu derivative which has a Mittag-Leffler kernel. The fractional differential equations are obtained from the classical ones by replacing the integer-order derivatives with fractional ones. Nowadays mathematical modeling is a very powerful tool in depicting the behavior of real-life problems. So, determining the solution of the fractional differential equation is a tough task for researchers because of its complex nature. To get rid of this problem, many numerical techniques and methods are derived and extended from the classical ones. In the fractional model of a physical system, the current stage is the function of the current stage as well as the previous stage. Fractional fluid dynamics traffic model can be helpful in shorting the continuum traffic flow [24]. Some of the numerical methods are Adomian decomposition method [30], predictor–corrector method [9,13], Legendre wavelets [17], GenocPsi polynomial [7], Laguerre polynomial [16], Legendre polynomial [23] and fractional differential transform method [12].

Nowadays, mathematical modeling in a fuzzy environment is so useful in many different fields. In every physical system, the phenomena are not always be put in the deterministic mathematical model. The parameters available in the model, initial and boundary conditions may absorb value from a finite interval. So, such a type of model can be studied in fuzzy mathematical modeling. Like COVID-19 model is studied with the fuzzy environment in article [5]. This article depicts the advantages of the fuzzy differential equation model over the non-fuzzy. The fuzzy modeling gives us the freedom to study those who are not deterministic in nature. Many numerical methods have been introduced to deal with a fuzzy fractional differential equation such as Adomian decomposition method [27], Laplace transforms method [28], implicit finite difference method [32], Homotopy analysis method for Volterra-Fredholm fuzzy integral equation [14] and fuzzy fractional diffusion equation [32]. Some literature on fuzzy concepts and theories can be found in [2,3,8]. Some recent research in the field of a fuzzy differential equation can be found in the literature [18,20,31]

The article is divided into the following sections:

The introduction to fuzzy concepts, fuzzy calculus is described in Section 2. Section 2 also includes the properties of Chebyshev polynomial which are used in this article and procedure of approximation of fuzzy function in terms of Chebyshev polynomial. The fuzzy model of the Fredholm–Volterra integral equation and the concerned method has been incorporated in Section 3. The numerical validation and accuracy of the method through numerical examples are given in Section 4. The concluding part of the whole article is added in Section 5.

2. Preliminaries

Some basic definitions and concepts are discussed here which are going to be used throughout the article. The definition related to fuzzy sets, fuzzy calculus, fractional derivative and properties of Chebyshev polynomial are discussed here.

2.1. Fuzzy sets [15,29]

The concept of fuzzy set was developed by Lotfi Zadeh. We need a non-empty set and a function called membership grade function to define a fuzzy set. The non-empty set is denoted by X membership grade function is represented by $u(\xi)$. The fuzzy set is a subset of $X \times [0, 1]$. Let us consider

$$A \subseteq \{(\xi, u(\xi)) : \xi \in X\} \tag{1}$$

where u is a function such that $u : \xi \longrightarrow [0, 1]$, then we say A is fuzzy set.

Definition 1. A mapping $\tilde{\iota}$ from the set of real numbers to the interval $[0, 1]$ is said to be fuzzy number if it follows some property given as below:

1. $\tilde{\iota}$ satisfies the property of upper semi continuity.
2. $\tilde{\iota}(\mu\xi$+(1-$\mu)\mu) \geq \min\{\tilde{\iota}(\xi), \tilde{\iota}(\mu)\}$ with $\mu \in [0, 1]$ and $\forall \xi, \mu \in R$.
3. $\tilde{\iota}$ satisfies the normality condition i.e., $\exists\ \mu_0 \in R$ s.t. $\tilde{\iota}(\mu_0) = 1$.
4. If we define the support of $\tilde{\iota}$ as

   $$supp(\tilde{\iota}) = \{\mu \in R : \tilde{\iota}(\mu) > 0\}.$$

   Then closure of this support must be compact.

The following definition defines the p-level set of fuzzy number $\tilde{\iota} \in R_F$ where R_F stands for all fuzzy numbers defined on real numbers

$$[\tilde{\iota}_s] = \begin{cases} \{\mu \in R : \tilde{\iota}(\mu) \geq s\}, & s \in (0, 1], \\ closure(supp(\tilde{\iota})), & s = 0. \end{cases} \tag{2}$$

The level set of a fuzzy number is a closed and bounded interval so we denote the p-level set by $[\tilde{\iota}_{(p)}] = [\tilde{\iota}^-(p), \tilde{\iota}^+(p)]$. Here terms $\tilde{\iota}^-(s)$ and $\tilde{\iota}^+(s)$ correspond to the left end point and right end point respectively. The equivalence in R and R_F can be seen from the following formula

$$\tilde{\iota}(z) = \begin{cases} 1, & z = \mu, \\ 0, & z \neq \mu. \end{cases} \tag{3}$$

The algebraic operations between fuzzy numbers can be defined as follows

1. $\tilde{\iota} \oplus \tilde{w} = (\tilde{\iota}^- + \tilde{w}^-, \tilde{\iota}^+ + \tilde{w}^+)$,
2. $\Lambda \odot \tilde{\iota} = \begin{cases} (\Lambda\tilde{\iota}^-, \Lambda\tilde{\iota}^+), & \Lambda \geq 0, \\ (\Lambda\tilde{\iota}^+, \Lambda\tilde{\iota}^-), & \Lambda < 0. \end{cases}$

We can also define the metric structure on set of fuzzy numbers R_F as

$$\Psi : R_F \times R_F \longrightarrow R^+ \cup 0$$

$$\Psi(\tilde{\iota}, \tilde{w}) = \sup_{s \in [0,1]} \max\{|\tilde{\iota}^-(s) - \tilde{w}^-(s)|, |\tilde{\iota}^+(s) - \tilde{w}^+(s)|\}. \tag{4}$$

Some properties followed by this metric are as follows

1. $\Psi(\tilde{\iota} \oplus \tilde{\xi}, \tilde{w} \oplus \tilde{\xi}) = \Psi(\tilde{\iota}, \tilde{w}), \forall \tilde{\iota}, \tilde{w}, \tilde{\xi} \in R_F$.
2. $\Psi(a \odot \tilde{\iota}, a \odot \tilde{w}) = |a|\Psi(\tilde{\iota}, \tilde{w}), \forall a \in R, \tilde{\iota}, \tilde{w} \in R_F$.
3. $\Psi(\tilde{\iota} \oplus \tilde{w}, \tilde{\xi} \oplus \tilde{\mu}) \leq \Psi(\tilde{\iota}, \tilde{\xi}) + \Psi(\tilde{w}, \tilde{\mu}), \forall \tilde{\iota}, \tilde{w}, \tilde{\xi}, \tilde{\mu} \in R_F$.
4. The space of all fuzzy numbers constitute a complete metric space.

Definition 2. Likewise with metric on space of fuzzy numbers, the norm on fuzzy number space is defined as

$$\|\tilde{\iota}\| = \Psi(\tilde{\iota}, \tilde{0}).$$

The function $\|.\| : R_F \longrightarrow R$ is consistent with all the properties of norm as one can verify.

Definition 3. Considering two fuzzy valued functions f_1 and f_2 from $[a, b]$ to R_F. The uniform distance between these fuzzy numbers is defined as

$$\Omega(f_1, f_2) = \sup_{\mu \in [a,b]} \Psi(f_1(\mu), f_2(\mu)). \tag{5}$$

Definition 4 (*Differentiability in Sense of Fuzzy Concepts*)**.** Let $\omega : (a, b) \longrightarrow R_F$ denote the fuzzy valued function and a point $\mu_0 \in (a, b)$. Then the differentiability of function ω is said to be exist if from the condition (1) and (2), one is satisfied.

1. The definition of 1-differentiability on open interval (a, b) is given by equation

   $$\lim_{l \to 0^+} \frac{\omega(\mu_0 + l) \ominus \omega(\mu_0)}{l} = \frac{\omega(\mu_0) \ominus \omega(\mu_0 - l)}{l} = \omega'(\mu_0) \tag{6}$$

   provided numerator of above terms exists.

2. Similarly, the definition of 2-differentiability is defined by the equation

$$\lim_{l\to 0^+}\frac{\omega(\mu_0\ominus\omega(\mu_0+l))}{-l}=\frac{\omega(\mu_0-l)\ominus\omega(\mu_0)}{-l}=\omega'(\mu_0). \tag{7}$$

Definition 5. The h-differentiability of function h from the interval (a,b) into the space of fuzzy numbers R_F at point μ_0 is given by

$$h'(\mu_0)=\lim_{l\to 0}\frac{h(\mu_0+l\ominus_\omega h(\mu_0))}{l}. \tag{8}$$

Definition 6. Let h be a fuzzy valued function, we say it is Riemann-Integrable if $\forall\ \epsilon>0\ \exists\ \delta>0$ such that corresponding to every partition $P=\{[\mu_1,\mu_2],\zeta\}$ of domain $[a,b]$

$$\Psi\Big(\sum_P^{*}(\mu_1-\mu_2)\odot h(\zeta)\Big)<\epsilon, \tag{9}$$

with norms $\Delta(P)<\delta$. Fuzzy Riemann-Integrability is denoted by the following expression

$$I=(FR)\int_a^b h(\mu)dy. \tag{10}$$

2.2. Definitions of fractional derivatives in Riemann–Liouville and Caputo sense [10,25]

Definition 7. The fractional integral in Riemann–Liouville sense of function $\xi(z)$ of order β can be defined as

$$I^\beta\xi(z)=\frac{1}{\Gamma(\beta)}\int_0^z(z-\Omega)^{\beta-1}\xi(\Omega)d\Omega,\quad z>0,\quad \beta\in R^+. \tag{11}$$

The Riemann–Liouville fractional derivative in terms of Riemann–Liouville integral operator is defined as

$$D_l^\beta\xi(z)=(\frac{d}{dz})^l(I^{l-\beta}\xi)(z),\quad (\beta>0,\quad l-1<\beta<l). \tag{12}$$

Definition 8. The derivative which is very useful from application point of view named as Caputo derivative can be represented by the following formula

$$D_c^\beta\xi(z)=\begin{cases}\frac{d^l\xi(z)}{dz^l}, & \beta=l\in N,\\ \frac{1}{\Gamma(\beta)}\int_0^z(z-\varsigma)^{l-\beta-1}\xi^l(\varsigma)d\varsigma, & l-1<\beta<l.\end{cases} \tag{13}$$

Here, l is an integer, $z>0$.

This derivative has a property common with the integer order derivative

$$D_c^\vartheta C=0, \tag{14}$$

where C is a constant and

$$D_c^\Omega z^\sigma=\begin{cases}0, & \sigma\in N\cup 0\quad \&\quad \sigma<\lceil\Omega\rceil\\ \dfrac{\Gamma\ (1+\sigma)}{\Gamma(1-\Omega+\sigma)}z^{-\Omega+\sigma}, & \sigma\in N\cup 0\quad \&\quad \sigma\geq\lceil\Omega\rceil or \sigma\notin N\quad and\quad \sigma>\lfloor\Omega\rfloor,\end{cases} \tag{15}$$

where $\lfloor\Omega\rfloor$ is floor function. Notice that

$$D_c^\Omega(C_1\xi_1(t)+C_2\xi_2(t))=C_1D_c^\Omega\xi_1(t)+C_2D_c^\Omega\xi_2(t), \tag{16}$$

with C_1 and C_2 are constants. The Riemann–Liouville integral and Caputo derivative operators satisfy the relation:

$$(I^\Omega D_c^\Omega\xi)(z)=\xi(z)-\sum_{k=0}^{l-1}\xi^k(0^+)\frac{z^k}{k!},\qquad l-1<\Omega\leq l. \tag{17}$$

2.3. Fuzzy Caputo fractional derivative [1,4]

The definitions of fractional operators are extended for fuzzy application which is as follows

Definition 9. The collection of all measurable function which are fuzzy in nature are denoted by $L_p^E[a,b]$. The notation $C^E[a,b]$ stands for all continuous fuzzy valued functions. Let $h(\xi)$ belong to the set $L_p^E \cap C^E$ then g-Caputo fuzzy fractional of this function given by

$$({}^g D_{a^+}^{\gamma} h)(\xi) = \lim_{h\to 0} \frac{\psi(\xi + k) \ominus_g \psi(\xi)}{k}, \tag{18}$$

where

$$\psi(\xi) = \frac{1}{\Gamma(1-\gamma)} \int_a^{\xi} (\xi - t)^{-\gamma} h(t) dt.$$

We can derive the following s-level set of this derivative

$$\begin{aligned} ({}^g D_{a^+}^{\gamma} h^-)(\xi, s) &= \frac{1}{\Gamma(1-\gamma)} \int_a^{\xi} (\xi - t)^{-\gamma} h^{-'}(t,s) dt \\ , ({}^g D_{a^+}^{\gamma} h^+)(\xi, s) &= \frac{1}{\Gamma(1-\gamma)} \int_a^{\xi} (\xi - t)^{-\gamma} h^{+'}(t,s) dt. \end{aligned} \tag{19}$$

If function belongs to the space of all absolute continuous function then the definition of fuzzy fractional derivative of this function is given by

$$({}^C D_{a^+}^{\gamma} h)(\xi, s) = \Big[\frac{1}{\Gamma(1-\gamma)} \int_a^{\xi} (\xi - t)^{-\gamma} h^{-'}(t) dt, \frac{1}{\Gamma(1-\gamma)} \int_a^{\xi} (\xi - t)^{-\gamma} h^{+'}(t) dt \Big]. \tag{20}$$

The above definition is for the first differentiability of function $h(\xi)$. Applying this concept for the second differentiability we have

$$({}^C D_{a^+}^{\gamma} h)(\xi, s) = \Big[\frac{1}{\Gamma(1-\gamma)} \int_a^{\xi} (\xi - t)^{-\gamma} h^{+'}(t) dt, \frac{1}{\Gamma(1-\gamma)} \int_a^{\xi} (\xi - t)^{-\gamma} h^{-'}(t) dt \Big]. \tag{21}$$

Lemma 1 ([22]). *A function is chosen in such a way that it satisfies $\tilde{h}(\xi) \in AC^E(0,b]$ and ${}^C D_{a^+}^{k\gamma} \tilde{h}(\xi) \in C^E(0,b]$ with $k = 0, 1, \ldots, n+1$, $0 < \gamma < 1$ then*

$$\begin{aligned} &[\tilde{h}(\xi)]^r = [\underline{h^r}(\xi), \overline{h^r}(\xi)] \\ &\underline{h^r}(\xi) = \sum_{j=0}^{n} \frac{\xi^{\gamma j}}{\Gamma(j\gamma+1)} {}^C D^{i\gamma} \underline{h^r}(0^+) + \frac{{}^C D^{(n+1)\gamma} h^{\gamma}(\xi_0)}{\Gamma(n\gamma+\gamma+1)} \xi^{(n+1)\gamma}, \\ &\overline{h^r}(\xi) = \sum_{j=0}^{n} \frac{\xi^{\gamma j}}{\Gamma(j\gamma+1)} {}^C D^{i\gamma} \overline{h^r}(0^+) + \frac{{}^C D^{(n+1)\gamma} h^{\gamma}(\xi_0)}{\Gamma(n\gamma+\gamma+1)} \xi^{(n+1)\gamma}, \end{aligned} \tag{22}$$

with ${}^C D^{i\gamma} \overline{h^r}(0^+) = {}^C D^{i\gamma} \overline{h^r}(\xi)|_{\xi=0}$ *and* ${}^C D^{i\gamma} \underline{h^r}(0^+) = {}^C D^{i\gamma} \underline{h^r}(\xi)|_{\xi=0}$.

2.4. Some basic properties of Chebyshev polynomials [11]

In this section, we are going to discuss the series representation of Chebyshev polynomial and their properties which will be used in proposed method. As we take our model on the domain [0, 1] so we shift these polynomials from the interval [−1, 1] to the interval [0, 1]. The following series representation corresponds to the Chebyshev polynomials of degree i

$$\psi_i(\xi) = i \sum_{k=0}^{i} \frac{(-1)^{-k+i}(k-1+i)! 2^{2k}}{2k!(-k+i)!} \xi^k, \tag{23}$$

where $i = 0, 1, \ldots$. As we know these polynomials are orthogonal on their interval $[-1, 1]$. So the orthogonal property is modified according to the shift of polynomials from the interval $[-1, 1]$ to $[0, 1]$. These shifted polynomials are orthogonal with respect to the weight function $\frac{1}{\sqrt{(\xi-\xi^2)}}$

$$\int_{-1}^{1} \frac{\psi_j(\xi)\psi_i(\xi)}{\sqrt{1-\xi^2}} = \begin{cases} 0, & m \neq l, \\ \pi, & l = m = 0, \\ \frac{\pi}{2} & m = l \neq 0. \end{cases} \tag{24}$$

2.5. Approximation of fuzzy function [11]

In this article, we are using Chebyshev polynomials which are shifted on interval [0, 1]. Considering the function $[\tilde{u}(\xi) \in L^E(0,1)] \cap C^E(0,1)$, then the approximation of this function is defined as follows

$$\tilde{u}(\xi) = \sum_{p=0}^{\infty} s_p \odot \phi_p(\xi), \tag{25}$$

where $\oplus$ stands for the fuzzy addition operation. The fuzzy coefficients s_p can be derived by the following formula

$$s_p = \frac{c_p}{\pi} \odot \int_0^1 \frac{\tilde{u}(\xi) \odot \phi_p(\xi)}{\sqrt{\xi - \xi^2}} d\xi, \tag{26}$$

where c_p are defined as

$$c_p = \begin{cases} 1 & p = 0 \\ 2 & p = 1, 2, \ldots \end{cases} \tag{27}$$

Restricting the infinite sum to the finite sum for the computational purpose and representing the series into matrix form, we have the following

$$\tilde{u}_m(\xi) = \sum_{p=0}^{m} s_p \odot \phi_p(\xi) = \Omega_m^T \odot \Psi_m(\xi), \tag{28}$$

where Ω_m^T and $\Psi_m(\xi)$ denote the coefficient vector and Chebyshev polynomial vector respectively.

$$\begin{aligned} &\Omega_m^T = [s_0, s_1, \ldots, s_p] \\ &\Psi_m(\xi) = [\phi_1(\xi), \phi_2(\xi), \ldots, \phi_m(\xi)]. \end{aligned} \tag{29}$$

Lemma 2. *Let $\tilde{u}(\xi) \in L^E(0,1] \cap C^E(0,1]$ and ${}^C D^\gamma \tilde{u}(\xi) \in C^E(0,1]$, $0 < \gamma < 1$, $\xi \in [0,1]$. If $\tilde{u}(\xi)$ is 1-differentiable, then*

$$\begin{aligned} &|\underline{\tilde{u}^r}(\xi) - \sum_{j=0}^{n} \frac{\xi^{\gamma j}}{\Gamma(j\gamma+1)} {}^C D^{i\gamma} \underline{\tilde{u}^r}(0^+)| \leq \underline{M}_\gamma^r \frac{\xi^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}, \\ &|\overline{\tilde{u}^r}(\xi) - \sum_{j=0}^{n} \frac{\xi^{\gamma j}}{\Gamma(j\gamma+1)} {}^C D^{i\gamma} \overline{\tilde{u}^r}(0^+)| \leq \overline{M}_\gamma^r \frac{\xi^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}, \end{aligned} \tag{30}$$

where $D^{(n+1)\gamma} \overline{\tilde{u}}(\xi_0) < \overline{M}_\gamma$ and $D^{(n+1)\gamma} \underline{\tilde{u}}(\xi_0) < \underline{M}_\gamma$.

Proof. Using Lemma 1 we get

$$\begin{aligned} &\underline{u^r}(\xi) = \sum_{j=0}^{n} \frac{\xi^{\gamma j}}{\Gamma(j\gamma+1)} {}^C D^{i\gamma} \underline{u^r}(0^+) + \frac{{}^C D^{(n+1)\gamma} u^\gamma(\xi_0)}{\Gamma(n\gamma+\gamma+1)} \xi^{(n+1)\gamma}, \\ &\overline{u^r}(\xi) = \sum_{j=0}^{n} \frac{\xi^{\gamma j}}{\Gamma(j\gamma+1)} {}^C D^{i\gamma} \overline{u^r}(0^+) + \frac{{}^C D^{(n+1)\gamma} u^\gamma(\xi_0)}{\Gamma(n\gamma+\gamma+1)} \xi^{(n+1)\gamma}. \end{aligned} \tag{31}$$

In consequence, we get

$$
\begin{aligned}
|\underline{u}^r(\xi) - \sum_{j=0}^{n} \frac{\xi^{\gamma j}}{\Gamma(j\gamma+1)} {}^C D^{i\gamma} \underline{u}^r(0^+)| &\le \frac{|{}^C D^{(n+1)\gamma} u^\gamma(\xi_0)|}{\Gamma(n\gamma+\gamma+1)} \xi^{(n+1)\gamma} \\
&\le \overline{M}_\gamma \frac{\xi^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}.
\end{aligned}
\tag{32}
$$

We can find the another expression like the similar way as discussed above.

Theorem 1. *Let the derivative of function $u(\xi)$ satisfy ${}^C D^\gamma \tilde{u}(\xi) \in L^E(0,1] \cap C^E(0,1]$. Let $\phi_k(\xi)|_{k=0,\dots,n}$ represent the basis of concerned space. The expression $\tilde{u}_n(\xi) = \Omega_n^T \odot \Psi_n(\xi)$ is the approximation in terms of Chebyshev polynomial and*

$$
\lim_{n\to\infty} D(\tilde{u}(\xi), \tilde{u}_n(\xi)) = 0. \tag{33}
$$

Proof. Picking a function h which is fuzzy in nature and $[\tilde{h}(\xi)]^r = [\underline{h}^r(\xi), \overline{h^r}(\xi)]$. The following expression is obtained with the help pf Taylor's formula which is derived in 1

$$
\begin{aligned}
|\underline{\tilde{u}}^r(\xi) - \underline{\tilde{h}}^r(\xi)| &\le \underline{M}_\gamma^r \frac{\xi^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}, \\
|\overline{\tilde{u}}^r(\xi) - \overline{\tilde{h}}^r(\xi)| &\le \overline{M}_\gamma^r \frac{\xi^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}.
\end{aligned}
\tag{34}
$$

As $\Omega_n^T \odot \Psi_n(\xi)$ is the best approximation of $\tilde{u}(\xi)$, we have

$$
\begin{aligned}
|\underline{\tilde{u}}^r(\xi) - \underline{\Omega}_n^T \odot \Psi_n(\xi)| &\le |\underline{\tilde{u}}^r(\xi) - \underline{\tilde{h}}^r(\xi)| \le \underline{M}_\gamma^r \frac{\xi^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}, \\
|\overline{\tilde{u}}^r(\xi) - \overline{\Omega}_n^T \odot \Psi_n(\xi)| &\le |\overline{\tilde{u}}^r(\xi) - \overline{\tilde{h}}^r(\xi)| \le \overline{M}_\gamma^r \frac{\xi^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}.
\end{aligned}
\tag{35}
$$

Thus

$$
\begin{aligned}
\|\underline{\tilde{u}}^r(\xi) - \underline{\Omega}_n^T \odot \Psi_n(\xi)\|^2 &\le \|\underline{\tilde{u}}^r(\xi) - \underline{\tilde{h}}^r(\xi)\|^2 \\
&\le \frac{\underline{M}_\gamma^{r\,2}}{(\Gamma(n\gamma+\gamma+1))^2} \int_0^1 \xi^{2(n+1)\gamma} d\xi, \\
&\le \frac{\underline{M}_\gamma^{r\,2}}{(\Gamma(n\gamma+\gamma+1))^2 (2(n+1)\gamma+1)^2}.
\end{aligned}
\tag{36}
$$

Similarly, we can obtain

$$
\begin{aligned}
\|\overline{\tilde{u}}^r(\xi) - \overline{\Omega}_n^T \odot \Psi_n(\xi)\|^2 &\le \|\overline{\tilde{u}}^r(\xi) - \overline{\tilde{h}}^r(\xi)\|^2 \\
&\le \frac{\overline{M}_\gamma^{r\,2}}{(\Gamma(n\gamma+\gamma+1))^2} \int_0^1 \xi^{2(n+1)\gamma} d\xi, \\
&\le \frac{\overline{M}_\gamma^{r\,2}}{(\Gamma(n\gamma+\gamma+1))^2 (2(n+1)\gamma+1)^2}.
\end{aligned}
\tag{37}
$$

As $n \to \infty$, the expressions in (40) and (41) tend to 0. The following conclusion is drawn

$$
\lim_{n\to\infty} D(\tilde{u}(\xi), \tilde{u}_n(\xi)) = 0
$$

2.6. Derivation of fuzzy operational matrix

Theorem 2. *If the Chebyshev polynomial vector is denoted by $\Psi_n(\xi)$ with $0 < \gamma < 1$, then,*

$$
{}_0^C D_x^\gamma \Psi_N(\xi) = Q^\gamma \Psi_N(\xi), \tag{38}
$$

where Q^{γ} stands for the $N \times N$ operational matrix of order γ

$$Q^{\gamma}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\gamma\rceil}^{\lceil\gamma\rceil}\varsigma_{\lceil\gamma\rceil,k,1} & \sum_{k=\lceil\gamma\rceil}^{\lceil\gamma\rceil}\varsigma_{\lceil\gamma\rceil,0,k} & \cdots & \sum_{k=\lceil\gamma\rceil}^{\lceil\gamma\rceil}\varsigma_{\lceil\gamma\rceil,m-1,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\gamma\rceil}^{i}\varsigma_{i,0,k} & \sum_{k=\lceil\gamma\rceil}^{i}\varsigma_{i,1,k} & \cdots & \sum_{k=\lceil\gamma\rceil}^{i}\varsigma_{i,m-1,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\gamma\rceil}^{N-1}\varsigma_{N-1,0,k} & \sum_{k=\lceil\gamma\rceil}^{N-1}\varsigma_{N-1,1,k} & \cdots & \sum_{k=\lceil\gamma\rceil}^{N-1}\varsigma_{N-1,N-1,k} \end{bmatrix},$$

where

$$\varsigma_{i,j,k}=\frac{2i\times\Gamma(-\gamma+k+\frac{1}{2})}{\Gamma(j+k-\gamma+1)\Gamma(1+k-\gamma-j)}\times\frac{(-1)^{k+i}(i-1+k)!}{c_j(-k+i)!\Gamma(k+\frac{1}{2})},$$

$i=\lceil\vartheta\rceil\cdots,N-1$ *and* $j=0,1,\ldots,N-1$.

Proof. Differentiating the approximation of $\psi_i(\xi)$, the following expression is obtained

$$\begin{aligned} {}_0^C D_x^{\gamma}\psi_i(\xi) &= i\sum_{k=0}^{i}\frac{(-1)^{i-k}(i+k-1)!2^{2k}}{2k!(i-k)!}\times {}_0^C D_x^{\gamma}\xi^k \\ &= i\sum_{k=\lceil\gamma\rceil}^{i}\frac{(-1)^{i-k}(i+k-1)!2^{2k}}{2k!(i-k)!}\times\frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)}\xi^{k-\gamma}. \end{aligned}$$

The term $\xi^{k-\gamma}$ is approximated in terms of Chebyshev polynomials as follows

$$\xi^{k-\gamma}=\sum_{j=0}^{N-1}\delta_{kj}\psi_j(\xi), \tag{39}$$

the unknown coefficients δ_{kj} can be easily derived by using the orthogonal property of Chebyshev polynomials

$$\delta_{kj}=\begin{cases} \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-\gamma+k)}{\Gamma(-\gamma+k+1)}, & j=0, \\ \frac{j}{\sqrt{\pi}}\sum_{s=0}^{j}\frac{(-1)^{-s+j}(-1+s+j)!2^{1+2s}}{(2s)!(-s+j)!}\times\frac{\Gamma(-\gamma+k+s\frac{1}{2})}{\Gamma(-\gamma+1+k+s)}, & j\neq 0. \end{cases} \tag{40}$$

The above derived expression is used for deriving the value of $\varsigma_{i,j,k}$ which ultimately gives $\rho_{i,j}=\sum_{\lceil\vartheta\rceil}^{i}\varsigma_{i,j,k}$ and that is how proof of theorem is completed.

3. Proposed algorithms

In this section we introduce and investigate a fractional Fredholm–Volterra integrodifferential equation in the fuzzy environment given by

$$\frac{d^{\beta}\tilde{u}(\xi)}{dt^{\beta}}=\mu_1\int_0^1\kappa_1(\xi,t)\tilde{u}(t)dt\oplus\mu_2\int_0^{\xi}\kappa_2(\xi,t)\tilde{u}(t)dt\oplus\tilde{f}(\xi), \tag{41}$$

with $0<\xi<1$. The initial condition is taken as follows

$$\tilde{u}(0)=\tilde{f}_1(\xi). \tag{42}$$

The function $\tilde{u}(\xi)$ which grasps the fuzzy value is a function whose value is to be determined. The functions $\kappa_1(\xi,t)$, $\kappa_2(\xi,t)$ and $\tilde{f}(\xi)$ are known functions. The model (41) can be fuzzified as follows for all value of $s \in [0,1]$

$$\begin{aligned}
&[\tilde{u}(\xi)]^s = [\underline{u}(\xi,s), \overline{u}(\xi,s)],\\
&\left[\int_0^1 \kappa_1(\xi,t)\tilde{u}(t,s)dt\right]^s = \left[\int_0^1 \kappa_1(\xi,t)\underline{u}(t,s)dt, \int_0^1 \kappa_1(\xi,t)\overline{u}(t,s)dt\right],\\
&\left[\int_0^\xi \kappa_2(\xi,t)\tilde{u}(t,s)dt\right]^s = \left[\int_0^\xi \kappa_2(\xi,t)\underline{u}(t,s)dt, \int_0^\xi \kappa_2(\xi,t)\overline{u}(t,s)dt\right],\\
&[\tilde{f}(\xi)] = [\underline{f}(\xi,s), \overline{f}(\xi,s)].
\end{aligned} \tag{43}$$

The lower bound and upper bound of model is depicted in the following equations

$$\begin{aligned}
\frac{d^\beta \underline{u}(\xi,s)}{dt^\beta} &= \mu_1 \int_0^1 \kappa_1(\xi,t)\underline{u}(t)dt \oplus \mu_2 \int_0^\xi \kappa_2(\xi,t)\underline{u}(t)dt \oplus \underline{f}(\xi,s),\\
&\underline{u}(\xi,s,0) = \underline{f}(\xi,s),
\end{aligned} \tag{44}$$

$$\begin{aligned}
\frac{d^\beta \overline{u}(\xi,s)}{dt^\beta} &= \mu_1 \int_0^1 \kappa_1(\xi,t)\overline{u}(t)dt \oplus \mu_2 \int_0^\xi \kappa_2(\xi,t)\overline{u}(t)dt \oplus \overline{f}(\xi,s),\\
&\overline{u}(\xi,s,0) = \overline{f}(\xi,s),
\end{aligned} \tag{45}$$

To derive the solution of model, the unknown function $\tilde{u}(\xi)$ is approximated with the help of Chebyshev polynomials in matrix form

$$\tilde{u}_m(\xi) = \sum_{j=0}^{m} \tilde{c}_j \odot \psi_j(\xi) = \tilde{\Omega}_m^T \odot \Psi_m(\xi), \tag{46}$$

the unknowns fuzzy numbers $\tilde{c}_j$ are to be determined. With the help of 2, the fractional differentiation of order β is imposed on Eq. (46)

$$\frac{\partial^\beta \tilde{u}(\xi,t)}{\partial t^\beta} = \tilde{\Omega}_m^T \odot Q^\beta \Psi_m(\xi). \tag{47}$$

The approximation of integration term can be done in following way

$$\begin{aligned}
\int_0^1 \kappa_1(\xi,t)\tilde{u}(t)dt &= \int_0^1 \kappa_1(\xi,t) \times \sum_{j=0}^{m} \tilde{c}_j \psi_j(t)dt\\
&= \int_0^1 \kappa_1(\xi,t) \times \sum_{j=0}^{m} \tilde{c}_j \sum_{k=0}^{j} \frac{j(-1)^{j-k}(j+k-1)!2^{2k}}{2k!(j-k)!} t^k dt,\\
&= \sum_{j=0}^{m}\sum_{k=0}^{j} \tilde{c}_j \frac{j(-1)^{j-k}(j+k-1)!2^{2k}}{2k!(j-k)!} \times \int_0^1 \kappa_1(\xi,t) \times t^k dt,\\
&= \Xi_j(\xi),
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
\int_0^\xi \kappa_2(\xi,t)\tilde{u}(t)dt &= \int_0^\xi \kappa_2(\xi,t) \times \sum_{j=0}^{m} \tilde{c}_j \psi_j(t)dt\\
&= \int_0^\xi \kappa_2(\xi,t) \times \sum_{j=0}^{m} \tilde{c}_j \sum_{k=0}^{j} \frac{j(-1)^{j-k}(j+k-1)!2^{2k}}{2k!(j-k)!} t^k dt,\\
&= \sum_{j=0}^{m}\sum_{k=0}^{j} \tilde{c}_j \frac{j(-1)^{j-k}(j+k-1)!2^{2k}}{2k!(j-k)!} \times \int_0^\xi \kappa_2(\xi,t) \times t^k dt,\\
&= \Lambda_j(\xi).
\end{aligned} \tag{49}$$

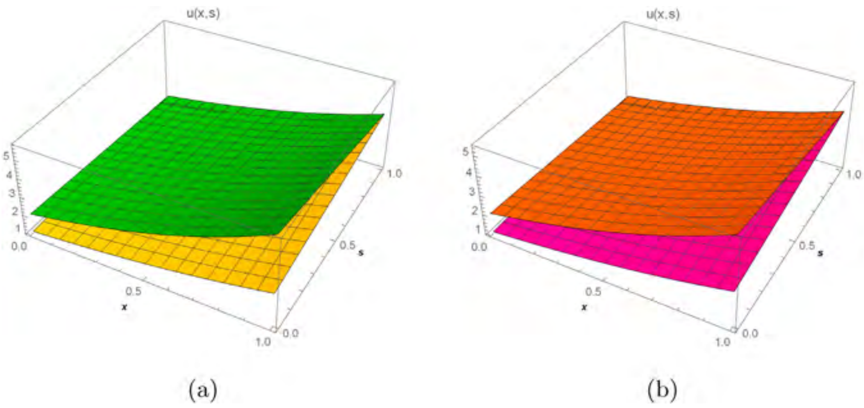

Fig. 1. Plots of $u(\xi, t, s)$ for $m = 6$ in case of numerical and exact solution for $t = 0.5$.

We have used a numerical scheme for evaluating the integral term presented in Eqs. (48) and (49). The fuzzy initial is also approximated in the following manner

$$\tilde{\Omega}_m^T \odot \Psi_m(0) = \tilde{f}(\xi). \tag{50}$$

The following residue is derived after putting the value of all derivatives and unknown functions in the concerned model (41)

$$\tilde{\xi}(\xi) = \tilde{\Omega}_m^T \odot Q^\beta \Psi_m(\xi) \ominus \mu_1 \Xi_j(\xi) \ominus \mu_2 \Lambda_j(\xi) \ominus \tilde{f}(\xi, s). \tag{51}$$

A system of fuzzy algebraic equation is determined after collocating Eq. (51) Chebyshev nodes. We get the solution of our model after solving the system of fuzzy algebraic equation [6].

4. Finding and validation of method

This section contains the numerical validation of derived method for the fractional fuzzy Fredholm–Volterra integral equation. All numerical calculations are performed on Wolfram Mathematica version-11.3.

Example 1. In this first example a simple fuzzy Fredholm–Volterra integrodifferential equation is picked up which is obtained from the model (41) with parameters value $\beta = 1$, $\mu_1 = 1$, $\mu_2 = 1$, $\kappa_1(\xi, t) = e^{\xi - t}$, and $\kappa_2(\xi, t) = \xi - t$

$$\frac{d\tilde{u}(\xi)}{d\xi} = \int_0^1 e^{\xi - t}\tilde{u}(t)dt \oplus \int_0^\xi (\xi - t)\tilde{u}(t)dt \oplus \tilde{f}(\xi), \tag{52}$$

The fuzzy initial condition is

$$\tilde{u}(0) = (1 + 0.5s, 2 - 0.5s). \tag{53}$$

The exact solution of (56)–(57) is $\tilde{u}(\xi, s) = (1 + 0.5s, 2 - 0.5s)e^\xi$. Fig. 1 represents the exact and analytical solution respectively. The variation of absolute error can be shown in Table 1. Both shows the accuracy and validity of presented method for fuzzy type integral equation.

Example 2. Now we consider the following fuzzy fractional Fredholm–Volterra integrodifferential equation that is obtained from the model (41) with parametric value $\beta = 0.9$, $\mu_1 = 1$, $\mu_2 = 1$, $\kappa_1(\xi, t) = e^{\xi - t}$, and $\kappa_2(\xi, t) = \xi - t$

$$\frac{d^\beta \tilde{u}(\xi)}{dt^\beta} = \mu_1 \int_0^1 \kappa_1(\xi, t)\tilde{u}(t)dt \oplus \mu_2 \int_0^\xi \kappa_2(\xi, t)\tilde{u}(t)dt \oplus \tilde{f}(\xi), \tag{54}$$

subject to the following fuzzy initial condition

$$\tilde{u}(\xi, 0) = 0. \tag{55}$$

Table 1
Exhibition of absolute errors at $t = 0.1$, $s = 0.1$ and for $m = 7$.

$\xi \downarrow$	Absolute error value for lower solution	Absolute error value for upper solution
0.1	3.2×10^{-10}	3.5×10^{-11}
0.2	1.0×10^{-11}	6.7×10^{-11}
0.3	7.1×10^{-10}	8.5×10^{-10}
0.4	3.8×10^{-10}	9.3×10^{-10}
0.5	5.5×10^{-10}	5.2×10^{-10}
0.6	4.6×10^{-10}	4.7×10^{-10}
0.7	1.9×10^{-10}	6.5×10^{-10}
0.8	7.9×10^{-10}	2.8×10^{-10}

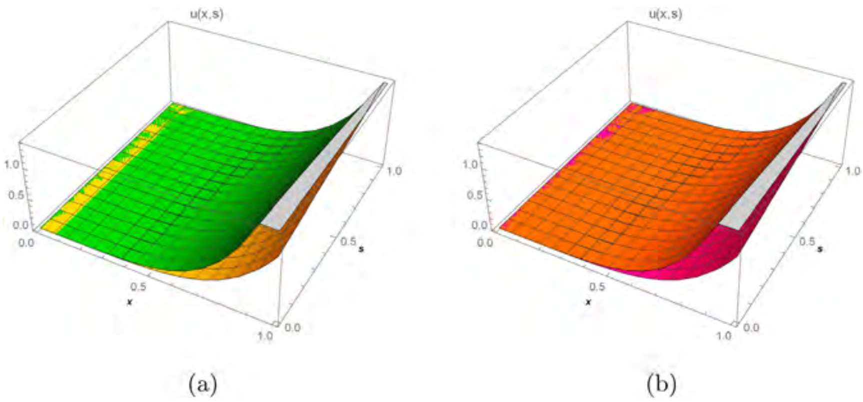

Fig. 2. Plots of $u(\xi, t, s)$ for $m = 4$ in case of numerical and exact solution for $t = 0.5$.

Table 2
Exhibition of absolute errors with $t = 0.1$, $s = 0.1$ and for $m = 7$.

$\xi \downarrow$	Absolute error value for lower solution	Absolute error value for upper solution
0.1	4.2×10^{-7}	8.6×10^{-7}
0.2	3.6×10^{-7}	4.7×10^{-7}
0.3	5.7×10^{-6}	5.5×10^{-6}
0.4	6.4×10^{-6}	1.9×10^{-6}
0.5	5.9×10^{-6}	2.9×10^{-6}
0.6	4.9×10^{-6}	9.2×10^{-6}
0.7	1.6×10^{-6}	3.4×10^{-6}
0.8	6.0×10^{-6}	5.8×10^{-6}

We choose the force function $\tilde{f}(\xi)$ such that exact solution of this problem is $\tilde{u}(\xi) = (1 + 0.5s, 2 - 0.5s)\xi^5$. The exact and analytical solution of this problem are depicted by Fig. 2. The analogy between both the figures shows the feasibility of method. The variation of absolute error is given in Table 2. This table shows that our numerical results are in good agreement with the exact ones.

Example 3. Considering another particular case in which $\beta = 0.9$, $\mu_1 = 1$, $\mu_2 = 1$, $\kappa_1(\xi, t) = \sinh(\xi - t)$, and $\kappa_2(\xi, t) = xt$

$$\frac{d^{0.9}\tilde{u}(\xi)}{dt^{0.9}} = \int_0^1 \sinh(\xi - t)\tilde{u}(t)dt \oplus \mu_2 \int_0^\xi xt\tilde{u}(t)dt \oplus \tilde{f}(\xi), \tag{56}$$

with the following fuzzy initial condition

$$\tilde{u}(\xi, 0) = 0. \tag{57}$$

We choose the force function $\tilde{f}(\xi)$ such that exact solution of this problem is $\tilde{u}(\xi) = (1 + 0.5s, 2 - 0.5s)\cosh(\xi)$.

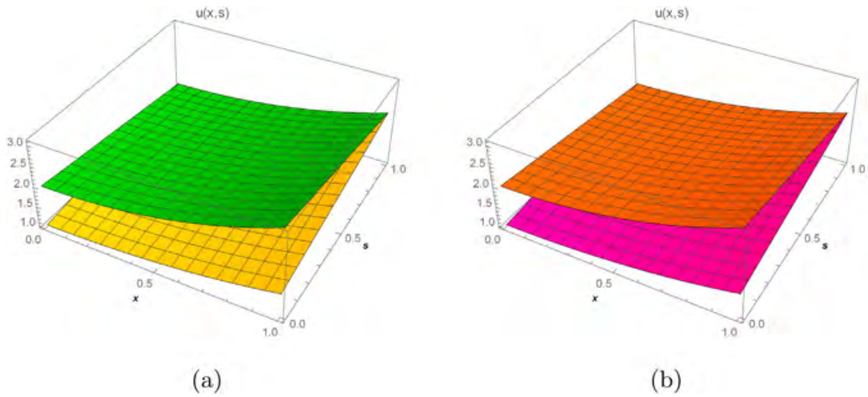

Fig. 3. Plots of $u(\xi, t, s)$ for $m = 4$ in case of numerical and exact solution for $t = 0.5$.

Table 3
Exhibition of absolute errors at $t = 0.1$, $s = 0.1$ and for $m = 7$.

$\xi \downarrow$	Absolute error for lower solution	Absolute error for upper solution
0.1	6.7×10^{-6}	8.6×10^{-6}
0.2	3.7×10^{-5}	5.3×10^{-5}
0.3	4.9×10^{-5}	6.2×10^{-5}
0.4	8.8×10^{-5}	7.4×10^{-5}
0.5	2.3×10^{-5}	1.3×10^{-5}
0.6	1.7×10^{-5}	4.1×10^{-5}
0.7	9.3×10^{-5}	7.6×10^{-5}
0.8	6.4×10^{-5}	4.2×10^{-5}

Fig. 3 depicts the exact and numerical solution 3D plots for example 3 respectively. These figures prove the feasibility of our method. The Table 3 is drawn for absolute error for this example respectively. We can conclude that the accuracy of our method is desirable.

5. Conclusion

This article is devoted to study of fuzzy fractional Fredholm Volterra integral equation. First, the introductory definitions and concepts of fuzzy calculus are discussed. Some lemmas and theorems are presented to deal with fuzzy model. The unknown fuzzy function is approximated in terms of shifted Chebyshev polynomials. The Chebyshev operational matrix has been derived to find out the numerical solution of model. The differential term was approximated by using Caputo fuzzy fractional operational matrix while the integral term was approximated by using Chebyshev spectral method. We solved particular cases of our model with this spectral method and demonstrated the validity of our method for fuzzy fractional Fredholm–Volterra integrodifferential equation with a desirable accuracy. Our contribution in the field of fuzzy fraction dynamical systems is that we have taken a mix model of Fredholm and Volterra integral term equipped with fractional differential term which is new of its kind. A complex dynamical system which is uncertain in nature and have property of integro differential equation, its behavior can be easily depicted by this method. This work can be extended to a new class of integral equation with fuzzy environment. Many physical models having integral equations in their model can be solved by this method. On other hand, integro-differential equation solutions with non-singular kernel like exponential and Mittag-Leffler can be derived by extending this method.

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