Journal of Statistics and Management Systems

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tsms20

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Journal of Statistics & Management Systems
ISSN 0972-0510 (Print), ISSN 2169-0014 (Online)
DOI : 10.1080/09720510.2021.1891699

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Abstract

The similarity measure in this paper is verified for Pythagorean fuzzy sets (PFSs). We point out that a similar measure for two PFSs is a Pythagorean fuzzy value (PFV). To calculate the similarity measure of two PFSs, it is logical that their similarity should be vague. Therefore, we introduce for the first time, as far as we know, the similarity measures for two PVFs is the same as PFV. For calculating PFV of similar measure, we first present the method for calculating Pythagorean similar measure for two PFV by using T-norm and S-norm and then extend it to calculate similar measure for two PFSs. Finally, a numerical example is provided to illustrate the validity and applicability of the presented decision-making method.

Subject Classification: *03B52, 03E72, 90B50.*

Keywords: *Similarity measure, Pythagorean fuzzy value (PFV), Pythagorean fuzzy sets (PFSs), Multiple criteria decision making (MCDM).*

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1. Introduction

In many multi-criteria decision making (MCDM) problems, the decision maker has to use vague (qualitative) values to decide. In the face of qualitative values, for the first time, Bellman and his associates introduced the theory of fuzzy sets (FSs) using mathematical modeling. In making decision with fuzzy sets, the decision maker considers only the degree of correctness of the option and cannot take into account the degree of incorrectness. Continuing this trend, Atanassov in [1] introduced the intuitionistic fuzzy sets (IFSs) theory which is a generalization from FSs which included both the degrees of membership and non-membership.

Atanassov and Gargov [2] presented an interval valued IFS (IVIFS) as a generalization and an intuitionist fuzzy sets (IFS) that makes use of interval value instead of a real number.

The intuitionistic set theory membership value and non-membership value must be less than or equal to one. For example, if the membership degree is equal to 0.4 and the non-membership degree is 0.7, it is easy to see that $0.4 + 0.7 \geq 1$. Therefore, the IFS cannot be used with these values. In 2014, Yager [3] regarding $(0.4)^2 + (0.7)^2 \leq 1$, introduced the sum of squares of membership and non-membership of equal to or less than 1 as a generalization of the IFSs. This new fuzzy set (Fs) was called the Pythagorean fuzzy sets (PFSs).

The PFS has many uses in a variety of problems, and especially in MCDM. In [4-10], aggregation operators and power aggregation operators such as division PF and subtraction PF, confidence PF weighted, generalized PF Einstein ordered weighted averaging, generalized PF Einstein weighted averaging, PF Einstein ordered weighted averaging, PF Einstein weighted averaging, interval valued PF weighted average and weighted geometric operators for different interval valued PFS have been extended and presented. A comparison method for decision making with score function is presented to recognize the PF positive ideal solution and the PF negative ideal solution has been proposed in [11].

In [12] some novel operational laws have been extended and discussed in real world MCDM problems. A connection of the complex numbers and the PFV has been presented and the idea of PF membership values and the linked idea of PF subsets have been elaborated. A discussion related to correlation coefficients for IFSs has been pointed out and then between two PFSs, a new weighted correlation coefficient has been formulated in [14]. Several novel probabilistic aggregation operator with PFS such as probabilistic PF weighted average operator, immediate probability

PF ordered weighted average operator and probabilistic PF weighted geometric operator have been developed in [15] and these operators are extended to interval valued PFS (IVPFS). In [16] PF 2-tuple linguistic data in MCDM problems has been investigated and arithmetic and geometric operations and PF 2-tuple linguistic aggregation operators are developed. In [17], the combination of the concepts of linguistic fuzzy set (LFS) and PFS is based on the new linguistic PFS method. The exponential operational laws in the PFS environment set [18] can be mentioned.

The similarity measures, is one of the important tools that can be used to determine the degree of similarity between two different types of values. Many researchers introduced the similarity measure to calculate the similarity between two FS, IFS and IVIFS [19-29]. Recently, the similarity measures for PFS have been introduced and examined by many scholars.

For instance, in [31], connection between the similarity measure, the distance measure, the inclusion measure and the entropy for PFSs have been investigated. In [32], multi criteria group decision-making (MCGDM) problems with interval valued PF (IVPF) and point operator-based similarity measures have been investigated. In [33], similarity measure has been used for MCGDM problem in PFV environment. According to point operators in [34], a series of similarity measures have been introduced. Base on cosine function in [35], some similarity measures are introduced for PFVs and PFSs. Some distance and similarity measures for PFVs, PFSs and their application in [36] are presented, which are based on the five parameters of PFSS viz: membership value, nonmembership value, hesitancy value, direction of commitment and strength of commitment on membership.

The motivation for writing this paper is that PFV is vague. So, to calculate their similarity measures, it is logical that their similarity should be vague. Therefore, in this paper, we propose for the first time, that as far as we know, the similarity measures for two PVFs is the same as PFV. Then we generalized it to the calculation of the similarity of the two PFS. Finally, the application of this method is elaborated to select the best option among the m options relative to the n criteria with PFV.

As a preparation, in Section 2, we first outline some definitions and some properties. In Section 3, similarity measure for PFSs and PFVs are verified. In Section 4, we present an algorithm for similarity measure PFSs. In Section 5, we provide the real world MCDM problem.

2. Definitions and some properties

In set theory, ambiguity in membership can be viewed from two perspectives: truth-membership and false-membership. Hence, these notions can be defined as follows.

Definition 2.1 : [12] Let $U = \{u_1, u_2, u_3, \ldots, u_n\}$ denote the discourse set. A PFS θ in U is demonstrated by

- A truth-membership function $\mu_\theta : U \to [0,1]$,
- A false-membership function $\tau_\theta : U \to [0,1]$,
- $\mu_\theta(u_i)$ are grades of the membership of u_i in θ,
- $\tau_\theta(u_i)$ are the negations of u_i in θ,
- $\mu_\theta^2(u_i) + \tau_\theta^2(u_i) \le 1$.

Simply expressed, $\mu_{\theta_i} := \mu_\theta(u_i)$ and $\tau_{\theta_i} := \tau_\theta(u_i)$, $i = 1, \ldots, n$. $M(U)$ stands for the set of all Pythagorean fuzzy subsets in U and for $\theta \in M(U)$, then we write it as

$$A = \{(u_i, M_\theta(u_i)) \mid u_i \in U\}, \tag{2.1}$$

where $M_\theta(u_i) = (\mu_{\theta_i}, \tau_{\theta_i})$ is PFV.

Definition 2.2 : [13] For two PFVs (μ_1, τ_1) and (μ_2, τ_2), we have

(1) $(\mu_1, \tau_1) = (\mu_2, \tau_2) \Leftrightarrow \mu_1 = \mu_2$ and $\tau_1 = \tau_2$.

(2) $(\mu_1, \tau_1) \preceq (\mu_2, \tau_2) \Leftrightarrow \mu_1 \le \mu_2$ and $\tau_1 \ge \tau_2$.

(3) $(\mu_1, \tau_1) \prec (\mu_2, \tau_2) \Leftrightarrow \mu_1 < \mu_2$ and $\tau_1 > \tau_2$.

Definition 2.3 : [13] Let $A = (\mu_1, \tau_1)$ and $B = (\mu_2, \tau_2)$ be two PFVs. We consider max and min of the two PFVs as fallows:

(1) $max\{(\mu_1, \tau_1), (\mu_2, \tau_2)\} = (\mu, \tau) \Leftrightarrow \mu = max\{\mu_1, \mu_2\}, \tau = min\{\tau_1, \tau_2\}$,

(2) $min\{(\mu_1, \tau_1), (\mu_2, \tau_2)\} = (\mu, \tau) \Leftrightarrow \mu = min\{\mu_1, \mu_2\}, \tau = max\{\tau_1, \tau_2\}$.

Lemma 2.4 : *The minimum and maximum of two PFVs, is a PFV.*

Proof : Suppose $A = (\mu_1, \tau_1)$ and $B = (\mu_2, \tau_2)$ be two PFVs. Let $p = (\mu, \tau)$, We should prove that $\mu^2 + \tau^2 \le 1$. We have $\mu = \min\{\mu_1, \mu_2\}$ and $\tau = \max\{\tau_1, \tau_2\}$. Then

$$\mu^2 + \tau^2 = (\min\{\mu_1, \mu_2\})^2 + (\max\{\tau_1, \tau_2\})^2. \tag{2.2}$$

Proof is evident when $\min\{\mu_1,\mu_2\}=\mu_1$ and $\max\{\tau_1,\tau_2\}=\tau_1$, or $\min\{\mu_1,\mu_2\}=\mu_2$ and $\max\{\tau_1,\tau_2\}=\tau_2$. If $\min\{\mu_1,\mu_2\}=\mu_1$ and $\max\{\tau_1,\tau_2\}=\tau_2$, we have

$$\begin{aligned} \tau^2+\mu^2 &= (\max\{\tau_1,\tau_2\})^2+(\min\{\mu_1,\mu_2\})^2 \\ &\le \tau_2^2+\mu_1^2 \le (1-\mu_2^2)+\mu_1^2 \\ &= 1-(\mu_1^2-\mu_2^2)\le 1. \end{aligned} \tag{2.2}$$

For $\max\{\tau_1,\tau_2\}=\tau_1$ and $\min\{\mu_1,\mu_2\}=\mu_2$, the proof is similar. □

Definition 2.5 : [37] Suppose $A=(\mu,\tau)$ be a PFN, then the score function of A will be stated as

$$sco(A) = \mu^2-\tau^2. \tag{2.3}$$

Definition 2.6 : [37] For two PFSs $\theta,\phi\in M(U)$, we say $\theta\subseteq\phi$ if and only if $M_\theta(u_i)\le M_\phi(u_i)$ for each $u_i\in U$, $i=1,\dots,n$.

Decision making is the process of selecting a possible course from all of the alternatives. An effective approach in order to select an eligible alternative based on our opinion is its similarity to the ideal point. Also in our study, we employ T-norm and S-norm as instrument for obtaining similarity measure. Hence, we can give definitions of similarity measure, T-norm and S-norm.

Definition 2.7 : [38] Let θ, ϕ and ρ be three sets. A real function $Sim(.,.)$ is called the similarity measure, if the next conditions are fulfilled

(1) $0\le Sim(\theta,\phi)\le 1$,

(2) $Sim(\theta,\phi)=Sim(\phi,\theta)$,

(3) $Sim(\theta,\phi)=1 \Leftrightarrow \theta=\phi$,

(4) If $\theta\subseteq\phi$, and $\phi\subseteq\rho$, then $Sim(\theta,\rho)\le min\{Sim(\theta,\phi), Sim(\phi,\rho)\}$.

Definition 2.8 : [40] Suppose $S:[0,1]\times[0,1]\mapsto[0,1]$. . A binary operation S is a continuous S-norm if the next conditions are fulfilled:

(S-1) S is associative and commutative,

(S-2) $S(x,0)=a$ for all $x\in[0,1]$,

(S-3) $S(x,y)\le S(z,t)$ whenever $x\le z$ and $y\le t$, for each $x,y,z,t\in[0,1]$.

Some of the basic S-norm are as follows:

$$S_1(x,y)=\max\{x,y\},\quad S_2(x,y)=\min\{x+y,1\},\quad S_3(x,y)=x+y-xy.$$

Definition 2.9 : [40] Suppose $T:[0,1]\times[0,1]\mapsto[0,1]$. A binary operation T is a continuous T-norm if the next conditions are met:

(T-1) T is associative and commutative,

(T-2) $T(x,1)=x$ for all $x\in[0,1]$,

(T-3) $T(x,y)\le T(z,t)$ whenever $x\le z$ and $y\le t$, for each $x,y,z,t\in[0,1]$.

Some of the basic T-norms are as follows:

$$T_1(x,y)=\min\{x,y\},\quad T_2(x,y)=\max\{x+y-1,0\},\quad T_3(x,y)=xy.$$

Proposition 2.10 : For any $(x,y)\in[0,1]\times[0,1]$; then $T(x,y)\le S(x,y)$.

Proof : $T(x,y)\le T(x,1)=x=S(x,0)\le S(x,y)$. □

Proposition 2.11 : $T(0,0)=S(0,0)=0$.

Proof : With proposition 2.10 and (S-2), $0\le T(0,0)\le S(0,0)=0;$; therefore,

$$S(0,0)=T(0,0)=0.$$

□

3. Pythagorean fuzzy sets and similarity measure

In this section, in order to determine Pythagorean similar measure between two PFSs, it is necessary that we first present the method for calculating Pythagorean similar measure for two PFVs.

Definition 3.1 : Let $x=(\mu_x,\tau_x)$ and $y=(\mu_y,\tau_y)$ be two PFVs in which ${\mu_x}^2+{\tau_x}^2\le 1$ and ${\mu_y}^2+{\tau_y}^2\le 1$. $Sim(x,y)_S^T$ is PFV of similarity measure based on S-norm and T-norm for the PFVs x and y if:

$$Sim(x,y)_S^T=(\mu_{Sim(x,y)},\tau_{Sim(x,y)}),\tag{3.1}$$

where $\mu_{Sim(x,y)}=\sqrt{1-S((\mu_x-\mu_y)^2,(\tau_x-\tau_y)^2)}$ is the truth-membership function of similarity and $\tau_{Sim(x,y)}=\sqrt{T((\mu_x-\mu_y)^2,(\tau_x-\tau_y)^2)}$ is the false-membership function of similarity x, y.

Proposition 3.2 : $\mu^2_{Sim(x,y)} + \tau^2_{Sim(x,y)} \leq 1.$.

Proof : With Proposition 2.10:

$$\mu^2_{Sim(x,y)} + \tau^2_{Sim(x,y)} = 1 - S((\mu_x - \mu_y)^2, (\tau_x - \tau_y)^2) + T((\mu_x - \mu_y)^2, (\tau_x - \tau_y)^2) \leq 1.$$

□

Theorem 3.3 : *Let* $x = (\mu_x, \tau_x)$, $y = (\mu_y, \tau_y)$ *and* $z = (\mu_z, \tau_z)$ *be three PFVs. Therefore,* $Sim(.,.)^T_S$ *corresponding to the formula (3.1)* () *is satisfied in the following property.*

(1) $(0,1) \preceq Sim(x,y)^T_S \preceq (1,0)$,

(2) $Sim(x,y)^T_S = Sim(y,x)^T_S$,

(3) $Sim(x,y)^T_S = (1,0) \Leftrightarrow x = y$,

(4) *If* $x \preceq y \preceq zl$; *then* $Sim(x,z)^T_S \preceq min\{Sim(x,y)^T_S, Sim(y,z)^T_S\}$.

Proof :

(1) According to ranking of PFVs, this proposition is true.

(2) With regard to Definition 3.1, this proposition is true.

(3) $Sim(x,y)^T_S = (1,0)$; then $\mu_{Sim(x,y)} = 1$ and $\tau_{Sim(x,y)} = 0$; hence $S((\mu_x - \mu_y)^2, (\tau_x - \tau_y)^2) = 0$ and $T((\mu_x - \mu_y)^2, (\tau_x - \tau_y)^2) = 0$. Therefore, it is obvious that with Proposition 2.11 $\mu_x = \mu_y$ and $\tau_x = \tau_y$ and this means that $x = y$.

(4) If $x \preceq y \preceq z$; then $\mu_x \leq \mu_y \leq \mu_z$, and $\tau_x \geq \tau_y \geq \tau_z$, then

- $(\mu_x - \mu_y)^2 \leq (\mu_x - \mu_z)^2$,
- $(\mu_z - \mu_y)^2 \leq (\mu_x - \mu_z)^2$,
- $(\tau_x - \tau_y)^2 \leq (\tau_x - \tau_z)^2$,
- $(\tau_z - \tau_y)^2 \leq (\tau_x - \tau_z)^2$.

Regarding (T-3) in Definition 2.9 and (S-3) in Definition 2.8,

- $S((\mu_x - \mu_y)^2, (\tau_x - \tau_y)^2) \leq S((\mu_x - \mu_z)^2, (\tau_x - \tau_z)^2)$,
- $S((\mu_z - \mu_y)^2, (\tau_z - \tau_y)^2) \leq S((\mu_x - \mu_z)^2, (\tau_x - \tau_z)^2)$,
- $T((\mu_x - \mu_y)^2, (\tau_x - \tau_y)^2) \leq T((\mu_x - \mu_z)^2, (\tau_x - \tau_z)^2)$,
- $T((\mu_z - \mu_y)^2, (\tau_z - \tau_y)^2) \leq T((\mu_x - \mu_z)^2, (\tau_x - \tau_z)^2)$.

Hence

- $\sqrt{1 - S((\mu_x - \mu_y)^2, (\tau_x - \tau_y)^2)} \geq \sqrt{1 - S((\mu_x - \mu_z)^2, (\tau_x - \tau_z)^2)}$,

- $\sqrt{1-S((\mu_z-\mu_y)^2,(\tau_z-\tau_y)^2)} \geq \sqrt{1-S((\mu_x-\mu_z)^2,(\tau_x-\tau_z)^2)}$,
- $\sqrt{T((\mu_x-\mu_y)^2,(\tau_x-\tau_y)^2)} \leq \sqrt{T((\mu_x-\mu_z)^2,(\tau_x-\tau_z)^2)}$,
- $\sqrt{T((\mu_z-\mu_y)^2,(\tau_z-\tau_y)^2)} \leq \sqrt{T((\mu_x-\mu_z)^2,(\tau_x-\tau_z)^2)}$,

with Definition 3.1,

- $\mu_{Sim(x,y)} \geq \mu_{Sim(x,z)}, \quad \mu_{Sim(y,z)} \geq \mu_{Sim(x,z)}$,
- $\tau_{Sim(x,y)} \leq \tau_{Sim(x,z)}, \quad \tau_{Sim(y,z)} \leq \tau_{Sim(x,z)}$,

then

- $\mu_{Sim(x,z)} \leq min\{\mu_{Sim(x,y)}, \mu_{Sim(y,z)}\}$,
- $\tau_{Sim(x,z)} \geq max\{\tau_{Sim(x,y)}, \tau_{Sim(y,z)}\}$,
- $(\mu_{Sim(x,z)}, \tau_{Sim(x,z)}) \preceq min\{(\mu_{Sim(x,y)}, \tau_{Sim(x,y)}), (\mu_{Sim(y,z)}, \tau_{Sim(y,z)})\}$,

and we have

$$Sim(x,z)_S^T \preceq min\{Sim(x,y)_S^T, Sim(y,z)_S^T\},$$

and proof will be completed. □

Example 3.4 : We are able to find similarity measures in Table 1 for PFVs $x=(0.9,0.3), y=(0.7,0.4)$ with various S-norms and T-norms presented in Definitions 2.9 and 2.8. In Table 2, we have shown a comparison for PFVs $x=(0.9,0.3)$, $y=(0.7,0.4)$ with our method via S_1-norms and T_1-norms and other suggested methods in [29, 30, 33, 35].

Table 1

Similarity measures for example 3.4.

$Sim(x,y)_{S_i}^{T_j}$	T_1	T_2	T_3
S_1	(0.98,0.1)	(0.98,0)	(0.98,0.02)
S_2	(0.97,0.1)	(0.97,0)	(0.97,0.02)
S_3	(0.97,0.1)	(0.97,0)	(0.97,0.02)

Table 2

Comparison of similarity measures for example 3.4.

Ref	Similarity Measure
Zhang [33]	0.67
Rani et al. [29] Method 1.	0.52
Method 2.	0.97
Wei & Wei [35] Method 1.	0.98
Method 2.	0.82
Peng & Dai [30]	(0.72,0.22)
$Sim(x,y)_{S_1}^{T_1}$	(0.98,0.1)

Example 3.5 : We can find similarity measures for PFVs $x=(1,0)$, $y=(0,1)$ with various S-norms and T-norms presented in Definitions 2.9 and 2.8.

$$Sim(x,y)_{S_i}^{T_j}=(0,1),\ i,j=1,2,3.$$

They have no similarities.

In Table 3, we shown a comparison for PFVs $x=(1,0)$, $y=(0,1)$ with our method and other suggested methods in [29, 30, 33, 35].

- In Tables 2 and 3, based on the proposed methods of Zhang [33], Rani et al. [29], Wei and Wei [35] and Peng and Dai [30], the similarity measure for two PFVs is a crisp value. With the help of Peng and

Table 3

Comparison of similarity measures for example 3.5.

Ref	Similarity Measure
Zhang [33]	0
Rani et al. [29] Method 1.	0
Method 2.	0
Wei & Wei [35] Method 1.	0
Method 2.	0
Peng & Dai [30]	(0, 1)
$Sim(x,y)_{S_1}^{T_1}$	(0, 1)

Dai method, like the method presented in this paper, the PFV for two PFVs is obtained. But the formula proposed by them for the measure of similarity will not work well when we have an equal truth-membership function or a false-membership function for two PFVs.

Definition 3.6 : Let $U = \{u_1, u_2, u_3, \ldots, u_n\}$ be discourse set and $\theta, \phi \in M(U)$ be two PFSs. Then, $Sim(\theta,\phi)_S^T$ is PFV of similarity measure based on S-norm and T-norm for the PFSs θ and ϕ if

$$Sim(\theta,\phi)_S^T = (\mu_{Sim(\theta,\phi)}, \tau_{Sim(\theta,\phi)}), \tag{3.2}$$

where the truth-membership function of similarity is

$$\mu_{Sim(\theta,\phi)} = \sqrt{\sum_{i=1}^{n} w_i\, \mu^2_{Sim(M_\theta(u_i), M_\phi(u_i))}},$$

and the false-membership function of similarity is

$$\tau_{Sim(\theta,\phi)} = \sqrt{\sum_{i=1}^{n} w_i\, \tau^2_{Sim(M_\theta(u_i), M_\phi(u_i))}},$$

and $w_i > 0$ is the weight of the element $u_i \in U, i = 1, 2, \ldots, n$ where $\sum_{i=1}^{n} w_i = 1$, and it depends on who the decision-maker is.

Proposition 3.7 : For two PFSs θ, ϕ in U, ${\mu_{S(\theta,\phi)}}^2 + {\tau_{S(\theta,\phi)}}^2 \leq 1.$.

Proof :

$$\begin{aligned}
{\mu_{S(\theta,\phi)}}^2 + {\tau_{S(\theta,\phi)}}^2 &= \sum_{i=1}^{n} w_i\, \mu^2_{Sim(M_\theta(u_i), M_\phi(u_i))} + \sum_{i=1}^{n} w_i\, \tau^2_{Sim(M_\theta(u_i), M_\phi(u_i))} \\
&= \sum_{i=1}^{n} w_i (\mu^2_{Sim(M_\theta(u_i), M_\phi(u_i))} + \tau^2_{Sim(M_\theta(u_i), M_\phi(u_i))}) \\
&\leq \sum_{i=1}^{n} w_i = 1.
\end{aligned}$$

□

Theorem 3.8 : *Let $\theta, \phi, \rho \in M(U)$ be three PFSs. Therefore, $Sim(\theta,\phi)_S^T$ corresponding to the formula (3.2) is satisfied in the following property.*

(1) $(0,1) \leq Sim(\theta,\phi)_S^T \leq (1,0)$,

(2) $Sim(\theta,\phi)_S^T = Sim(\phi,\theta)_S^T$,

(3) $Sim(\theta,\phi)_S^T = (1,0) \Leftrightarrow \theta = \phi$,
(4) If $\theta \subseteq \phi \subseteq \rho$, then $Sim(\theta,\rho)_S^T \leq min\{Sim(\theta,\phi)_S^T, Sim(\phi,\rho)_S^T\}$.

Proof.

(1) This proposition is true according to ranking of PFVs, Definitions 3.1 and 3.6.

(2) With regard to Definition 3.6, this proposition is true.

(3) $Sim(\theta,\phi)_S^T = (1,0)$; then $\mu_{Sim(\theta,\phi)} = 1$ and $\tau_{Sim(\theta,\phi)} = 0$; hence

$$\sum_{i=1}^{n} w_i \mu^2_{Sim(M_\theta(u_i),M_\phi(u_i))} = 1,$$

and

$$\sum_{i=1}^{n} w_i \tau^2_{Sim(M_\theta(u_i),M_\phi(u_i))} = 0.$$

Therefore, it is obvious that with Proposition 2.11,

$$\mu^2_{Sim(M_\theta(u_i),M_\phi(u_i))} = 1, \tau^2_{Sim(M_\theta(u_i),M_\phi(u_i))} = 0,$$

and this signifies that $x = y$.

(4) If $\theta \subseteq \phi \subseteq \rho$; then $M_\theta(u_i) \preceq M_\phi(u_i) \preceq M_\rho(u_i)$, then with case 4 of Theorem 3.3,

$$Sim(M_\theta(u_i), M_\rho(u_i))_S^T \preceq min\{Sim(M_\theta(u_i), M_\phi(u_i))_S^T, Sim(M_\phi(u_i), M_\rho(u_i))_S^T\},$$

hence with Definition 3.1

$$\mu_{Sim(M_\theta(u_i),M_\rho(u_i))} \leq \mu_{Sim(M_\theta(u_i),M_\phi(u_i))}, \ \mu_{Sim(M_\theta(u_i),M_\rho(u_i))} \leq \mu_{Sim(M_\phi(u_i),M_\rho(u_i))},$$

$$\tau_{Sim(M_\theta(u_i),M_\rho(u_i))} \geq \tau_{Sim(M_\theta(u_i),M_\phi(u_i))}, \ \tau_{Sim(M_\theta(u_i),M_\rho(u_i))} \geq \tau_{Sim(M_\phi(u_i),M_\rho(u_i))}.$$

Then,

$$\sqrt{\sum_{i=1}^{n} w_i \mu^2_{Sim(M_\theta(u_i),M_\rho(u_i))}} \leq \sqrt{\sum_{i=1}^{n} w_i \mu^2_{Sim(M_\theta(u_i),M_\phi(u_i))}},$$

$$\sqrt{\sum_{i=1}^{n} w_i \mu^2_{Sim(M_\theta(u_i),M_\rho(u_i))}} \leq \sqrt{\sum_{i=1}^{n} w_i \mu^2_{Sim(M_\phi(u_i),M_\rho(u_i))}},$$

$$\sqrt{\sum_{i=1}^{n} w_i \tau^2_{Sim(M_\theta(u_i),M_\rho(u_i))}} \geq \sqrt{\sum_{i=1}^{n} w_i \tau^2_{Sim(M_\theta(u_i),M_\phi(u_i))}},$$

$$\sqrt{\sum_{i=1}^{n} w_i \tau^2_{Sim(M_\theta(u_i),M_\rho(u_i))}} \geq \sqrt{\sum_{i=1}^{n} w_i \tau^2_{Sim(M_\phi(u_i),M_\rho(u_i))}}.$$

And we have the result will be

$$\mu_{Sim(\theta,\rho)} \leq \mu_{Sim(\theta,\phi)},\ \ \mu_{Sim(\theta,\rho)} \leq \mu_{Sim(\phi,\rho)},$$

$$\tau_{Sim(\theta,\rho)} \geq \tau_{Sim(\theta,\phi)},\ \ \tau_{Sim(\theta,\rho)} \geq \mu_{Sim(\phi,\rho)},$$

and

$$\mu_{Sim(\theta,\rho)} \leq min\{\mu_{Sim(\theta,\phi)}, \mu_{Sim(\phi,\rho)}\},$$

$$\tau_{Sim(\theta,\rho)} \geq max\{\tau_{Sim(\theta,\phi)}, \mu_{Sim(\phi,\rho)}\}.$$

Therefore, $Sim(\theta,\rho)^T_S \leq min\{Sim(\theta,\phi)^T_S, Sim(\phi,\rho)^T_S\}$ and proof is completed.

□

4. Modified method to MCDM problems with Pythagorean fuzzy sets

In this section, we develop the Topsis method for MCDM problems featuring PFSs. This modified method is expressed under an algorithm in the following part.

In the MCDM problem, the following conditions are considered:

- $C = \{t_1, \ldots, t_n\}$ is a finite set of criteria,
- $A = \{r_1, \ldots, r_m\}$ is a discrete set of m alternatives,
- $W = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of all criteria, which fulfills

$$\sum_{j=1}^{n} w_j = 1,\ \ 0 \leq w_j \leq 1.$$

We denote the evaluation values of the alternative r_i $(i = 1,2,\ldots,m)$ with respect to the criterion t_j $(j = 1,2,\ldots,n)$ by PFV $M(t_j(r_i)) = (\alpha_{ij}, \beta_{ij})$, and $R = [M(t_j(r_i))]_{m\times n}$ is a Pythagorean fuzzy decision matrix.

Therefore, the MCDM problem with PFV can be represented as the following matrix form:

$$R = \begin{matrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{matrix} \begin{bmatrix} (\alpha_{11}, \beta_{11}) & (\alpha_{12}, \beta_{12}) & \cdots & (\alpha_{1n}, \beta_{1n}) \\ (\alpha_{21}, \beta_{21}) & (\alpha_{22}, \beta_{22}) & & (\alpha_{2n}, \beta_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha_{m1}, \beta_{m1}) & (\alpha_{m2}, \beta_{m2}) & \cdots & (\alpha_{mn}, \beta_{mn}) \end{bmatrix} \tag{4.1}$$

where

- α_{ij} represents the value of the alternative r_i in terms of the the criterion t_j.
- β_{ij} indicates the negations value of alternative r_i in terms of the criterion t_j.

The suggested approach is based on: ***The most suitable alternative is to have the greatest similarity with the positive ideal (PI) and the least similarity with a negative ideal (NI)***.

For this purpose, we first define the PI Pythagorean fuzzy set and the NI Pythagorean fuzzy set for the Pythagorean fuzzy matrix 4.1 by defining the maximum and minimum of the two Pythagorean fuzzy values presented in Definition 2.3 as follows:

$$PI = \{(t_1, (\overline{\alpha_1}, \overline{\beta_1})), (t_2, (\overline{\alpha_2}, \overline{\beta_2})), \ldots, (t_n, (\overline{\alpha_n}, \overline{\beta_n}))\}, \tag{4.2}$$

where

$$\overline{\alpha_j} = \max_i \alpha_{ij}, \ \overline{\beta_j} = \min_j \beta_{ij}.$$

And

$$NI = \{(t_1, (\underline{\alpha_1}, \underline{\beta_1})), (t_2, (\underline{\alpha_2}, \underline{\beta_2})), \ldots, (t_n, (\underline{\alpha_n}, \underline{\beta_n}))\}, \tag{4.3}$$

where

$$\underline{\alpha_j} = \min_i \alpha_{ij}, \ \underline{\beta_j} = \max_j \beta_{ij}.$$

Using Eq. (3.2), the similarity between the alternative PFS r_i and the Pythagorean fuzzy set *PI* and Pythagorean fuzzy set *NI* can be obtained as follows:

$$Sim(r_i, PI)_S^T = (\mu_{Sim(r_i, PI)}, \tau_{Sim(r_i, PI)}), \tag{4.4}$$

where

$$\mu_{Sim(r_i,PI)} = \sqrt{\sum_{j=1}^{n} w_j\, \mu^2_{Sim((\alpha_{ij},\beta_{ij}),\,(\overline{\alpha_j},\overline{\beta_j}))}}\ ,\ \tau_{Sim(r_i,PI)} = \sqrt{\sum_{j=1}^{n} w_j\, \tau^2_{Sim((\alpha_{ij},\beta_{ij}),\,(\overline{\alpha_j},\overline{\beta_j}))}}\ .$$

And

$$Sim(r_i, NI)_S^T = (\mu_{Sim(r_i,NI)}, \tau_{Sim(r_i,NI)}), \tag{4.5}$$

where

$$\mu_{Sim(r_i,NI)} = \sqrt{\sum_{j=1}^{n} w_j\, \mu^2_{Sim((\alpha_{ij},\beta_{ij}),(\underline{\alpha_j},\underline{\beta_j}))}}\ ,\ \tau_{Sim(r_i,NI)} = \sqrt{\sum_{j=1}^{n} w_j\, \tau^2_{Sim((\alpha_{ij},\beta_{ij}),(\underline{\alpha_j},\underline{\beta_j}))}}\ .$$

By calculating the similarity measure with equations 4.4 and 4.5, the ranking order of the *m* alternatives $r_1, r_2, \ldots, r_m$ is obtained using

$$D_i = 1 + D_i^+ - D_i^-,$$

in which $D_i^+ = sco(Sim(r_i, PI)_S^T)$ and $D_i^- = sco(Sim(r_i, NI)_S^T)$.

The above approach can be summarized in the Algorithm *1* in four steps as follows:

Algorithm 1 : It is modified algorithm of MCDM problem by similarity measure in the Pythagorean fuzzy environment.

Step 1 : Make the decision matrix with PFVs.

Step 2 : Create the Pythagorean fuzzy set *PI* and Pythagorean fuzzy set *NI*, utilizing equations (4.2) and (4.3).

Step 3 : Use Eq. (3.2) and weight vector of all criteria to compute the similarity measure between the alternative r_i and the Pythagorean fuzzy *PI* as well as similarity between the alternative r_i and the Pythagorean fuzzy *NI*.

Step 4 : Obtain the similarity measure with equations 4.4 and 4.5. The ranking order of *m* alternatives $r_1, r_2, \ldots, r_m$ will be calculated, using

$$D_i = 1 + D_i^+ - D_i^-,$$

in which $D_i^+ = sco(Sim(r_i, PI)_S^T)$ and $D_i^- = sco(Sim(r_i, NI)_S^T)$.

5. Application examples

A practical example in this section of MCDM problem will be considered to evaluate the quality of service between domestic airlines [37] to illustrate the proposed approach and conduct a comparison analysis.

In this MCDM problem, we have four alternatives referring to Taiwanese domestic airlines, which are shown below:

- UN IAir r_1,
- Daily Air r_2,
- Mandarin r_3,
- Transasia r_4.

The alternatives (Taiwanese domestic airlines) referred to above, are evaluated based on the following criteria:

- Booking and ticketing service t_1,
- Responsiveness t_2,
- Cabin service t_3,
- Check-in and boarding process t_4.

On the basis of the above mentioned concepts, the steps of this method for working with PFVs are shown below.

Step 1 : In Table 4, Pythagorean fuzzy decision matrix is displayed.

Table 4

Pythagorean fuzzy decision Matrix.

	t_1	t_2	t_3	t_4
r_1	(0.9,0.3)	(0.7,0.6)	(0.5,0.8)	(0.6,0.3)
r_2	(0.4,0.7)	(0.9,0.2)	(0.8,0.1)	(0.5,0.3)
r_3	(0.8,0.4)	(0.7,0.5)	(0.6,0.2)	(0.7,0.4)
r_4	(0.7,0.2)	(0.8,0.2)	(0.8,0.4)	(0.6,0.6)

Step 2 : The Pythagorean fuzzy set PI and Pythagorean fuzzy set NI utilizing equations (4.2) and (4.3) have been created.

$$PI = \{(t_1,(0.9,0.2)), (t_2,(0.9,0.2)), (t_3,(0.8,0.1)), (t_4,(0.7,0.3))\},$$

Table 5

Results obtained by the Pythagorean fuzzy approach.

	$Sim(r_i, PI)_S^T$	$sco(Sim(r_i, PI)_S^T)$	$Sim(r_i, NI)_S^T$	$sco(Sim(r_i, NI)_S^T)$	D_i
r_1	(0.885720, 0.203715)	0.743	(0.969536, 0.162788)	0.9135	0.8295
r_2	(0.975961, 0.193649)	0.915	(0.875214, 0.203715)	0.7245	1.1905
r_3	(0.977241, 0.122474)	0.94	(0.915150, 0.164317)	0.8105	1.1295
r_4	(0.968246, 0.050000)	0.935	(0.929516, 0.234521)	0.8165	1.1185

$$NI = \{(t_1,(0.4,0.7)), (t_2,(0.7,0.6)), (t_3,(0.5,0.8)), (t_4,(0.5,0.6))\}.$$

Step 3 : Using equation (3.2) and weight vector $W = (0.15, 0.25, 0.35, 0.25)$, the similarity between the alternative r_i and the Pythagorean fuzzy PI is calculated as well as the similarity between the alternative r_i and the Pythagorean fuzzy NI. We have the results in Table 5.

Step 4 : By calculating the similarity measure with equations 4.4 and 4.5 ranking order of the 4 alternatives r_1, r_2, r_3, r_4, we have:

$$D_1 = 0.8295, D_2 = 1.1905, D_3 = 1.1295, D_4 = 1.1185.$$

The comparison between our method with the other two methods is given in the Table 6.

Table 6

Indexes calculation.

Method	**Order**
Yager [3]	$r_2 > r_4 > r_3 > r_1$
Zhang [37]	$r_2 > r_3 > r_4 > r_1$
Our metod	$r_2 > r_3 > r_4 > r_1$

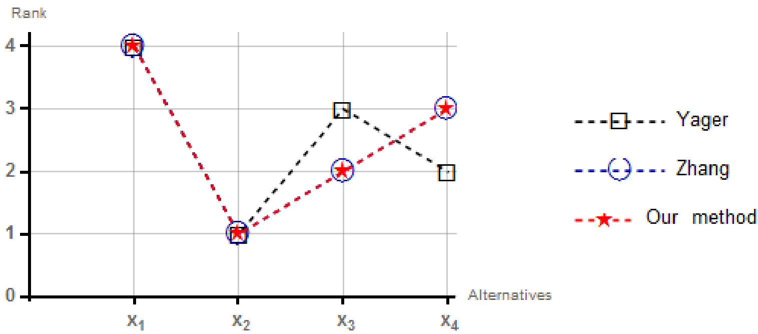

Fig. 1

The rankings of all alternatives.

The ranking of the four alternatives obtained by the suggested method is equivalent to the result by Zhang's method, as illustrated in Fig. 1.

In this example, we have used $T(a,b) = \min\{a,b\}$ and $S(a,b) = \max\{a,b\}$.

6. Conclusion

We have successfully introduced a new definition for similarity measure for Pythagorean collections, which is also a PFV. We define the similarity measure of the two PFVs with a PFV, using T-norm, and S-norm and we generalized it to PFS. This method is used for decision making with Pythagoras value, using similarity measure. The advantages of this method include:

- Calculating the similarity size in the form of PFSs takes more information into decision making, and we lose a lot of information, if converted to a real number.
- Decision making using the similarity measure is possible with fewer steps.

This modified approach can be used in other MCDM models such as AHP, PROMETHEE, ELECTRE and VIKOR.

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Received August, 2020

Revised December, 2020