# Nonlinear Dynamics Analysis of a Large Flexible Slender Truss-Structure Carrying a Manipulator for On-Orbit Assembly 

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#### Abstract

Nonlinear dynamic of a flexible slender truss-structure mounted manipulator for on-orbit assembly, which can be simplified as a beam-rotating link interaction system, is theoretically investigated. The governing partial differential equations (PDEs) of beam with time-varying coefficients is established by using the D'Alembert principle incorporated with the moment balance method where the beam is of a Euler-Bernoulli type and the influence of slope is considered. Such system is a typical parametrically excited system. The multiple scales method is used to determine the approximate solution and the conditions of the primary resonance ( $\omega_{1} \approx \omega_{\text {ref }}$ ) and sub-harmonic resonance ( $\omega_{1} \approx 2 \omega_{\text {ref }}, \omega_{1} \approx 3 \omega_{\text {ref }}$ and $\omega_{1} \approx 4 \omega_{\text {ref }}$ ) are obtained. In addition, the nonlinear response, stability and bifurcations for primary and sub-harmonic resonance conditions have also been investigated by varying system parameters. Moreover, the results of some specific conditions by the perturbation analysis are compared with the numerical solution and are found to be in good agreement. This work has certain guiding significance for autonomous on-orbit assembly task and the method can be extended to the more general three-dimensional case.


INDEX TERMS On-orbit assembly, flexible slender truss-structure mounted manipulator, parametric excitation, method of multiple scales, primary resonance, sub-harmonic resonance.

## I. INTRODUCTION

Future space exploration puts forward some new requirements of space structure, such as establishment of large space solar power plants to cope with energy depletion which is huge volume (from thousands of meters or even dozens of kilometers). To meet the requirements of such space missions, the space structures will be constructed too large to be launched and deployed as a whole [1]-[5]. It is identified as one of the most appealing solutions in which the manipulator is mounted on the long truss-structures to assemble or maintain the several adjacent blocks [6]. One problem of great concern is that of low-frequency structure may be easily excited by high-frequency robotic and hardly damped out in space environment due to the low-damping characteristics of the flexible structures [2], [7]. The vibration may cause inaccuracy of manipulator positioning, and more seriously, the premature fatigue failure of flexible structure [8] and

[^0]it could be reduced by improving the dynamic model of the system. Therefore, it is significant to conduct studies to apprehend dynamic characteristics of such system and find some structural parameter design criterion to minimize the vibration amplitude.

In this paper, a typical flexible slender truss-structure mounted manipulator (FSTMM) for on-orbit assembly, as shown in Fig.1(a), is studied. During the robot is assembling the truss structure, the robot is mounted on the long truss to assemble the next module of the truss. To focus on the fundamental issues of the dynamic problem, the long truss structure can be simplified as a flexible beam [3], [9] and only the first link of manipulator is considered. Such system is a typical beam-rotating link interaction system. A very limited work of such system has been reported. For example, the speed exclusion zone of a wind turbine, which was regarded as a typical cantilever beam structure attached with a rotating unbalanced mass, was investigated to prevent tower resonance [10]. The nonlinear dynamic behavior of a non-ideal unbalanced motor in a simple cantilever beam


FIGURE 1. (a) Concept picture for FSTMM for on-orbit assembly; (b) Schematic of the first link of manipulator mounted on a flexible beam at an arbitrary position.
system was investigated and the results indicate there appears the jump phenomenon, namely the Sommerfeld effect [11], [12]. A model using RLC circuits with variable capacitance based on the saturation phenomenon is used to control the vibration of a hinged-hinged beam supporting unbalanced machine [13]. A nonlinear dynamical model of a robot manipulator consisting of a flexible cantilever beam and rigid second link is derived using a Lagrange equation and a Lyapunov-based feedback control law is then introduced to suppressing bending vibrations in the flexible link [14].

The above-mentioned studies on beam-rotating link interaction system were contrived based on neglecting the effect of slope of beam, in which the terms with respect to rotating link in PDEs of beam is only appear as external excitation. In other words, the excitations due to rotating link only appear as an inhomogeneous term in governing differential equations of beam. Such system is an external excitation system. In this approach, the external resonance, i.e., primary resonance, may just happen. However, such mathematical model is not found apt to explore nonlinear dynamic behaviors of onorbit assembly system in which space robotic can operate with a high-frequency on a low-frequency structure. Based on above analyses, by involving the slope effect of beam into the problem, a more precise and rigorous formulation will be established, where time-dependent coefficients due to rigid link emerge in PDEs of beam and the concept of parametrically excited systems are exposed to discussion.

The problem of parametric resonance arises in many branches of physics and engineering. For example, pendulum with a moving support system [15], [16], two-link flexible manipulator [17], [18], flexible structures such as beams and plates under periodic motion or loads [19]-[22], etc. In contrast to the external excitations in which a small excitation


FIGURE 2. A typical primary resonance for beam ( $\omega_{1} \approx 1$ ).
produces a large response only if the frequency of the excitation is close to a linear natural frequency, however, a small parametric excitation may produce a large response when the frequency of the excitation is away from the linear natural frequencies of the system [23]. Considerable attention has been dedicated to parametrically excited systems, and there exists a unique type of resonance named principal parametric resonances. In such situation, some undesired phenomena may be occurred such as amplitude sharp increases or jumps when the parameters (e.g., the rotational speed of manipulator in robotic operations) approach one of several different critical values. So, the nature of parametric excitation can be risky to structures.

In this paper, the nonlinear dynamic of a FSTMM, which can be simplified as beam-rotating link interaction system, is theoretically studied, as shown in Fig.1. An appropriate nonlinear PDEs with time-varying coefficients is established by using D'Alembert's principle incorporated with the moment balance method [21], [24]-[27]. The approximate solution is then obtained by using a single-mode discretization via the Galerkin's method with those obtained by directly applying the method of multiple scales. Finally, the nonlinear response, stability and bifurcations for primary and sub-harmonic resonance conditions have been investigated by varying system parameters. The time response for some specific conditions is obtained by numerical solution which is used to verify the correctness of the perturbation method. Among them, the main contributions of this paper as follows:

1. A more precise and rigorous PDEs with time-varying coefficients of a beam-rotating link interaction system is established in which the slope of the beam is considered. Such a system is a typical parametrically excited system.
2. The frequency response conditions of the primary resonance $\left(\omega_{1} \approx 1\right)$ and sub-harmonic resonance $\left(\omega_{1} \approx 2\right.$, $\omega_{1} \approx 3$ and $\omega_{1} \approx 4$ ) are obtained by the first-order multiple scales method.
3. The nonlinear response, stability and bifurcations for primary and sub-harmonic resonance conditions have been
investigated by varying system parameters, namely the mass of link, the mounting position of the rigid link on the beam and damping of the beam, which may provide some design principles for avoiding large-value vibration of the flexible truss.

The rest of this paper is organized as follows. In Section II, the mathematical model of a beam-rotating link interaction system is derived. In Section III, the multiple scales method is employed to calculate the nonlinear solution and analyze its stability. In Section IV, the numerical simulation and discussion are carried out to show the nonlinear dynamic behavior of the system. Finally, some conclusions are drawn in Section V.

## II. EQUATIONS OF DYNAMICS

The schematic diagram of a typical on-orbit assembly of truss structure installation by manipulator is presented in Fig.1(a). As shown in Fig.1(a), the manipulator is mounted on a highly flexible truss to install another short truss and after completion, the manipulator will be moved forward and repeat this process. In addition, to focus on the fundamental issues of the parametric resonance problem introduced by assembly manipulator, the effect of attitude motion and the complex space environments such as sunlight pressure and thermal shock are neglected. In order to simplify the modeling, only the first link of the manipulator is considered and the flexible truss is simplified as a slender cantilever beam in this paper, as shown in Fig. 1(b).

As for the slender beams or low-order modes of beams, the rotational inertia and shear deformation can be neglected, in other word, the theory of Euler-Bernoulli is adopted in subsequent derivation. In addition, the attention of the paper is limited to planar motions and the approach can be extended to the more general three-dimensional case. The influence of gravity is ignored. For convenience, the nomenclatures used in the subsequent derivation are shown in TABLE 1.

Here, D'Alembert's principle is used to derivate the dynamic equation of motion. According to the EulerBernoulli theory, the bending moment at any cross-section $s$ from point $O$ can be expressed as [25], [28]

$$
\begin{equation*}
M(s, t)=E I k(s, t)=E I v^{\prime \prime}\left(1+\frac{1}{2} v^{\prime 2}\right) \tag{1}
\end{equation*}
$$

where the prime denotes differentiation with respect to $s$ and the curvature of the beam $k(s, t)$ can be expressed as

$$
\begin{equation*}
k(s, t)=\frac{\partial \phi(s, t)}{\partial s}=\phi^{\prime}(s, t) \tag{2}
\end{equation*}
$$

Here, the slope of beam, $\phi(s, t)$, can be written in terms of beam elastic displacements as [25]

$$
\begin{equation*}
\sin \phi=v^{\prime} \text { or } \cos \phi=1-\frac{1}{2} v^{\prime 2} \tag{3}
\end{equation*}
$$

In light of D'Alembert's principle, the following moment equilibrium relation exists in the beam micro-segment:

$$
\begin{equation*}
M^{\prime \prime}-M_{\zeta u}^{\prime \prime}-M_{\zeta v}^{\prime \prime}=0 \tag{4}
\end{equation*}
$$

TABLE 1. Nomenclatures.

| Symbol | Quantity |
| :--- | :--- |
| $\rho$ | Mass per unit length of link |
| $A$ | Area of cross section of link |
| $L$ | Length of flexible link |
| $h$ | Thickness of link |
| $b$ | Width of link |
| $E$ | Young's modulus of material of link |
| $I$ | Moment of inertia of link |
| $m_{b}$ | Mass of mounting base of rigid manipulator including |
| $m_{l}$ | servomotor |
| $p$ | Mass of rigid manipulator |
| Distance from mass center of the manipulator to the |  |
| $c$ | central axis of motor |
| $s, \xi, \zeta$ | Coefficient of viscous damping of beam |
| $s_{b}$ | Installation porition of rigid manipulator on beam |
| $u(s, t)$ | Longitudinal displacement of the beam at a distance $s$ |
| $v(s, t)$ | Transverse displacement of the beam at a distance $s$ |
| $\phi(s, t)$ | Slope of the beam at a distance $s$ |
| $\theta(t)$ | Angular displacement of the manipulator |
| $\omega_{l}$ | Excitation frequency caused by the rotation of the <br> manipulator |

where $M_{\zeta u}$ and $M_{\zeta v}$ are the moment in the $x$ and $y$ directions respectively, which can be expressed as:

$$
\begin{align*}
M_{\zeta u}= & \int_{s}^{L}\left\{-\left[\rho A+\left(m_{b}+m_{l}\right) \delta\left(\zeta-s_{b}\right)\right] \ddot{u}\right. \\
& +m_{l} p[(\ddot{\theta}+\ddot{\phi}) \sin (\theta+\phi) \\
& \left.\left.-(\dot{\theta}+\dot{\phi})^{2} \cos (\theta+\phi)\right] \delta\left(\zeta-s_{b}\right)\right\} \int_{s}^{\zeta} \sin \phi d \eta d \zeta  \tag{5}\\
M_{\zeta v}= & \int_{s}^{L}\left\{-\left[\rho A+\left(m_{b}+m_{l}\right) \delta\left(\zeta-s_{b}\right)\right] \ddot{v}-c \dot{v}\right. \\
& +m_{l} p[(\ddot{\theta}+\ddot{\phi}) \cos (\theta+\phi) \\
& \left.\left.+(\dot{\theta}+\dot{\phi})^{2} \sin (\theta+\phi)\right] \delta\left(\zeta-s_{b}\right)\right\} \int_{s}^{\zeta} \cos \phi d \eta d \zeta \tag{6}
\end{align*}
$$

where a dot is used over a symbol to represent the time derivative and $\delta$ is the Delta function, here, it can be noted that the moment $M_{\zeta u}$ and $M_{\zeta \nu}$ takes into account both the slope of the beam $\phi$ and the rotation angle $\theta$ of the manipulator which introduce the couple terms into PDEs. In addition, the simplified viscous damping of beam $c$ is only considered and more accurate damping model such as the material damping due to internal features of the beam material can be obtained from References [29], [30].

Since $\theta$ is determined by the operation tasks and are timevarying, it is difficult to give an analytical solution of system about arbitrary operation tasks. Thus, in order to compute the steady-state response of the flexible structure, we assumed


FIGURE 3. Time responses, phase portraits and Poincare's map corresponding to points $\mathbf{Q}, \mathrm{M}$ and N as shown in Fig. 2 with different initial conditions.
that the manipulator rotates with a constant angular velocity in this paper, i.e., $\ddot{\theta}=0$.

On the other hand, the inextensibility condition of a cantilever beam can be denoted as [25]

$$
\begin{equation*}
v^{\prime 2}+\left(1+u^{\prime}\right)^{2}=1 \tag{7}
\end{equation*}
$$

or, more conveniently, the displacement $u$ in the longitudinal direction can be described in terms of displacement $v$ as

$$
\begin{equation*}
u(s, t)=s-\int_{0}^{s} \cos \phi(\xi, t) d \xi \tag{8}
\end{equation*}
$$

By substituting Eq. (1), (5)~(7) into Eq. (4) and differentiating the resulting equation twice with respect to $s$ and applying Leibnitz's rules, one may obtain the following governing differential equation of motion:

$$
\begin{aligned}
& E I\left(v^{\prime \prime \prime}+\frac{1}{2} v^{\prime 2} v^{\prime \prime \prime}+v^{\prime \prime 3}+v^{\prime} v^{\prime \prime} v^{\prime \prime \prime}\right) \\
& +\left(1-\frac{1}{2} v^{\prime 2}\right)\left\{\left[\rho A+\left(m_{b}+m_{l}\right) \delta\left(\zeta-s_{b}\right)\right] \ddot{v}+c \dot{v}\right. \\
& \left.-m_{l} p\left[(\ddot{\theta}+\ddot{\phi}) \cos (\theta+\phi)+(\dot{\theta}+\dot{\phi})^{2} \sin (\theta+\phi)\right] \delta\left(s-s_{b}\right)\right\} \\
& +v^{\prime} v^{\prime \prime} \int_{s}^{L}\left\{\left[\rho A+\left(m_{b}+m_{l}\right) \delta\left(\zeta-s_{b}\right)\right] \ddot{v}+c_{1} \dot{v}\right. \\
& \left.-m_{l} p\left[(\ddot{\theta}+\ddot{\phi}) \cos (\theta+\phi)+(\dot{\theta}+\dot{\phi})^{2} \sin (\theta+\phi)\right] \delta\left(\zeta-s_{b}\right)\right\} d \zeta \\
& +v^{\prime}\left\{\left[\rho A+\left(m_{b}+m_{l}\right) \delta\left(\zeta-s_{b}\right)\right] \int_{0}^{s}\left(\dot{v}^{\prime 2}+v^{\prime} \ddot{v^{\prime}}\right) d \xi\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.-m_{l} p\left[(\ddot{\theta}+\ddot{\phi}) \sin (\theta+\phi)-(\dot{\theta}+\dot{\phi})^{2} \cos (\theta+\phi)\right] \delta\left(s-s_{b}\right)\right\} \\
& +v^{\prime \prime} \int_{s}^{L}\left\{-\left[\rho A+\left(m_{b}+m_{l}\right) \delta\left(\zeta-s_{b}\right)\right] \int_{0}^{s}\left(\dot{v}^{\prime 2}+v^{\prime} \ddot{v}^{\prime}\right) d \xi\right. \\
& \left.+m_{l} p[\ddot{\theta}+\ddot{\phi}) \sin (\theta+\phi)-(\dot{\theta}+\dot{\phi})^{2} \cos (\theta+\phi)\right] \\
& \left.\times \delta\left(\zeta-s_{b}\right)\right\} d \zeta=0 \tag{9}
\end{align*}
$$

It can be noted that this equation can be reduced to that of [19] by substituting $m_{l}$ equal to zero in Eq.(9). One can observe that the PDE of motion of this system including the inertial term and the elastic part. Based on the Galerkin's method, the transverse deformation of the beam can be written as

$$
\begin{equation*}
v(s, t)=r \varphi(s) q(t) \tag{10}
\end{equation*}
$$

where $r$ is the scaling factor; $q(t)$ is the time modulation, and $\varphi(s)$ assumed as a linear combination of classical cantilever mode function which can been given by [12], [29]

$$
\begin{align*}
\varphi_{n}(s)= & \sin \beta_{n} s-\sinh \beta_{n} s \\
& +\frac{\sin \beta_{n} L+\sinh \beta_{n} L}{\cos \beta_{n} L-\cosh \beta_{n} L}\left(\cos \beta_{n} s-\cosh \beta_{n} s\right) \\
\cos \beta_{n} & +\cosh \beta_{n}+1=0 \tag{11}
\end{align*}
$$

here, in order to simplify the computational complexity, only first-order modes are included.

Using the dimensionless variables and parameters defined by

$$
\bar{m}_{b}=\frac{m_{b}}{\rho A L}, \bar{m}_{l}=\frac{m_{l}}{\rho A L}, \bar{p}=\frac{p}{L}
$$



FIGURE 4. Basins of attraction for $\omega_{1}=1$ and $\omega_{1}=1.2$ key as in Fig.2.

$$
\begin{equation*}
\bar{\xi}=\frac{\xi}{L}, \bar{\zeta}=\frac{\zeta}{L}, \bar{s}_{b}=\frac{s_{b}}{L}, \bar{t}=\omega_{r e f} t \tag{12}
\end{equation*}
$$

By substituting Eq. (3), (10) and (12) into the Eq. (9), one may obtain the nondimensional second-order ODEs of motion, which can be expressed as

$$
\begin{align*}
& \ddot{q}+2 \varepsilon \mu \dot{q}+q+\varepsilon\left(H_{3} q^{3}+h_{1} q+h_{2} q^{2}+h_{3} q^{3}+\alpha_{1} \dot{q}+\alpha_{2} \ddot{q}\right. \\
& \left.\quad+\alpha_{3} \dot{q}^{2}+\alpha_{4} q \ddot{q}+\alpha_{5} q \dot{q}^{2}+\alpha_{6} q^{2} \dot{q}+\alpha_{7} q^{2} \ddot{q}+f\right)=0 \tag{13}
\end{align*}
$$

The expression for $\omega_{\text {ref }}$ in Eq. (12) and the coefficients (i.e., $\mu, H_{3}, h_{1}, h_{2}, h_{3}, \alpha_{1}, \ldots, \alpha_{7}, f$ ) in Eq. (13) are given in the APPENDIX A.

Here, a small book keeping parameter $\varepsilon$ is used to make the order of all the coefficients (i.e., $\mu, H_{3}, h_{1}, h_{2}, h_{3}, \alpha_{1}$, $\left.\ldots, \alpha_{7}, f\right)$ less than one. The coefficients of non-dimensional temporal equation of Eq. (13) can be classified as: (1) constant coefficient terms: cubic geometric nonlinear stiffness term $\left(H_{3} q^{3}\right)$ and linear damping term ( $2 \varepsilon \mu \dot{q}$ ); (2) nonlinear parametric excitation terms with time-varying coefficient: the linear stiffness term $\left(h_{1} q\right)$, nonlinear stiffness term $\left(h_{2} q^{2}+\right.$ $h_{3} q^{3}$ ), nonlinear damping term $\left(\alpha_{1} \dot{q}+\alpha_{5} q \dot{q}^{2}\right) \alpha_{2}$, nonlinear inertial terms $\left(\alpha_{2} \ddot{q}+\alpha_{3} \dot{q}^{2}+\alpha_{4} \ddot{q}+\alpha_{5} q \dot{q}^{2}+\alpha_{7} q^{2} \ddot{q}\right)$ and (3) nonlinear forced term $f$. One can observe that nonlinear forced term $f$ appears as an external excitation in equation


FIGURE 5. Frequency response of beam for primary resonance case with different $c$.


FIGURE 6. Frequency response of beam for primary resonance case with different $\bar{m}_{I}$.


FIGURE 7. Frequency response of beam for primary resonance case with different $\overline{\boldsymbol{s}}_{\boldsymbol{b}}$.
governing the motion of the system in which a small excitation produces a large response, i. e., primary resonance, only if the frequency of the excitation is close to a linear natural frequency. However, except primary resonance, time-varying
coefficient terms appear as parametric excitation which can produce a large response when the frequency of the excitation is away from the linear natural frequencies of the system, i . e., principal parametric resonances. So, the nonlinear system expressed in Eq. (13) is a combination of nonlinear forced and parametrically excited systems. Moreover, it can be noted that the Eq. (13) is the set of multiple complex nonlinear terms, and it is difficult to find a closed form solution. Therefore, the approximate solution can be obtained by the perturbation method in Section III.

## III. SOLUTION PROCEDURE

An approximate solution of the Eq. (13) can be obtained by a number of perturbation techniques. Here we use the method of multiple scales [23].

We express the solution in terms of different time scales as

$$
\begin{equation*}
q(\tau, \varepsilon)=q_{0}\left(T_{0}, T_{1}\right)+\varepsilon q_{1}\left(T_{0}, T_{1}\right)+\cdots \tag{14}
\end{equation*}
$$

where $T_{0}=\tau, T_{1}=\varepsilon \tau$ and using chain rule, time derivatives in terms of $T_{0}, T_{1}$ become

$$
\begin{align*}
\frac{d}{d \tau} & =D_{0}+\varepsilon D_{1}+\varepsilon^{2} D_{2}+\cdots \\
\frac{d^{2}}{d \tau^{2}} & =D_{0}^{2}+2 D_{0} D_{1} \varepsilon+D_{1}+\left(D_{1}^{2}+2 D_{0} D_{2}\right) \varepsilon^{2}+\cdots \tag{15}
\end{align*}
$$

here, $D_{0}=\frac{\partial}{\partial T_{0}}, D_{1}=\frac{\partial}{\partial T_{1}}$.
Substituting (14), (15) into (13) and equating the coefficients of $\varepsilon^{0}$ and $\varepsilon^{1}$ on the both sides, we obtain

$$
\begin{align*}
\varepsilon^{0}: & D_{0}^{2} q_{0}+q_{0} \\
& =0  \tag{16}\\
\varepsilon^{1}: & D_{0}^{2} q_{1}+q_{1} \\
= & -2 D_{0} D_{1} q_{0}-2 \mu D_{0} q_{0}-H_{3} q_{0}^{3}-\hbar_{1} \omega_{1}^{2} \cos \omega_{1} t q_{0} \\
& -\hbar_{2} \omega_{1}^{2} \sin \omega_{1} \tau q_{0}^{2}-\hbar_{3} \omega_{1}^{2} \cos \omega_{1} \tau q_{0}^{3} \\
& -a_{1} \omega_{1} \sin \omega_{1} \tau D_{0} q_{0}-a_{2} \cos \omega_{1} \tau D_{0}^{2} q_{0} \\
& -a_{3} \sin \omega_{1} \tau\left(D_{0} q_{0}\right)^{2}-\left(a_{41}+a_{42} \sin \omega_{1} \tau\right) q_{0} D_{0}^{2} q_{0} \\
& -\left(a_{51}+a_{52} \cos \omega_{1} \tau\right) q_{0}\left(D_{0} q_{0}\right)^{2} \\
& -\left(a_{61}+a_{62} \omega_{1} \sin \omega_{1} \tau\right) q_{0}^{2} D_{0} q_{0} \\
& -\left(a_{71}+a_{72} \cos \omega_{1} \tau\right) q_{0}^{2} D_{0}^{2} q_{0}-f \omega_{1}^{2} \sin \omega_{1} \tau \tag{17}
\end{align*}
$$

here, $\omega_{1}=\dot{\theta} / \omega_{r e f}$ which can be called the frequency of excitation caused by the rotation of the manipulator.

The solution to Eq. (16) takes the form

$$
\begin{equation*}
q_{0}=A\left(T_{1}\right) e^{i T_{0}}+\bar{A}\left(T_{1}\right) e^{-i T_{0}} \tag{18}
\end{equation*}
$$

where $i$ is the imaginary unit, $A$ is the complex amplitude, and $\bar{A}$ is its complex conjugate.

Substituting Eq. (18) into (17) and using Euler's formula and grouping the exponential terms, we obtain

$$
\begin{align*}
& D_{0}^{2} q_{1}+q_{1}=S T_{1} e^{i T_{0}}+S T_{2} e^{i \omega_{1} T_{0}}+S T_{3} e^{i\left(1-\omega_{1}\right) T_{0}} \\
& \quad+S T_{4} e^{i\left(2-\omega_{1}\right) T_{0}}+S T_{5} e^{i\left(3-\omega_{1}\right) T_{0}}+N S T+c c \tag{19}
\end{align*}
$$

where $S T_{1}, S T_{2}, \ldots, S T_{5}$ and $N S T$ represents the secular and non-secular generating terms of first-order, respectively, which can be obtained by

$$
\begin{align*}
S T_{1}= & -2 i D_{1} A-2 \mu i A-3 H_{3} A^{2} \bar{A} \\
& -a_{51} A^{2} \bar{A}-i a_{61} A^{2} \bar{A}+3 a_{71} A^{2} \bar{A}  \tag{20}\\
S T_{2}= & i h_{2} \omega_{1}^{2} A \bar{A}+a_{3} i A \bar{A}-a_{42} i A \bar{A}+\frac{1}{2} f \omega_{1}^{2} i  \tag{21}\\
S T_{3}= & \frac{1}{2}\left(-h_{1} \omega_{1}^{2} \bar{A}-3 h_{3} \omega_{1}^{2} A \bar{A}^{2}+a_{1} \omega_{1} \bar{A}+a_{2} \bar{A}\right. \\
& \left.-a_{52} A \bar{A}^{2}+a_{62} \omega_{1} A \bar{A}^{2}+3 a_{72} A \bar{A}^{2}\right)  \tag{22}\\
S T_{4}= & \frac{1}{2} i\left(-h_{2} \omega_{1}^{2}+a_{3}+a_{42}\right) A^{2}  \tag{23}\\
S T_{5}= & \frac{1}{2}\left(-h_{3} \omega_{1}^{2}+a_{52}+a_{62} \omega_{1}+a_{72}\right) A^{3}  \tag{24}\\
N S T= & a_{41} A \bar{A}+a_{41} A^{2} e^{2 i T_{0}}+\left(-H_{3} A^{3}+a_{51} A^{3}\right. \\
& \left.-a_{61} i A^{3}+a_{71} A^{3}\right) e^{3 i T_{0}} \frac{1}{2}\left(-h_{1} \omega_{1}^{2} A-3 h_{3} \omega_{1}^{2} A^{2} \bar{A}\right. \\
& \left.-a_{1} \omega_{1} A+a_{2} A-a_{62} \omega_{1} A^{2} \bar{A}+a_{72} A^{2} \bar{A}\right) e^{i\left(1+\omega_{1}\right) T_{0}} \\
& +\frac{1}{2} i\left(h_{2} \omega_{1}^{2}-a_{4}-\frac{1}{2} a_{5,2}\right) A^{2} e^{i\left(2+\omega_{1}\right) T_{0}} \\
& +\frac{1}{2}\left(-h_{3} \omega_{1}^{2}+a_{52}-a_{62} \omega_{1}+a_{72}\right) A^{3} e^{i\left(3+\omega_{1}\right) T_{0}} \tag{25}
\end{align*}
$$

One may observe that any solution of Eq. (19) will contain secular or small divisor terms when $\omega_{1} \approx 1, \omega_{1} \approx 2, \omega_{1} \approx$ 3 and $\omega_{1} \approx 4$. From Eq. (19), when the frequency of excitation $\omega_{1}$ is near equal to the linear natural frequency of system $\omega_{r e f}$, the nonlinear forced and parametrically excitation terms will lead to the primary resonance. In addition, when the frequency of excitation $\omega_{1}$ is near equal to 2,3 or 4 times of the linear natural frequency of the system $\omega_{\text {ref }}$, i. e., $\omega_{1} \approx 2$, $\omega_{1} \approx 3$ and $\omega_{1} \approx 4$, the nonlinear parametrically excitation terms will lead to the sub-harmonic resonances. These two cases will be investigated in Section 3.1 and 3.2, respectively.

## A. PRIMARY RESONANCE

In this part, the case when the frequency of excitation $\omega_{1}$ near equal to the linear natural frequency of system $\omega_{\text {ref }}$, i. e., $\omega_{1} \approx$ 1 , is considered to be the primary resonance.

A detuning parameter $\sigma$ is introduced which quantitatively describes the nearness of $\omega_{1}$ to 1 and can be expressed as

$$
\begin{equation*}
\omega_{1}=1+\varepsilon \sigma, \sigma=O(1) \tag{26}
\end{equation*}
$$

Substituting Eq. (26) into (19) and one can obtained a secular terms as

$$
\begin{align*}
& -2 i D_{1} A-2 \mu i A-3 H_{3} A^{2} \bar{A}-a_{51} A^{2} \bar{A}-i a_{61} A^{2} \bar{A} \\
& +3 a_{71} A^{2} \bar{A}+i\left(h_{2} \omega_{1}^{2} A \bar{A}+a_{3} A \bar{A}-a_{42} A \bar{A}+\frac{1}{2} f \omega_{1}^{2}\right) e^{i \sigma T_{1}} \\
& +\frac{1}{2} i\left(-h_{2} \omega_{1}^{2}+a_{3}+a_{42}\right) A^{2} e^{-i \sigma T_{1}}=0 \tag{27}
\end{align*}
$$

The secular term must be eliminated since it results in an unbounded growth of the response of the model, which is
inconsistent with the physical system. The complex amplitude $A$ can be assumed as:

$$
\begin{equation*}
A=\frac{1}{2} a\left(T_{1}\right) e^{i \beta\left(T_{1}\right)} \tag{28}
\end{equation*}
$$

in which $a$ and $\beta$ are the steady state amplitude and phases of the motion. Also, $a$ and $\beta$ are the real functions of $T_{1}$.

Considering $\vartheta=\sigma T_{1}-\beta$ and separating the real and imaginary parts of Eq. (27) would give:

$$
\begin{equation*}
a^{\prime}=-\mu a-\frac{1}{8} a_{61} a^{3}+\frac{1}{8}\left(h_{2} \omega_{1}^{2} a^{2}+3 a_{3} a^{2}+4 f \omega_{1}^{2}\right) \cos \vartheta \tag{29}
\end{equation*}
$$

$$
\begin{align*}
a \vartheta^{\prime}= & a \sigma+\frac{1}{8}\left(3 a_{71}-3 H_{3}-a_{51}\right) a^{3} \\
& -\frac{1}{8}\left(3 h_{2} \omega_{1}^{2} a^{2}+a_{3} a^{2}+4 f \omega_{1}^{2}\right) \sin \vartheta \tag{30}
\end{align*}
$$

where the prime denotes differentiation with respect to $T_{1}$. One may observe that the Eq. (29) (30) only have nontrivial solutions. The steady state response $\left(a_{0}, \vartheta_{0}\right)$ can be obtained by making $a^{\prime}=\gamma^{\prime}=0$. The amplitude-frequency relationship can be solved by the algebraic equation which obtained by eliminating the $\gamma$ from Eq. (29) (30).
In addition, the stability of the steady state response can be determined by investigating the eigenvalues of the Jacobian matrix obtained by perturbing (29) and (30), then the Jacobian matrix $\boldsymbol{J}_{\mathbf{1}}$ can be written as follow:

$$
\begin{align*}
J_{1}= & {\left[\begin{array}{ll}
J_{1,11} & J_{1,12} \\
J_{1,21} & J_{1,22}
\end{array}\right] } \\
J_{1,11}= & -\mu-\frac{3}{8} a_{61} a_{0}^{2}+\frac{1}{4}\left(h_{2} \omega_{1}^{2}+3 a_{3}\right) a_{0} \cos \vartheta_{0} \\
J_{1,12}= & -\frac{1}{8}\left(h_{2} \omega_{1}^{2} a^{2}+3 a_{3} a^{2}+4 f \omega_{1}^{2}\right) \sin \vartheta_{0} \\
J_{1,21}= & \sigma-\frac{3}{8}\left(3 a_{71}-3 H_{3}-a_{51}\right) a_{0}^{2} \\
& -\frac{1}{4}\left(3 h_{2} \omega_{1}^{2}+a_{3}\right) a_{0} \sin \vartheta_{0} \\
J_{1,22}= & -\frac{1}{8}\left(3 h_{2} \omega_{1}^{2} a^{2}+a_{3} a^{2}+4 f \omega_{1}^{2}\right) \cos \vartheta_{0} \tag{31}
\end{align*}
$$

The stability of the steady state solutions of the system are now decided by the nature of eigenvalues of matrixes of Eq. (31). If all the eigenvalues have negative or zero real parts, the steady state solutions are stable. Finally, the first approximation solution in steady state can be expressed as

$$
\begin{equation*}
q=a \cos (\tau+\varepsilon \sigma \tau-\vartheta)+O(\varepsilon) \tag{32}
\end{equation*}
$$

## B. SUB-HARMONIC RESONANCES

To be economically feasible, the rotational speed of the manipulator has to be reasonably fast. On the other hand, the linear natural frequency $\omega_{\text {ref }}$ of highly flexible truss foundation may be very low. In such a case, the coupling behavior between the low-frequency structures and high-frequency robot may be excited high order resonance conditions. In this part, the sub-harmonic resonance is investigated when the excitation frequency caused by the uniform rotation of the manipulator nearly equal to 2,3 and 4 times of the system.

To analysis the first condition of super-harmonic resonances, one could introduce the detuning parameter to be:

$$
\begin{equation*}
\omega_{1}=2+\varepsilon \sigma, \sigma=O(1) \tag{33}
\end{equation*}
$$

Substituting Eq. (33) into Eq. (19) and the secular term is obtained as

$$
\begin{align*}
& -2 i D_{1} A-2 \mu i A-3 H_{3} A^{2} \bar{A}-i a_{61} A^{2} \bar{A}+3 a_{71} A^{2} \bar{A} \\
& +\frac{1}{2}\left(-h_{1} \omega_{1}^{2} \bar{A}-3 \hbar_{3} \omega_{1}^{2} A \bar{A}^{2}+a_{1} \omega_{1} \bar{A}+a_{2} \bar{A}\right. \\
& \left.-a_{52} A \bar{A}^{2}+a_{62} \omega_{1} A \bar{A}^{2}+3 a_{72} A \bar{A}^{2}\right) e^{i \sigma T_{1}} \\
& +\frac{1}{2}\left(-h_{3} \omega_{1}^{2}+a_{52}+a_{62} \omega_{1}+a_{72}\right) A^{3} e^{-i \sigma T_{1}}=0 \tag{34}
\end{align*}
$$

We assumed $\vartheta=\sigma T_{1}-2 \beta$ and separate the real and imaginary parts of Eq. (34) as

$$
\begin{align*}
a^{\prime}= & -\mu a-\frac{1}{8} a_{61} a^{3}+\frac{1}{8}\left(2\left(-h_{1} \omega_{1}^{2}+a_{1} \omega_{1}+a_{2}\right) a\right. \\
& \left.+\left(-h_{3} \omega_{1}^{2}-a_{52}+a_{72}\right) a^{3}\right) \sin \vartheta  \tag{35}\\
a \vartheta^{\prime}= & a \sigma+\frac{3}{8}\left(3 a_{71}-3 H_{3}-a_{51}\right) a^{3} \\
& +\frac{1}{4}\left(2\left(-h_{1} \omega_{1}^{2}+a_{1} \omega_{1}+a_{2}\right) a\right. \\
& \left.+\left(-2 h_{3} \omega_{1}^{2}+a_{62} \omega_{1}+2 a_{72}\right) a^{3}\right) \cos \vartheta \tag{36}
\end{align*}
$$

Unlikely with primary resonance which only have trivial solution in Eq. (29) and (30), it is worth noting that the Eq. (35), (36) have both trivial and nontrivial responses. The $a^{\prime}$, $\gamma^{\prime}$ are assumed to be zeros to find the steady-state solution of the system. Similar to the primary resonance, the Jacobian matrix can be used to determine its stability and can be written as

$$
\begin{align*}
J_{2}= & {\left[\begin{array}{ll}
J_{2,11} & J_{2,12} \\
J_{2,21} & J_{2,22}
\end{array}\right] } \\
J_{2,11}= & -\mu-\frac{3}{8} a_{61} a_{0}^{2}+\frac{1}{8}\left(-2 h_{1} \omega_{1}^{2}+2 a_{1} \omega_{1}\right. \\
& \left.+2 a_{2}+3\left(-h_{3} \omega_{1}^{2}-a_{52}+a_{72}\right) a_{0}^{2}\right) \sin \vartheta_{0} \\
J_{2,12}= & \frac{1}{8}\left(2\left(-h_{1} \omega_{1}^{2}+a_{1} \omega_{1}+a_{2}\right) a_{0}\right. \\
& \left.+\left(-h_{3} \omega_{1}^{2}-a_{52}+a_{72}\right) a_{0}^{3}\right) \cos \vartheta_{0} \\
J_{2,21}= & \sigma+\frac{9}{8}\left(3 a_{71}-3 H_{3}-a_{51}\right) a_{0}^{2}+\frac{1}{4}\left(-2 h_{1} \omega_{1}^{2}+2 a_{1} \omega_{1}\right. \\
& \left.+2 a_{2}+3\left(-2 h_{3} \omega_{1}^{2}+a_{62} \omega_{1}+2 a_{72}\right) a_{0}^{2}\right) \cos \vartheta_{0} \\
J_{2,22}= & \frac{1}{8}\left(2\left(-h_{1} \omega_{1}^{2}+a_{1} \omega_{1}+a_{2}\right) a_{0}\right. \\
& \left.+\left(-h_{3} \omega_{1}^{2}-a_{52}+a_{72}\right) a_{0}^{3}\right) \sin \vartheta_{0} \tag{37}
\end{align*}
$$

Then, the first approximation solution of super-harmonic resonances of the system can be expressed as

$$
\begin{equation*}
q_{0}=a \cos \left(\frac{1}{2}\left(\omega_{1} \tau-\vartheta\right)\right)+O(\varepsilon) \tag{38}
\end{equation*}
$$

The second super-harmonic resonance is then captured when $\omega_{1} \approx 3$. For this case, the detuning parameter and the corresponding secular term are derived as

$$
\begin{equation*}
\omega_{1}=3+\varepsilon \sigma_{1}, \sigma_{1}=O(1) \tag{39}
\end{equation*}
$$



FIGURE 8. Frequency response of beam for super-harmonic case ( $\omega_{1} \approx 2$ ) with different $\bar{m}_{I}$ and $\bar{s}_{\boldsymbol{b}} ; \boldsymbol{c}=\mathbf{0} .15$.

$$
\begin{gather*}
-2 i D_{1} A-2 \mu i A-3 H_{3} A^{2} \bar{A}-i a_{61} A^{2} \bar{A}+3 a_{71} A^{2} \bar{A} \\
\quad+\frac{1}{2} i\left(-h_{2} \omega_{1}^{2}+a_{3}+a_{42}\right) \bar{A}^{2} e^{i \sigma T_{1}}=0 \tag{40}
\end{gather*}
$$

Substituting $A=\frac{1}{2} a\left(T_{1}\right) e^{i \beta\left(T_{1}\right)}$ into Eq. (40) and separating the real and imaginary parts, one obtains

$$
\begin{align*}
a^{\prime}= & -\mu a-\frac{1}{8} a_{61} a^{3}+\frac{1}{8}\left(-h_{2} \omega_{1}^{2}+a_{3}+a_{42}\right) a^{2} \cos \vartheta  \tag{41}\\
a \dot{\vartheta}= & a \sigma+\frac{3}{8}\left(3 a_{71}-3 H_{3}-a_{51}\right) a^{3} \\
& -\frac{3}{8}\left(-h_{2} \omega_{1}^{2}+a_{3}+a_{42}\right) a^{2} \sin \vartheta \tag{42}
\end{align*}
$$

here, $\vartheta=\sigma_{1} T_{1}-3 \beta$ and one can observe that the solution for the second super-harmonic resonance have both trivial and nontrivial solutions. Similarly, Jacobian matrix and the first approximation solution of second super-harmonic resonances can be expressed as

$$
J_{3}=\left[\begin{array}{ll}
J_{3,11} & J_{3,12} \\
J_{3,21} & J_{3,22}
\end{array}\right]
$$

$J_{3,11}=-\mu-\frac{3}{8} a_{61} a_{0}^{2}+\frac{1}{4}\left(-h_{2} \omega_{1}^{2}+a_{3}+a_{42}\right) a_{0} \cos \vartheta_{0}$
$J_{3,12}=-\frac{1}{8}\left(-h_{2} \omega_{1}^{2}+a_{3}+a_{42}\right) a_{0}^{2} \sin \vartheta_{0}$
$J_{3,21}=\sigma+\frac{9}{8}\left(3 a_{71}-3 H_{3}-a_{51}\right) a_{0}^{2}$

$$
\begin{align*}
& -\frac{6}{8}\left(-h_{2} \omega_{1}^{2}+a_{3}+a_{42}\right) a_{0} \sin \vartheta_{0} \\
J_{3,22}= & -\frac{3}{8}\left(-h_{2} \omega_{1}^{2}+a_{3}+a_{42}\right) a_{0}^{2} \cos \vartheta_{0} \\
q_{0}= & a \cos \left(\frac{1}{3}\left(\omega_{1} \tau-\vartheta\right)\right)+O(\varepsilon) \tag{44}
\end{align*}
$$

In similar approach, the corresponding results could be obtained for the third super-harmonic resonance when $\omega_{1} \approx 4$

$$
\begin{align*}
& \omega_{1}=4+\varepsilon \sigma_{1}, \sigma_{1}=O(1)  \tag{45}\\
& -2 i D_{1} A-2 \mu i A-3 H_{3} A^{2} \bar{A}-i a_{61} A^{2} \bar{A}+3 a_{71} A^{2} \bar{A} \\
& \quad+\frac{1}{2}\left(-h_{3} \omega_{1}^{2}+a_{52}+a_{62} \omega_{1}+a_{72}\right) A^{3} e^{i \sigma_{1} T_{1}}=0 \tag{46}
\end{align*}
$$

Separating the real and imaginary parts and considering $\vartheta=\sigma_{1} T_{1}-4 \beta$, one can obtain

$$
\begin{align*}
\dot{a}= & -\mu a-\frac{1}{8} a_{61} a^{3} \\
& +\frac{1}{16}\left(-h_{3} \omega_{1}^{2}+a_{52}+a_{62} \omega_{1}+a_{72}\right) a^{3} \sin \vartheta  \tag{47}\\
a \dot{\vartheta}= & a \sigma-\frac{1}{2}\left(3 H_{3}+a_{51}-3 a_{71}\right) a^{3} \\
& +\frac{1}{4}\left(-h_{3} \omega_{1}^{2}+a_{52}+a_{62} \omega_{1}+a_{72}\right) a^{3} \cos \vartheta \tag{48}
\end{align*}
$$

Similar to the previous case, here for both trivial and nontrivial responses, the system will be stable if the real part


FIGURE 9. Frequency response of beam for super-harmonic case ( $\omega_{1} \approx 2$ ) with different $c ; \bar{m}_{I}=0.4$ and $\bar{s}_{b}=0.5$.
of all the eigenvalues of the Jacobian matrix is negative. The Jacobian matrix is given by

$$
\begin{align*}
J_{4}= & {\left[\begin{array}{ll}
J_{4,11} & J_{4,12} \\
J_{4,21} & J_{4,22}
\end{array}\right] } \\
J_{4,11}= & -\mu-\frac{3}{8} a_{61} a_{0}^{2} \\
& +\frac{3}{16}\left(-h_{3} \omega_{1}^{2}+a_{52}+a_{62} \omega_{1}+a_{72}\right) a_{0}^{2} \sin \vartheta_{0} \\
J_{4,12}= & \frac{1}{16}\left(-h_{3} \omega_{1}^{2}+a_{52}+a_{62} \omega_{1}+a_{72}\right) a_{0}^{3} \cos \vartheta_{0} \\
J_{4,12}= & \sigma-\frac{3}{2}\left(3 H_{3}+a_{51}-3 a_{71}\right) a_{0}^{2} \\
& -\frac{3}{4}\left(-h_{3} \omega_{1}^{2}+a_{52}+a_{62} \omega_{1}+a_{72}\right) a_{0}^{2} \cos \vartheta_{0} \\
J_{4,12}= & -\frac{1}{4}\left(-h_{3} \omega_{1}^{2}+a_{52}+a_{62} \omega_{1}+a_{72}\right) a_{0}^{3} \sin \vartheta_{0} \tag{49}
\end{align*}
$$

Then, the first-order approximation solution of third superharmonic resonances can be obtained

$$
\begin{equation*}
q_{0}=a \cos \left(\frac{1}{4}\left(\omega_{1} \tau-\vartheta\right)\right)+O(\varepsilon) \tag{50}
\end{equation*}
$$

## IV. NUMERICAL RESULTS AND DISCUSSIONS

In present work, a metal beam with length $L=10 \mathrm{~m}$, width $b=0.5 \mathrm{~m}$, depth $h=0.04 \mathrm{~m}$, the Elastic modulus $E=1.62 \times 10^{11} \mathrm{~N} / \mathrm{m}^{2}$ and mass density $\rho=$ $7830 \mathrm{~kg} / \mathrm{m}^{3}$ is considered to simulate highly flexible slender truss-structure. The scaling factor $r$ is assumed to be 0.1 . The non-linear response, stability and bifurcations for primary and sub-harmonic resonance conditions with varying system parameters, namely the mass of link $\bar{m}_{l}$, the mounting position of the rigid link on the beam $\bar{s}_{b}$ and damping of beam $c$ (here, we assumed $\bar{m}_{b}=0.5 \bar{m}_{l}$ and $\bar{p}=0.5$ for convenient calculation), are investigated. Although low-damping characteristics exists in the space environment, the damping of flexible base can be achieved by installing damping structure on trusses. In the following subsection, the simple resonance
condition is discussed in Section 4.1. Sub-harmonic resonance condition is studied in Section 4.2. In all frequency response curves, the real line contains a stable solution, and the imaginary line represents an unstable solution.

## A. PRIMARY RESONANCE

This section presents the simple resonance when the frequency of excitation $\omega_{1}$ is near equal to the linear natural frequency of the system $\omega_{r e f}$, i. e. $\omega_{1} \approx 1$, has to be investigated.

Fig. 2 shows the dynamic response of flexible beam with rigid manipulator when $\bar{s}_{b}=0.5, \bar{m}_{l}=0.2$ and $c=0.15 \mathrm{~N} / \mathrm{s}$. Comparing with amplitude-frequency response curve without considering the effect of slope [12], the curve obtained by presented method exhibits spring softening behavior of the steady-state response. One can observe that the manipulator will eventually oscillate with a steady-state response corresponding to the excitation frequency $\left(\omega_{1}\right)$ since there is no trivial solution. In addition, the steady-state amplitude of the system will vary as: $A \rightarrow B \rightarrow C$ as the frequency increasing. Meanwhile, it is experienced a jump down phenomenon at point $B$ which may lead to system failure. If the system does not fail at point $B$, the vibration amplitude will suddenly change to point $C$ and then with further increase of frequency, the amplitude will decrease. Conversely, if the frequency decreases, the steady-state amplitude of the system will experience from the point $C \rightarrow D \rightarrow E \rightarrow A$. A jump up phenomena of saddle-node (SN) bifurcation point $D$ may occur when $\omega_{1}=1.078$.

In order to evaluate the accuracy of the perturbation results, the response curve obtained by the method of multiple scales can be compared with the time response obtained by numerically solving the Eq. (29) and (30). Fig. 3 shows the time responses, phase portraits and Poincare' map corresponding to points $Q, M$ and $N$ marked in Fig.2. It can be observed that the steady-state response obtained by the fourth-order Runge-Kutta method is in good agreement with the results determined by the multiple scales method.

In addition, the bistable region exists after the $\mathrm{S}-\mathrm{N}$ bifurcation point $D$, which means a single frequency value has two stable response amplitudes as shown in Fig.2. In this area, the final steady state response of the system is determined by the initial condition and a slight change of initial conditions may cause a significant change in the response amplitude of the structure [19]. Therefore, it is necessary to investigate the relationship between the steady state amplitude of the system and different initial conditions. The basins of attraction with $\omega_{1}=1$ and $\omega_{1}=1.2$ are shown in Fig. 4 (i) and (ii), respectively. One can be observed that there has only one steady state responses when $\omega_{1}=1$, however, a bistable region exists(i.e., two stable response amplitudes) when $\omega_{1}=$ 1.2 which can be obtained by solving Eq. (29) (30) using numerically solving methods and such phenomenon also can be confirmed by the time response of points $Q, M, N$ in Fig.3.


FIGURE 10. Time responses, phase portraits and Poincare's map for the point $A, M, H$ and $U$ marked in Fig.8(ii).


FIGURE 11. Frequency response of beam for super-harmonic case ( $\omega_{1} \approx 3$ ) with different, $\bar{m}_{I} \overline{\boldsymbol{s}}_{\boldsymbol{b}}$ and $\boldsymbol{c}$.

The parameter sensitivity analysis of the primary resonance is carried out and the influence of different parameters on the steady-state responses are studied in Figs.5, 6 and 7.

Fig. 5 shows the frequency response curve for three different values of viscous damping of the beam with mass of rigid manipulator ratio $\bar{m}_{l}$ and installation position of manipulator


FIGURE 12. Time response, phase portraits and Poincare's map corresponding to point $A, B, C$ and $D$ key as in Fig. 11 (iv).
ratio $\bar{s}_{b}$ equal to 0.4 and 0.5 , respectively. Increasing the level of damping will reduce the vibration amplitude of the beam. Similarly, Fig.6, 7 represent the frequency response of beam for primary resonance case with different $\bar{m}_{l}$ and $\bar{s}_{b}$, respectively. One can observe that increasing viscous damping of the beam, decreasing the mass of the manipulator and making the distance of installation position of manipulator close to fixed end tend to alleviate the nonlinear effect of system.

## B. SUB-HARMONIC RESONANCE

In this case, the sub-harmonic resonance is investigated when the excitation frequency caused by the uniform rotation of the manipulator nearly equal to 2,3 and 4 times of the system.

Fig. 8 and Fig. 9 show variations in the frequency response of curves depicted in series for super-harmonic case ( $\omega_{1} \approx 2$ ) with different manipulator mass ratio $\bar{m}_{l}$, installation position of manipulator ratio $\bar{s}_{b}$ and damping coefficient of beam $c$. Unlike primary resonance, the system here has both trivial and nontrivial responses. In addition, two subcritical pitchfork bifurcations are existed at points $S P_{1}$ and $S P_{2}$. As shown in Fig. 8 (ii), the system response of this kind can be classified into three different regions: (a) the system consistently
exhibits a stable equilibrium region (trivial response) when $\omega_{1}<\omega_{S P 1}$ (zone I), (b) stationary oscillation region for $\omega_{S P 1}<\omega_{1}<\omega_{S P 2}$ (zone II) and (c) bistable regions for $\omega_{1}>\omega_{S P 2}$ (zone III), where $\omega_{S P 1}$ and $\omega_{S P 2}$ are the excitation frequency value at points $S P_{1}$ and $S P_{2}$, respectively.

In the stable equilibrium region, i. e., zone $I$, the system will calm down to zero solution stability regardless of the initial value. As the $\omega_{1}$ increases, trivial solution loses its stability and enters a branch of stable nontrivial solution through subcritical pitchfork bifurcation point $S P_{1}$. In the stationary oscillation region, there exist an unstable trivial solution and a stable non-trivial solution, which makes the system oscillate regardless of the initial state (or any sudden disturbance). After this short transition, conditionally static equilibrium state or stationary periodic oscillation will occur according to the initial state. Therefore, only under the sudden disturbance of $\omega_{1}>\omega_{S P 1}$, the system will have steady state oscillation.

Like in the previous case, the time responses, phase portraits and Poincare' map under different excitation frequencies and initial conditions are obtained by numerically solving the Eq. (35) and (36). Fig. 10 shows the time response, phase portraits and Poincare' map for the point $A, M, H$ and $U$


FIGURE 13. Frequency response of beam for super-harmonic case ( $\omega_{1} \approx 4$ ) with different, $\bar{m}_{\boldsymbol{l}} \overline{\boldsymbol{s}}_{\boldsymbol{b}}$ and $\boldsymbol{c}$.
marked in Fig.8(ii) with $\bar{m}_{l}=0.8$ and $\bar{s}_{b}=0.5$ for different initial conditions. One can be observed that point $A$ in region I eventually tends to zero solution stability. The point $M$ in region II exhibits supercritical pitchfork bifurcation and eventually oscillate with a steady-state vibration amplitude at point $N$. The system response is going to a static equilibrium state (zero solution) when the initial conditions at point $U$ (initial displacement and initial velocity, etc.) are small and the system response jumps to the limit of non-trivial branch vibration curve when the initial conditions (point $H$ ) are large which suggests that the completely different steadystate responses may be obtained when some initial conditions have slight deviations in zone III.

The parameter sensitivity study is also carried out for the sub-harmonic resonance $\left(\omega_{1} \approx 2\right)$ and the effects of different parameters on the steady-state responses are investigated. From Fig.8, one can also observe that increasing the mass level of rigid manipulator, as well as increasing the distance between the installation position and the fixed end will increase the zone II which may lead to catastrophic failure of the system. However, Fig. 9 shows that the frequency response of the beam is weakly affected by damping.

Fig. 11 shows variations in the frequency response of curves depicted in series for super-harmonic case ( $\omega_{1} \approx$ 3) with different manipulator mass ratio $\bar{m}_{l}$ and installation position of manipulator ratio $\bar{s}_{b}$ for $c=0.15 \mathrm{~N} / \mathrm{m}$ and $\bar{p}=0.5$. As shown in Fig.11(i), there exist both trivial and nontrivial solutions and the trivial solutions is always stable. The system response of this kind can be classified into two different
regions by the saddle node bifurcation point ( $S N$ ): (a) unconditionally static equilibrium state (only trivial solution) when $\omega_{1}<\omega_{S N}$ in zone I and (b) conditionally static equilibrium state or stationary periodic oscillation (bistable regions both have trivial and nontrivial solution) depending on the initial state when $\omega_{1}>\omega_{S N}$ in zone II. From Fig.11, the jump down phenomenon can be observed at point SN with such a frequency response topology. Obviously, the steady-state oscillation of the system is transmitted only in region II, i.e., $\omega_{1}>\omega_{S N}$ where $\omega_{S N}$ is the excitation frequency value at points $S N$.

From Fig.11, one may observe that with decreasing $\bar{m}_{l}$ and $\bar{s}_{b}$, as well as increasing $c$, the frequency value saddle of node bifurcation point is shifted away from 3, which means small mass can achieve larger unconditionally static equilibrium region, i.e., zone I. However, it may be noted that the distance of trivial and non-trivial curves increases in which the system may easier to fail due to sudden jump at points SN from non-trivial branch to static equilibrium state when decrease of velocity of rigid link. Although a high level of damping is helpful for structural vibration attenuation.

From Fig.12, the vibration behavior of typical working conditions for super-harmonic case ( $\omega_{1} \approx 3$ ) can be observed by the time response, phase portraits and Poincare's map marked at points $A, B, C$ and $D$ in Fig.11(iv) with different initial conditions by solving the Eq. (41) and (42). It is observed that for the point $A\left(\omega_{1}=2.9\right)$, the trajectories in the phase plane goes toward the trivial fixed point with time
tends to infinity which also verify that vibration will decay to static equilibrium regardless of initial conditions in the zone I. For critical point $B$, the transient response of the system with initial condition $(a(0)=0.4177, \vartheta(0)=3)$ gives a beating type phenomenon in the early stage and after a long period of time eventually reached stable periodic oscillations. One can also find that the transition process for system, from initial conditions to the stationary periodic oscillation near the critical point, is more complex than that far away from the critical point. Similar to super-harmonic case $\left(\omega_{1} \approx\right.$ 2 ), the trajectories of the point $C$ and $D$ at $\omega_{1} \approx 3.2(\mathrm{in}$ bistable regions) with different initial conditions go toward the static equilibrium state and stationary periodic oscillation, respectively.

The frequency response curves of super-harmonic case $\left(\omega_{1} \approx 4\right)$ with different manipulator ratio $\bar{m}_{l}$ and installation position of manipulator ratio $\bar{s}_{b}$ are also depicted in Fig.13. One can observe that the shape of super-harmonic ( $\omega_{1} \approx 4$ ) frequency response curves is similar to the $\omega_{1} \approx 3$. The response of system can also be divided into two different regions according to saddle node bifurcation point (SN): (a) unconditionally static equilibrium state and (b) conditionally static equilibrium state or stationary periodic oscillation, depending on the initial state. In addition, decreasing the level of $\bar{m}_{l}$ and $\bar{s}_{b}$ decreases, as well as, increasing $c$, the saddle node bifurcation point is located not only at larger amplitude, but also at higher frequency. As a consequence, the possible jump down phenomenon of the stable non-trivial response may occur at a higher frequency and have a larger jump down height for small mass manipulator or the installation position closing to the fixed end at a large damping case or with a high damping.

## V. CONCLUSION

Nonlinear dynamic of a flexible slender truss-structure mounted manipulator for on-orbit assembly, which can be simplified as a beam-rotating link interaction system, is theoretically investigated for primary and sub-harmonic resonance. The PDEs of system is established by using the D'Alembert principle incorporated with the moment balance method in which the slope of beam is considered. Such system is a typical parametrically excited system. The multiple scales method is used to solve the second-order ODEs which is reduced by the Galerkin's method with a single mode approach from obtained PDEs. Then, the non-linear response, stability and bifurcations for primary and all sub-harmonic resonance conditions have been investigated by varying system parameters. By inspecting the results of this analysis, the following conclusions could be made:
1). When the excitation frequency of manipulator $\omega_{1}$ near equal to the linear natural frequency of flexible trussstructure $\omega_{r e f}$, there will exist simple resonance condition and only have nontrivial steady state solution. The system resonance could be classified into monostable and bistable region by saddle-node ( SN ) bifurcation point. The steady state responses of the system in bistable region are determined
by the initial condition and a slight change of initial conditions may cause a significant change. On the other hand, increasing the level of damping, as well as decreasing the mass of rigid manipulator and installation position of manipulator close to fixed end will reduce steady state responses of beam.
2). Due to low-frequency flexible truss-structures and highfrequency manipulator, parametric excitations of manipulator may excite high order resonance conditions of flexible trussstructures. For the first sub-harmonic resonance case ( $\omega_{1} \approx$ 2), the response has both trivial and non-trivial solution. In addition, the system response at steady-state could be classified into three different regions though two subcritical pitchfork bifurcation points: unconditionally stable equilibrium region (zone I), unconditionally stationary periodic oscillation (zone II) and conditionally static equilibrium state or stationary periodic oscillation (zone III). Moreover, one can observe that increasing the $m_{l}$ and $s_{b}$ will increase the zone II which may lead to catastrophic failure of the system. The frequency response of the beam is weakly affected by damping.
3). For the second and third sub-harmonic resonance case ( $\omega_{1} \approx 3$ and $\omega_{1} \approx 4$ ), trivial response existed along with nontrivial response. The trivial solution is always stable and the steady-state response of system could be classified into two regions by the saddle node bifurcation point (SN): (a) unconditionally static equilibrium state and (b) conditionally static equilibrium state or stationary periodic oscillation (bistable region). In addition, decreasing the level of $\bar{m}_{l}$ and $\bar{s}_{b}$, as well as, increasing $c$, the saddle node bifurcation point is located not only at larger amplitude, but also at higher frequency. As a consequence, the possible jump down phenomenon of the stable non-trivial response can occur at a higher frequency and have a larger jump down height for small mass manipulator or the installation position is close to the fixed end. Although a high level of damping is helpful for structural vibration attenuation.

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## APPENDIX A

$$
\begin{aligned}
\mu & =\frac{1}{2 \varepsilon} \mu_{1}, H_{3}=\frac{1}{\varepsilon}\left(H_{31}+H_{32}+H_{33}\right), \\
h_{1} & =\frac{1}{\varepsilon} h_{11} \cos \theta \dot{\theta}^{2}, \\
h_{2} & =\frac{1}{\varepsilon} h_{21} \sin \theta \dot{\theta}^{2}, h_{3}=\frac{1}{\varepsilon} h_{31} \cos \theta \dot{\theta}^{2}, a_{1}=\frac{1}{\varepsilon} a_{11} \sin \theta \dot{\theta}^{2}, \\
\alpha_{2} & =\frac{1}{\varepsilon} \alpha_{21} \sin \theta \dot{\theta}, \alpha_{3}=\frac{1}{\varepsilon} \alpha_{31} \cos \theta, \alpha_{4}=\frac{1}{\varepsilon} \alpha_{41} \sin \theta, \\
\alpha_{5} & =\frac{1}{\varepsilon}\left(\alpha_{51}+\alpha_{52} \sin \theta\right), \alpha_{6}=\frac{1}{\varepsilon}\left(\alpha_{61}+\alpha_{62} \sin \theta \dot{\theta}\right), \\
\alpha_{7} & =\frac{1}{\varepsilon}\left(\alpha_{71}+\alpha_{72} \cos \theta\right), f=\frac{1}{\varepsilon} f_{1} \sin \theta \dot{\theta}^{2},
\end{aligned}
$$

$$
\begin{aligned}
& \bar{m}_{b l}=\bar{m}_{b}+\bar{m}_{l} \\
& \omega_{\text {ref }}=\sqrt{\left(\frac{E I}{\rho A L^{4}}\right) \frac{H_{1}}{H_{21}+\bar{m}_{b l} H_{22}}} \\
& H_{21}=\int_{0}^{1}(\varphi(\bar{s}))^{2} d \bar{s}, H_{22}=\int_{0}^{1}(\varphi(\bar{s}))^{2} \delta\left(\bar{\zeta}-\bar{s}_{b}\right) d \bar{s} \\
& \mu_{1}=\left(\frac{1}{H_{21}+\bar{m}_{b l} H_{22}}\right)\left(\frac{c}{\rho A \omega_{r e f}}\right) \int_{0}^{1}(\varphi(\bar{s}))^{2} d \bar{s} \\
& H_{31}=\frac{1}{2} \frac{E I}{\rho A \omega_{r e f}^{2} L^{4}}\left(\frac{r}{L}\right)^{2} \int_{0}^{1} \frac{d^{4} \varphi(\bar{s})}{d \bar{s}^{4}}\left(\frac{d \varphi(\bar{s})}{d \bar{s}}\right)^{2} \varphi(\bar{s}) d \bar{s}, \\
& H_{32}=\frac{E I}{\rho A \omega_{r e f}^{2} L^{4}}\left(\frac{r}{L}\right)^{2} \int_{0}^{1}\left(\frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}}\right)^{3} \varphi(\bar{s}) d \bar{s} q^{3}, \\
& H_{33}=\frac{3 E I}{\rho A \omega_{r e f}^{2} L^{4}}\left(\frac{r}{L}\right)^{2} \int_{0}^{1} \frac{d^{3} \varphi(\bar{s})}{d \bar{s}^{3}} \frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}} \frac{d \varphi(\bar{s})}{d \bar{s}} \varphi(\bar{s}) d \bar{s} \\
& h_{11}=-\frac{\bar{m}_{l} \bar{p}}{H_{21}+\bar{m}_{b l} H_{21}}\left[\int_{0}^{1}\left(\frac{\partial \varphi(\bar{s})}{\partial \bar{s}} \varphi(\bar{s}) \delta\left(\bar{s}-\bar{s}_{b}\right)\right) d \bar{s}\right. \\
& \left.+\int_{0}^{1}\left(\frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}}\right) \varphi(\bar{s}) \int_{\bar{s}}^{1} \delta\left(\bar{\zeta}-\bar{s}_{b}\right) d \bar{\zeta} d \bar{s}\right] \\
& h_{21}=\left(\frac{r}{L}\right)\left(\frac{\bar{m}_{l} \bar{p}}{H_{21}+\bar{m}_{b l} H_{21}}\right)\left[-\int_{0}^{1} \frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}} \frac{d \varphi(\bar{s})}{d \bar{s}} \varphi(\bar{s})\right. \\
& \times \int_{\bar{s}}^{1} \delta\left(\bar{\zeta}-\bar{s}_{b}\right) d \bar{\zeta} d \bar{s} \\
& \left.+\int_{0}^{1}\left(\frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}}\right) \varphi(\bar{s}) \int_{\bar{s}}^{1} \frac{d \varphi(\bar{\zeta})}{d \bar{s}} \delta\left(\bar{\zeta}-\bar{s}_{b}\right) d \bar{\zeta} d \bar{s}\right] \\
& h_{31}=\left(\frac{r}{L}\right)^{2}\left(\frac{\bar{m}_{l} \bar{p}}{H_{21}+\bar{m}_{b l} H_{21}}\right) \\
& \times\left[-\int_{0}^{1} \frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}} \frac{d \varphi(\bar{s})}{d \bar{s}} \varphi(\bar{s}) \int_{\bar{s}}^{1} \frac{d \varphi(\bar{\zeta})}{d \bar{\zeta}} \delta\left(\bar{\zeta}-\bar{s}_{b}\right)\right. \\
& \times d \bar{\zeta} d \bar{s}+\frac{1}{2} \int_{0}^{1}\left(\frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}}\right) \varphi(\bar{s}) \int_{\bar{s}}^{1}\left(\frac{d \varphi(\bar{\zeta})}{d \bar{\zeta}}\right)^{2} \\
& \left.\times \delta\left(\bar{\zeta}-\bar{s}_{b}\right) d \bar{\zeta} d \bar{s}\right] \\
& \alpha_{11}=-\left(\frac{2 \bar{m}_{l} \bar{p}}{H_{21}+\bar{m}_{b l} H_{21}}\right) \int_{0}^{1}\left(\frac{\partial \varphi(\bar{s})}{\partial \bar{s}} \varphi(\bar{s}) \delta\left(\bar{s}-\bar{s}_{b}\right)\right) d \bar{s} \\
& \alpha_{21}=-\left(\frac{\bar{m}_{l} \bar{p}}{H_{21}+\bar{m}_{b l} H_{21}}\right) \int_{0}^{1}\left(\frac{\partial \varphi(\bar{s})}{\partial \bar{s}} \varphi(\bar{s}) \delta\left(\bar{s}-\bar{s}_{b}\right)\right) d \bar{s}
\end{aligned}
$$

$$
\alpha_{31}=-\left(\frac{r}{L}\right)\left(\frac{\bar{m}_{l} \bar{p}}{H_{21}+\bar{m}_{b l} H_{21}}\right) \int_{0}^{1}\left(\left(\frac{\partial \varphi(\bar{s})}{\partial \bar{s}}\right)^{2}\right.
$$

$$
\left.\varphi(\bar{s}) \delta\left(\bar{s}-\bar{s}_{b}\right)\right) d \bar{s}
$$

$$
\begin{aligned}
\alpha_{41}= & \left(\frac{r}{L}\right)^{2}\left(\frac{1}{H_{21}+\bar{m}_{b l} H_{21}}\right) \int_{0}^{1} \frac{d \varphi(\bar{s})}{d \bar{s}} \varphi(\bar{s}) \\
& \times\left[1+\left(\bar{m}_{b}+\bar{m}_{l}\right) \delta\left(\bar{s}-\bar{s}_{b}\right)\right] \int_{0}^{\bar{s}} \frac{d^{2} \varphi(\bar{\xi})}{d \bar{\xi}^{2}} d \bar{\xi} d \bar{s}
\end{aligned}
$$

$$
\alpha_{42}=\left(\frac{r}{L}\right)\left(\frac{\bar{m}_{l} \bar{p}}{H_{21}+\bar{m}_{b l} H_{21}}\right) \int_{0}^{1} \frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}}
$$

$$
\times \varphi(\bar{s}) \int_{\bar{s}}^{1} \frac{d \varphi(\bar{\zeta})}{d \bar{s}} \delta\left(\bar{\zeta}-\bar{s}_{b}\right) d \bar{\zeta} d \bar{s}
$$

$$
\alpha_{51}=\left(\frac{r}{L}\right)^{2}\left(\frac{1}{H_{21}+\bar{m}_{b l} H_{21}}\right)\left[\int_{0}^{1} \frac{d \varphi(\bar{s})}{d \bar{s}} \varphi(\bar{s})\right.
$$

$$
\times\left[1+\left(\bar{m}_{b}+\bar{m}_{l}\right) \delta\left(\bar{s}-\bar{s}_{b}\right)\right] \int_{0}^{\bar{s}} \frac{d^{2} \varphi(\bar{\xi})}{d \bar{\xi}^{2}} d \bar{\xi} d \bar{s}
$$

$$
-\int_{0}^{1} \frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}} \varphi(\bar{s}) \int_{\bar{s}}^{1}\left[1+\left(\bar{m}_{b}+\bar{m}_{l}\right) \delta\left(\bar{\zeta}-\bar{s}_{b}\right)\right]
$$

$$
\times \int_{0}^{\bar{\zeta}}\left(\frac{d \varphi(\bar{\xi})}{d \bar{\xi}}\right)^{2} d \bar{\xi} d \bar{\zeta} d \bar{s}
$$

$$
\alpha_{52}=-\left(\frac{r}{L}\right)^{2}\left(\frac{\bar{m}_{l} \bar{p}}{H_{21}+\bar{m}_{b l} H_{21}}\right)
$$

$$
\times\left[\int_{0}^{1}\left(\frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}}\right) \varphi(\bar{s}) \int_{\bar{s}}^{1}\left(\frac{d \varphi(\bar{\zeta})}{d \bar{\zeta}}\right)^{2}\right.
$$

$$
\times \delta\left(\bar{\zeta}-\bar{s}_{b}\right) d \bar{\zeta} d \bar{s}+\int_{0}^{1}\left(\left(\frac{\partial \varphi(\bar{s})}{\partial \bar{s}}\right)^{3}\right.
$$

$$
\left.\left.\times \varphi(\bar{s}) \delta\left(\bar{s}-\bar{s}_{b}\right)\right) d \bar{s}\right]
$$

$$
\alpha_{61}=\left(\frac{r}{L}\right)^{2}\left(\frac{1}{H_{21}+\bar{m}_{b l} H_{21}}\right)\left(\frac{c}{\rho A \omega_{r e f}}\right)
$$

$$
\times\left[\int_{0}^{1} \frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}} \frac{d \varphi(\bar{s})}{d \bar{s}} \varphi(\bar{s}) \int_{\bar{s}}^{1} \varphi(\bar{\zeta}) d \bar{\zeta} d \bar{s}\right.
$$

$$
\left.-\frac{1}{2} \int_{0}^{1}\left(\left(\frac{\partial \varphi(\bar{s})}{\partial \bar{s}}\right)^{2} \varphi^{2}(\bar{s}) \delta\left(\bar{s}-\bar{s}_{b}\right)\right) d \bar{s}\right]
$$

$$
\alpha_{62}=\left(\frac{r}{L}\right)^{2}\left(\frac{\bar{m}_{l} \bar{p}}{H_{21}+\bar{m}_{b l} H_{21}}\right) \times\left[2 \int_{0}^{1}\left(\frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}}\right)\right.
$$

$$
\begin{aligned}
& \times \varphi(\bar{s}) \int_{\bar{s}}^{1}\left(\frac{d \varphi(\bar{s})}{d \bar{s}}\right)^{2} \delta\left(\bar{\zeta}-\bar{s}_{b}\right) d \bar{\zeta} d \bar{s} \\
& -\int_{0}^{1}\left(\left(\frac{\partial \varphi(\bar{s})}{\partial \bar{s}}\right)^{3} \varphi(\bar{s}) \delta\left(\bar{s}-\bar{s}_{b}\right)\right) d \bar{s} \\
& \left.-2 \int_{0}^{1} \frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}} \frac{d \varphi(\bar{s})}{d \bar{s}} \varphi(\bar{s}) \int_{\bar{s}}^{1} \frac{d \varphi(\bar{\zeta})}{d \bar{\zeta}} \delta\left(\bar{\zeta}-\bar{s}_{b}\right) d \bar{\zeta}\right] \\
& \alpha_{71}=\left(\frac{r}{L}\right)^{2}\left(\frac{1}{H_{21}+\bar{m}_{b l} H_{21}}\right)\left[\int_{0}^{1} \frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}} \frac{d \varphi(\bar{s})}{d \bar{s}} \varphi(\bar{s})\right. \\
& \times \int_{\bar{s}}^{1}\left(1+\bar{m}_{b l} \delta\left(\bar{\zeta}-\bar{s}_{b}\right)\right) \varphi(\bar{\zeta}) d \bar{\zeta} d \bar{s} \\
& -\frac{1}{2} \int_{0}^{1}\left(\left(1+\bar{m}_{b l} \times \delta\left(\bar{\zeta}-\bar{s}_{b}\right)\right)\left(\frac{\partial \varphi(\bar{s})}{\partial \bar{s}}\right)^{2} \varphi^{2}(\bar{s})\right) \\
& \times d \bar{s}-\int_{0}^{1}\left(\frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}}\right) \times \varphi(\bar{s}) \int_{\bar{s}}^{1}\left(1+\bar{m}_{b l} \delta\left(\bar{\zeta}-\bar{s}_{b}\right)\right) \\
& \left.\times \int_{0}^{\bar{\zeta}}\left(\frac{d \varphi(\bar{\xi})}{d \bar{\xi}}\right)^{2} d \bar{\xi} d \bar{\zeta} d \bar{s}\right] \\
& \alpha_{72}=\left(\frac{r}{L}\right)^{2}\left(\frac{\bar{m}_{l} \bar{p}}{H_{21}+\bar{m}_{b l} H_{21}}\right) \\
& {\left[-\frac{1}{2} \int_{0}^{1}\left(\left(\frac{\partial \varphi(\bar{s})}{\partial \bar{s}}\right)^{3} \varphi(\bar{s}) \delta\left(\bar{s}-\bar{s}_{b}\right)\right) d \bar{s}\right.} \\
& -\int_{0}^{1} \frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}} \frac{d \varphi(\bar{s})}{d \bar{s}} \varphi(\bar{s}) \int_{\bar{s}}^{1} \frac{d \varphi(\bar{s})}{d \bar{s}} \delta\left(\bar{\zeta}-\bar{s}_{b}\right) d \bar{\zeta} d \bar{s} \\
& \left.+\int_{0}^{1}\left(\frac{d^{2} \varphi(\bar{s})}{d \bar{s}^{2}}\right) \varphi(\bar{s}) \int_{\bar{s}}^{1}\left(\frac{d \varphi(\bar{\zeta})}{d \bar{\zeta}}\right)^{2} \delta\left(\bar{\zeta}-\bar{s}_{b}\right) d \bar{\zeta} d \bar{s}\right] \\
& f_{1}=-\left(\frac{L}{r}\right)\left(\frac{\bar{m}_{l} \bar{p}}{H_{21}+\left(\bar{m}_{b}+\bar{m}_{l}\right) H_{21}}\right) \int_{0}^{1} \varphi(\bar{s}) \delta\left(\bar{s}-\bar{s}_{b}\right) d \bar{s}
\end{aligned}
$$

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