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ORIGINAL ARTICLE

Fractal soliton solutions for the fractal-fractional shallow water wave equation arising in ocean engineering

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Abstract The generalized shallow water wave equation is an important mathematical model that is used to elaborate ocean engineering, weather simulations, tsunami prediction and tidal currents. In this work, the generalized (3 + 1)-dimensional fractal-fractional shallow water wave equation (FFSWWE) is investigated where fractal-fractional derivative is taken in the conformable derivative sense. Some new fractal soliton solutions of FFSWWE are successfully derived by the fractal-fractional variational wave method (FFVWM), which is a new mathematical technology. This new method has the advantages of being simple, efficient, and direct. The 3D graphics that describe these new fractal soliton solutions that were obtained are tremendously important for improving our understanding of physical oceanography.

1. Introduction

Fractal is a very common phenomenon in the real worlds [1]. Since the birth of human beings, people have begun to use fractal structures in their daily life. More than 2,000 years ago, during the Spring and Autumn Period and the Warring States Period, the Chinese people built the "Yun Wen Tong Jin" which was used by nobles to furnish wine vessels at ceremonies or feasts. The Yun Wen Tong Jin is a structure of ancient fractal geometry. In recent decades, fractal theory has been applied to modern technology and engineering techniques, such as ocean engineering, space science, material science, biological engineering and so on [2–4].

In this paper, we mainly investigate the generalized (3 + 1)-dimensional fractal-fractional shallow water wave equation (FFSWWE) as follows

$$\frac{\partial\phi}{\partial x^{3\delta}\partial y^{\delta}} - a^{\delta}\frac{\partial\phi}{\partial x^{\delta}}\frac{\partial\phi}{\partial x^{\delta}\partial y^{\delta}} + b^{\delta}\frac{\partial\phi}{\partial y^{\delta}}\frac{\partial\phi}{\partial x^{2\delta}} + c^{\delta}\frac{\partial\phi}{\partial y^{\delta}\partial t^{\delta}} - \frac{\partial\phi}{\partial x^{\delta}\partial z^{\delta}} = 0, \tag{1.1}$$

where δ is called fractal dimension, $\partial\phi/\partial x^{\delta}, \partial\phi/\partial y^{\delta}$ and $\partial\phi/\partial t^{\delta}$ are conformable operators (CO) [5–7]. Eq. (1.1) is an important physical equation in ocean science that is used to describe ocean currents, tsunami motions with irregular boundaries or in fractal media. Therefore, it is very meaningful and necessary for us to study of the fractal soliton solutions of

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Eq. (1.1). These obtained new fractal solition solutions are very helpful to study the physical oceanography.

1.1. *Literature*

When $\delta = 1$, Eq. (1.1) is the classical (3 + 1)-dimensional shallow water wave equation (CSWWE) is presented as follows

$$\frac{\partial\phi}{\partial x^3\partial y} - a\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial x\partial y} + b\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x^2} + c\frac{\partial\phi}{\partial y\partial t} - \frac{\partial\phi}{\partial x\partial z} = 0, \tag{1.2}$$

where a, b, c are constants. In [8], Kumar et al. studied the analytical soliton solution of Eq. (1.2) by using exponential rational function method (ERFM) with parameters $a = 3, b = -3, c = 1$. In [9], Abdul-Majid Wazwaz gained the different types of soliton solutions of the CSWWE via three distinct methods, such as tanh-coth method (TCM), Exp-function method (EFM) and Hirota bilinear method (HBM). In [10], D.Kumar and S.Kumar obtained the periodic solitary wave solutions of the CSWWE by using the Lie symmetry technology (LST). In [11], Zhang et al. obtained the lump solutions and rational solutions of the CSWWE by employing the bilinear operator method (BOM) with different constants. In addition, the approximate analytical solution of CSWWE can also be obtained by using other powerful analytical technologies [12–15].

1.2. *Main contributions*

Fractional derivatives have many different types of definitions, among which the most popular are Riemann–Liouville fractional derivative [16–18], Caputo fractional derivative [19–24], Yang-Abdel-Aty-Cattani fractional derivative [25], Yang's local fractional derivative [26–28], Atangana-Baleanu fractional derivative [29,30], fractal derivative [31], Atangana-Baleanu-Riemann derivative [32], He's fractal derivative [33], Hilfer derivative [34–36] and so on [37–41]. However, the conformable fractional derivative is one of the most important definitions. In this work, the generalized (3 + 1)-dimensional fractal-fractional shallow water wave equation (FFSWWE) is described by the conformable operator [42] for the first time. The FFSWWE is studied by fractal variational perspective, and a new and efficient mathematical scheme is established to gain its different types of fractal soliton solutions, which is called fractal-fractional variational wave method (FFVWM). The advantage of this new method is that it is simple, easy to operate, takes only three steps, and then gets good results. The main structure of the article is summarized as follows: The definition and properties of conformable fractional derivative are introduced in Section 2; In Section 3, The fractal-fractional variational wave method (FFVWM) is described; In Section 4, we gain the fractal soliton solutions of FFSWWE by employing the FFVWM; These properties of fractal soliton solutions are discussed in Section 5; Conclusion is given in Section 6.

2. Conformable fractional derivative

In this section, we review some concepts on the conformable operator.

Definition 2.1. Suppose a function $\phi : [0, \infty) \to R$. The conformable fractional derivative of ϕ of order δ is given as [43]

$$\frac{\partial\phi(\eta)}{\partial\eta^\delta} = \lim_{\varepsilon\to 0}\frac{\phi(\eta + \varepsilon\eta^{1-\delta}) - \phi(\eta)}{\varepsilon}, \tag{2.1}$$

for all $\eta > 0$, fractal dimension $\delta \in (0, 1)$.

Theorem 2.1. Let $\delta \in (0, 1]$ and M, N be δ-differentiable at a point $x^\delta > 0$. We have the following relations

$$(1)\,\frac{\partial\left(f^\delta M(x^\delta) \pm g^\delta N(x^\delta)\right)}{\partial x^\delta} = f^\delta\frac{\partial M(t^\delta)}{\partial x^\delta} \pm g^\delta\frac{\partial N(x^\delta)}{\partial x^\delta}, \tag{2.2}$$

$$(2)\frac{\partial(M(x^\delta)N(x^\delta))}{\partial x^\delta} = N\left(x^\theta\right)\frac{\partial(M(x^\delta)}{\partial x^\delta} + M\left(x^\delta\right)\frac{\partial(N(x^\delta)}{\partial x^\delta}, \tag{2.3}$$

$$(3)\,\frac{\partial(M(x^\delta)/N(x^\delta))}{\partial x^\delta} = \frac{N(x^\delta)\frac{\partial(M\left(x^\delta\right)}{\partial x^\delta} - M(x^\delta)\frac{\partial(N\left(x^\delta\right)}{\partial x^\delta}}{N^2\left(t^\delta\right)}. \tag{2.4}$$

Theorem 2.2. Suppose $0 < \delta < 1$ and $n \in \{1, 2, 3, \ldots\}$ then we have

$$\frac{\partial\phi}{\partial\eta^{n\delta}} = \underbrace{\frac{\partial}{\partial\eta^\delta}\frac{\partial}{\partial\eta^\delta}\frac{\partial}{\partial\eta^\delta}\cdots\frac{\partial}{\partial\eta^\delta}}_{n-times}\phi. \tag{2.5}$$

3. Novel methodology

Consider the following (3 + 1)-dimensional fractal-fractional wave equation as follows

$$\Upsilon\left(\frac{\partial\phi}{\partial t^\delta\partial x^\delta}, \frac{\partial\phi}{\partial x^\delta\partial y^\delta}, \frac{\partial\phi}{\partial x^\delta\partial z^\delta}, \frac{\partial\phi}{\partial x^{2\delta}\partial y^\delta}, \frac{\partial\phi}{\partial x^{3\delta}\partial y^\delta}\right) = 0, \tag{3.1}$$

where $\partial\phi/\partial x^\delta, \partial\phi/\partial y^\delta, \partial\phi/\partial z^\delta$ and $\partial\phi/\partial t^\delta$ are conformable fractional derivatives. **Step.1.** The fractal-fractional wave transformation (FFWT) is given as

$$\Phi = \phi\left(\omega^\delta\right) \tag{3.2}$$

and

$$\omega^\delta = \frac{d^\delta}{\delta}x^\delta + \frac{e^\delta}{\delta}y^\delta + \frac{f^\delta}{\delta}z^\delta + \frac{g^\delta}{\delta}t^\delta. \tag{3.3}$$

Apply the chain rule of conformable operator, and get

$$\Upsilon\left(d^\theta g^\theta\frac{\partial\Phi}{\partial\omega^{2\delta}}, d^\delta e^\delta\frac{\partial\Phi}{\partial\omega^{2\delta}}, d^\delta g^\delta\frac{\partial\Phi}{\partial\omega^{2\delta}}, d^{2\delta}e^\delta\frac{\partial\Phi}{\partial\omega^{3\delta}}, d^{3\delta}e^\delta\frac{\partial\Phi}{\partial\omega^{4\delta}}\right) = 0. \tag{3.4}$$

Integral Eq. (3.4) once, we gain

$$\Upsilon\left(d^\theta g^\theta\frac{\partial\Phi}{\partial\omega^\delta}, d^\delta e^\delta\frac{\partial\Phi}{\partial\omega^\delta}, d^\delta g^\delta\frac{\partial\Phi}{\partial\omega^\delta}, d^{2\delta}e^\delta\frac{\partial\Phi}{\partial\omega^{2\delta}}, d^{3\delta}e^\delta\frac{\partial\Phi}{\partial\omega^{3\delta}}\right) = 0. \tag{3.5}$$

Step.2. We assume

$$F = \frac{\partial\phi}{\partial\omega^\delta}. \tag{3.6}$$

In the view of Eq. (3.5)and Eq. (3.6), we get following form

$$\Upsilon\left(\left(d^{\theta}g^{\theta}, d^{\delta}e^{\delta}, d^{\delta}g^{\delta}\right)F, d^{2\delta}e^{\delta}\frac{\partial F}{\partial\omega^{\delta}}, d^{3\delta}e^{\delta}\frac{\partial F}{\partial\omega^{2\delta}}\right) = 0. \tag{3.7}$$

The fractal variational principle (FVP) of Eq. (3.7) is constructed as

$$H(\omega) = \int_{0^{\delta}}^{\infty^{\delta}} K d\omega^{\delta}, \tag{3.8}$$

where K is a fractal trail function. Suppose the fractal soliton solutions of Eq. (3.7) are the following forms

$$F(\omega^{\delta}) = A^{\delta}sech_{\delta}(B^{\delta}\omega^{\delta}), \tag{3.9}$$

$$F(\omega^{\delta}) = A^{\delta}cosh_{\delta}(B^{\delta}\omega^{\delta}), \tag{3.10}$$

$$F(\omega^{\delta}) = A^{\delta}tanh_{\delta}(B^{\delta}\omega^{\delta}), \tag{3.11}$$

$$F(\omega^{\delta}) = A^{\delta}sech_{\delta}^{2}(B^{\delta}\omega^{\delta}), \tag{3.12}$$

......,

where A^{δ} and B^{δ} are fractal constants on fractal space. Substituting above equations into Eq. (3.8), have

$$\frac{\partial H}{\partial A^{\delta}} = 0 \tag{3.13}$$

and

$$\frac{\partial H}{\partial B^{\delta}} = 0. \tag{3.14}$$

In view of Eq. (3.13) and Eq. (3.14), A^{θ} and B^{θ} are successfully constructed. **Step.3.** Thus, the fractal soliton solution of Eq. (3.1) is obtained as

$$\phi = \int F(\omega^{\delta})d\omega^{\delta}. \tag{3.15}$$

4. Fractal soliton solutions of FFSWWE

The generalized (3 + 1)-dimensional fractal-fractional shallow water wave equation (FFSWWE) is given as

$$\frac{\partial\phi}{\partial x^{3\delta}\partial y^{\delta}} - a^{\delta}\frac{\partial\phi}{\partial x^{\delta}}\frac{\partial\phi}{\partial x^{\delta}\partial y^{\delta}} + b^{\delta}\frac{\partial\phi}{\partial y^{\delta}}\frac{\partial\phi}{\partial x^{2\delta}} + c^{\delta}\frac{\partial\phi}{\partial y^{\delta}\partial t^{\delta}} - \frac{\partial\phi}{\partial x^{\delta}\partial z^{\delta}} = 0. \tag{4.1}$$

Use the FFWT as follows

$$\Phi = \phi(\omega^{\delta}) \tag{4.2}$$

and

$$\omega^{\delta} = \frac{d^{\delta}}{\delta}x^{\delta} + \frac{e^{\delta}}{\delta}y^{\delta} + \frac{f^{\delta}}{\delta}z^{\delta} + \frac{g^{\delta}}{\delta}t^{\delta}. \tag{4.3}$$

Eq. (4.1) is written into its partner by using Eq. (4.2) and Eq. (4.3) as

$$d^{3\delta}e^{\delta}\frac{\partial\Phi}{\partial\omega^{4\delta}} - a^{\delta}d^{2\delta}e^{\delta}\frac{\partial\Phi}{\partial\omega^{\delta}}\frac{\partial\Phi}{\partial\omega^{2\delta}} + b^{\delta}d^{2\delta}e^{\delta}\frac{\partial\Phi}{\partial\omega^{\delta}}\frac{\partial\Phi}{\partial\omega^{2\delta}} + c^{\delta}e^{\delta}g^{\delta}\frac{\partial\Phi}{\partial\omega^{2\delta}} - d^{\delta}f^{\delta}\frac{\partial\Phi}{\partial\omega^{2\delta}} = 0. \tag{4.4}$$

Eq. (4.4) can be changed into the following form by integration

$$d^{3\delta}e^{\delta}\frac{\partial\Phi}{\partial\omega^{3\delta}} + \frac{b^{\delta}d^{2\delta}e^{\delta} - a^{\delta}d^{2\delta}e^{\delta}}{2}\left(\frac{\partial\Phi}{\partial\omega^{\delta}}\right)^{2} + \left(c^{\delta}e^{\delta}g^{\delta} - d^{\delta}f^{\delta}\right)\frac{\partial\Phi}{\partial\omega^{\delta}} = 0. \tag{4.5}$$

We assume

$$F = \frac{\partial\Phi}{\partial\omega^{\delta}}. \tag{4.6}$$

Thus, Eq. (4.5) is further simplified into the following form

$$\frac{\partial F}{\partial\omega^{2\delta}} + \frac{b^{\delta} - a^{\delta}}{2d^{\delta}}F^{2} + \frac{\left(c^{\delta}e^{\delta}g^{\delta} - d^{\delta}f^{\delta}\right)}{d^{3\delta}e^{\delta}}F = 0. \tag{4.7}$$

The FVP of Eq. (4.7) is successfully gained as

$$H = \int_{0^{\delta}}^{\infty^{\delta}}\left\{-\frac{1}{2}\left(\frac{\partial F}{\partial\omega^{\delta}}\right)^{2} + \frac{b^{\delta} - a^{\delta}}{6d^{\delta}}F^{3} + \frac{\left(c^{\delta}e^{\delta}g^{\delta} - d^{\delta}f^{\delta}\right)}{2d^{3\delta}e^{\delta}}F^{2}\right\}d\omega^{\delta}. \tag{4.8}$$

Type.I. Let

$$F(\omega^{\delta}) = A^{\delta}sech_{\delta}(\omega^{\delta}). \tag{4.9}$$

Combining Eq. (4.8) and Eq. (4.9), we have the following relationship

$$\begin{aligned} H &= \int_{0^{\delta}}^{\infty^{\delta}}\left\{-\tfrac{1}{2}\left(-A^{\delta}sech_{\delta}(\omega^{\delta})tanh_{\delta}(\omega^{\delta})\right)^{2} + \tfrac{b^{\delta}-a^{\delta}}{6d^{\delta}}\left(A^{\delta}sech_{\delta}(\omega^{\delta})\right)^{3} + \tfrac{\left(c^{\delta}e^{\delta}g^{\delta}-d^{\delta}f^{\delta}\right)}{2d^{3\delta}e^{\delta}}\left(A^{\delta}sech_{\delta}(\omega^{\delta})\right)^{2}\right\}d\omega^{\delta} \\ &= -\frac{A^{2\delta}\left(A^{\delta}\pi a^{\delta}d^{2\delta}e^{\delta} - A^{\delta}\pi b^{\delta}d^{2\delta}e^{\delta} + 4d^{3\delta}e^{\delta} - 12c^{\delta}e^{\delta}g^{\delta} + 12d^{\delta}f^{\delta}\right)}{24d^{3\delta}e^{\delta}}. \end{aligned} \tag{4.10}$$

Then, get

$$\frac{\partial H}{\partial A^{\delta}} = -\frac{A^{\delta}\left(A^{\delta}\pi a^{\delta}d^{2\delta}e^{\delta} - A^{\delta}\pi b^{\delta}d^{2\delta}e^{\delta} + 4d^{3\delta}e^{\delta} - 12c^{\delta}e^{\delta}g^{\delta} + 12d^{\delta}f^{\delta}\right)}{12d^{3\delta}e^{\delta}} - \frac{A^{2\delta}\left(\pi a^{\delta}d^{2\delta}e^{\delta} - \pi b^{\delta}d^{2\delta}e^{\delta}\right)}{24d^{3\delta}e^{\delta}} = 0. \tag{4.11}$$

By Eq. (4.11), the parameter A^{δ} is easily found as

$$A^{\delta} = \frac{-8d^{3\delta}e^{\delta} + 24c^{\delta}e^{\delta}g^{\delta} - 24d^{\delta}f^{\delta}}{3\pi d^{2\delta}e^{\delta}\left(a^{\delta} - b^{\delta}\right)}. \tag{4.12}$$

So, have

$$F(\omega^{\delta}) = \frac{-8d^{3\delta}e^{\delta} + 24c^{\delta}e^{\delta}g^{\delta} - 24d^{\delta}f^{\delta}}{3\pi d^{2\delta}e^{\delta}\left(a^{\delta} - b^{\delta}\right)}sech_{\delta}(\omega^{\delta}). \tag{4.13}$$

Hence, the first type of fractal soliton solution of Eq. (4.1) is obtained as

$$\begin{aligned} \phi &= \int\left\{\frac{-8d^{3\delta}e^{\delta}+24c^{\delta}e^{\delta}g^{\delta}-24d^{\delta}f^{\delta}}{3\pi d^{2\delta}e^{\delta}\left(a^{\delta}-b^{\delta}\right)}sech_{\delta}(\omega^{\delta})\right\}d\omega^{\delta} \\ &= \frac{-8d^{3\delta}e^{\delta}+24c^{\delta}e^{\delta}g^{\delta}-24d^{\delta}f^{\delta}}{3\pi d^{2\delta}e^{\delta}\left(a^{\delta}-b^{\delta}\right)}arctanh_{\delta}(sinh_{\delta}(\omega^{\delta})) \\ &= \frac{-8d^{3\delta}e^{\delta}+24c^{\delta}e^{\delta}g^{\delta}-24d^{\delta}f^{\delta}}{3\pi d^{2\delta}e^{\delta}\left(a^{\delta}-b^{\delta}\right)}arctanh_{\delta}\left(sinh_{\delta}\left(\frac{d^{\delta}}{\delta}x^{\delta} + \frac{e^{\delta}}{\delta}y^{\delta} + \frac{f^{\delta}}{\delta}z^{\delta} + \frac{g^{\delta}}{\delta}t^{\delta}\right)\right). \end{aligned} \tag{4.14}$$

When $\delta \to 1$, we have

$$\lim_{\delta\to1}\phi=\lim_{\delta\to1}\left\{\frac{-8d^{3\delta}e^{\delta}+24c^{\delta}e^{\delta}g^{\delta}-24d^{\delta}f^{\delta}}{3\pi d^{2\delta}e^{\delta}\left(a^{\delta}-b^{\delta}\right)}arctanh_{\delta}\left(sinh_{\delta}\left(\frac{d^{\delta}}{\delta}x^{\delta}+\frac{e^{\delta}}{\delta}y^{\delta}+\frac{f^{\delta}}{\delta}z^{\delta}+\frac{g^{\delta}}{\delta}t^{\delta}\right)\right)\right\}$$
$$=\frac{-8d^{3}e+24ceg-24df}{3\pi d^{2}e(a-b)}arctanh(sinh(dx+ey+fz+gt)). \tag{4.15}$$

Type.II. Suppose the second type of fractal soliton solution of Eq. (4.1) is the below form

$$F\left(\omega^{\delta}\right)=A^{\delta}sech_{\delta}^{2}\left(\omega^{\delta}\right). \tag{4.16}$$

As the same technology, we get

$$H=\int_{0^{\delta}}^{\infty^{\delta}}\left\{-\frac{1}{2}\left(-2A^{\delta}sech_{\delta}^{2}(\omega^{\delta})tanh_{\delta}(\omega^{\delta})\right)^{2}+\frac{b^{\delta}-a^{\delta}}{6d^{\delta}}\left(A^{\delta}sech_{\delta}^{2}(\omega^{\delta})\right)^{3}+\frac{\left(c^{\delta}e^{\delta}g^{\delta}-d^{\delta}f^{\delta}\right)}{2d^{3\delta}e^{\delta}}\left(A^{\delta}sech_{\delta}^{2}(\omega^{\delta})\right)^{2}\right\}d\omega^{\delta}$$
$$=-\frac{A^{2\delta}\left(4A^{\delta}a^{\delta}d^{2\delta}e^{\delta}-4A^{\delta}b^{\delta}d^{2\delta}e^{\delta}+12d^{3\delta}e^{\delta}-15c^{\delta}e^{\delta}g^{\delta}+15d^{\delta}f^{\delta}\right)}{45d^{3\delta}e^{\delta}}. \tag{4.17}$$

Then, have

$$\frac{\partial H}{\partial A^{\delta}}=-\frac{2A^{\delta}\left(4A^{\delta}a^{\delta}d^{2\delta}e^{\delta}-4A^{\delta}b^{\delta}d^{2\delta}e^{\delta}+12d^{3\delta}e^{\delta}-15c^{\delta}e^{\delta}g^{\delta}+15d^{\delta}f^{\delta}\right)}{45d^{3\delta}e^{\delta}}-\frac{A^{2\delta}\left(4a^{\delta}d^{2\delta}e^{\delta}-4b^{\delta}d^{2\delta}e^{\delta}\right)}{45d^{3\delta}e^{\delta}}=0. \tag{4.18}$$

Taking Eq. (4.18), we get the A^{δ} as follows

$$A^{\delta}=\frac{-12d^{3\delta}e^{\delta}+15c^{\delta}e^{\delta}g^{\delta}-15d^{\delta}f^{\delta}}{2d^{2\delta}e^{\delta}\left(a^{\delta}-b^{\delta}\right)}. \tag{4.19}$$

Thus, we get the F as

$$F\left(\omega^{\delta}\right)=\frac{-12d^{3\delta}e^{\delta}+15c^{\delta}e^{\delta}g^{\delta}-15d^{\delta}f^{\delta}}{2d^{2\delta}e^{\delta}\left(a^{\delta}-b^{\delta}\right)}sech_{\delta}^{2}\left(\omega^{\delta}\right). \tag{4.20}$$

Consequently, the second type of fractal soliton solution is gained

$$\phi=\int\left\{\frac{-12d^{3\delta}e^{\delta}+15c^{\delta}e^{\delta}g^{\delta}-15d^{\delta}f^{\delta}}{2d^{2\delta}e^{\delta}\left(a^{\delta}-b^{\delta}\right)}sech_{\delta}^{2}(\omega^{\delta})\right\}d\omega^{\delta}$$
$$=\frac{-12d^{3\delta}e^{\delta}+15c^{\delta}e^{\delta}g^{\delta}-15d^{\delta}f^{\delta}}{2d^{2\delta}e^{\delta}\left(a^{\delta}-b^{\delta}\right)}\frac{sinh_{\delta}\left(\frac{d^{\delta}}{\delta}x^{\delta}+\frac{e^{\delta}}{\delta}y^{\delta}+\frac{f^{\delta}}{\delta}z^{\delta}+\frac{g^{\delta}}{\delta}t^{\delta}\right)}{cosh_{\delta}\left(\frac{d^{\delta}}{\delta}x^{\delta}+\frac{e^{\delta}}{\delta}y^{\delta}+\frac{f^{\delta}}{\delta}z^{\delta}+\frac{g^{\delta}}{\delta}t^{\delta}\right)}. \tag{4.21}$$

When $\delta\to1$, get

$$\lim_{\delta\to1}\phi=\lim_{\delta\to1}\left\{\frac{-12d^{3\delta}e^{\delta}+15c^{\delta}e^{\delta}g^{\delta}-15d^{\delta}f^{\delta}}{2d^{2\delta}e^{\delta}\left(a^{\delta}-b^{\delta}\right)}\frac{sinh_{\delta}\left(\frac{d^{\delta}}{\delta}x^{\delta}+\frac{e^{\delta}}{\delta}y^{\delta}+\frac{f^{\delta}}{\delta}z^{\delta}+\frac{g^{\delta}}{\delta}t^{\delta}\right)}{cosh_{\delta}\left(\frac{d^{\delta}}{\delta}x^{\delta}+\frac{e^{\delta}}{\delta}y^{\delta}+\frac{f^{\delta}}{\delta}z^{\delta}+\frac{g^{\delta}}{\delta}t^{\delta}\right)}\right\}$$
$$=\frac{-12d^{3}e+15ceg-15df}{2d^{2}e(a-b)}\frac{sinh(dx+ey+fz+gt)}{cosh(dx+ey+fz+gt)}. \tag{4.22}$$

5. Results and discussion

The fractal soliton solutions of generalized (3 + 1)-dimensional fractal-fractional shallow water wave equation (FFSWWE) are obtained by the fractal-fractional variational method (FFVM). These characteristics of obtained fractal soliton solutions are illustrated in the following 3D graphs. In Fig. 1, the corresponding 3D graph of Eq. (4.14) with parameters $a^{\theta}=1,b^{\theta}=2,c^{\theta}=2,d^{\theta}=1,e^{\theta}=2,f^{\theta}=1,g^{\theta}=1$ for different fractal dimensional $\delta=ln2/ln3$ and $\delta=0.8$ at $y^{\delta}=0,z^{\delta}=0$.

Remark 5.1. By observing figure.1, we conclude that when we determine the values of fractal parameters, the shape of the fractal solitary wave depends on the fractal dimension δ. Therefore, we can control the shape of solitary waves by changing the fractal dimension.

In Fig. 2, we plot the fractal soliton at $t^{\delta}=0.1$ and $t^{\delta}=0.3$ with $a^{\delta}=2,b^{\delta}=3,c^{\delta}=4,d^{\delta}=2,e^{\delta}=3,f^{\delta}=5,g^{\delta}=1,\quad\delta=0.9$.

Remark 5.2. By observing Fig. 2, we can find the motion law of the fractal solitary wave. The fractal solitary wave propagates along the negative direction of the x-axis with time variation. It is very useful for studying the dynamics of fractal solitary waves.

In Fig. 3, the corresponding 3D graph of Eq. (4.21) with different parameters $a^{\theta}=2,b^{\theta}=1,c^{\theta}=3,d^{\theta}=2,e^{\theta}=1,f^{\theta}=4,g^{\theta}=1$ and $a^{\theta}=3,b^{\theta}=2,c^{\theta}=1,d^{\theta}=3,e^{\theta}=2,f^{\theta}=5,g^{\theta}=1$ for fractal dimensional $\delta=ln2/ln3$.

In Fig. 4, we present the fractal soliton at $t^{\delta}=1$ and $t^{\delta}=2$ with $a^{\delta}=3,b^{\delta}=2,c^{\delta}=1,d^{\delta}=3,e^{\delta}=2,f^{\delta}=5,g^{\delta}=1,\delta=0.8$.

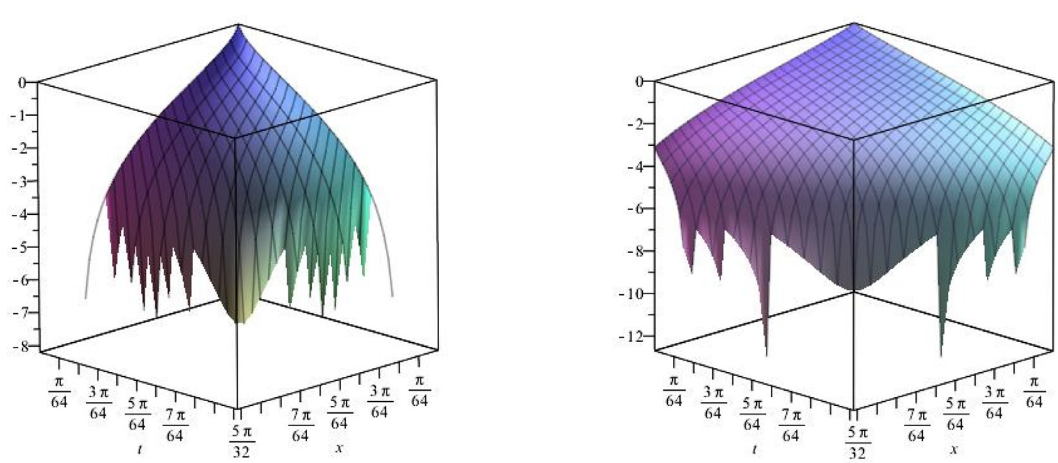

Fig. 1 3D graph of Eq. (4.14) with fractal dimension $\delta=ln2/ln3$ and $\delta=0.8$.

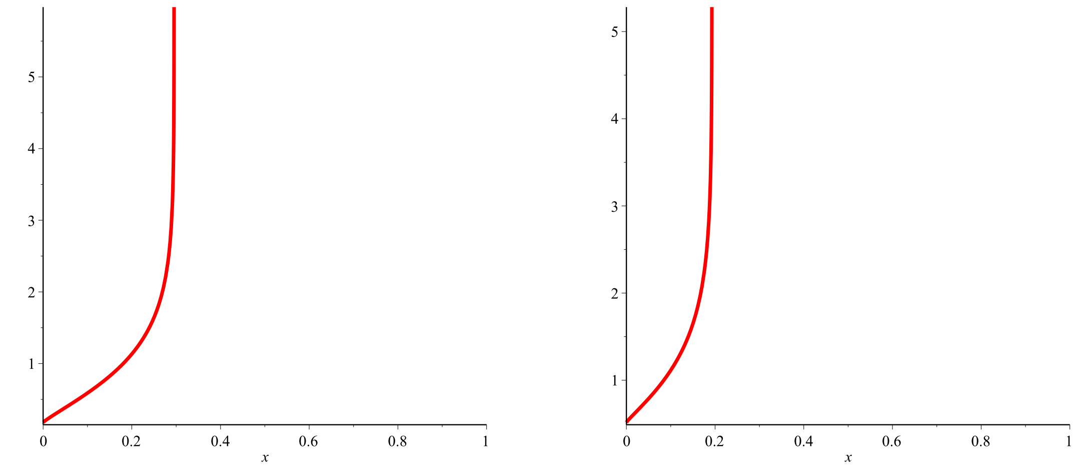

Fig. 2 3D graph of Eq. (4.14) with fractal dimension $\delta = 0.9$ at $t^{\delta} = 0.1$ and $t^{\delta} = 0.3$.

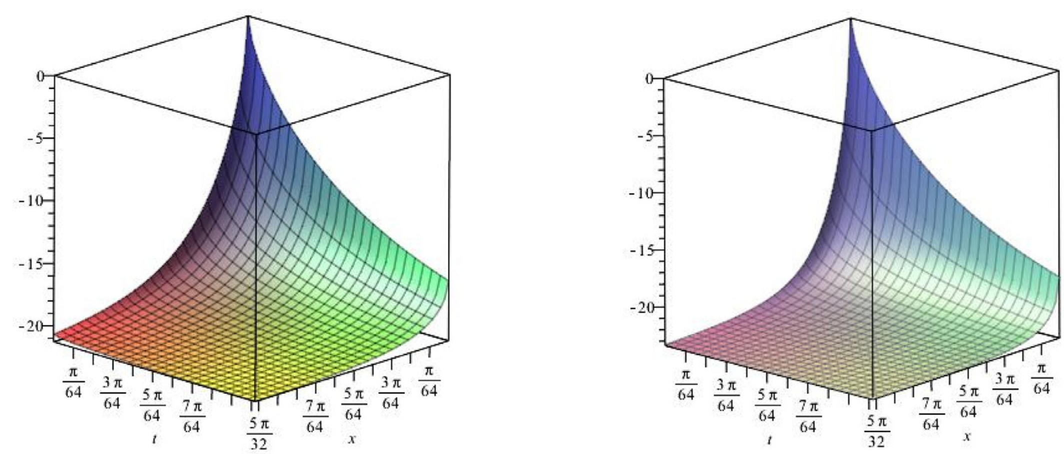

Fig. 3 3D graph of Eq. (4.21) with different parameters at fractal dimension $\delta = ln2/ln3$.

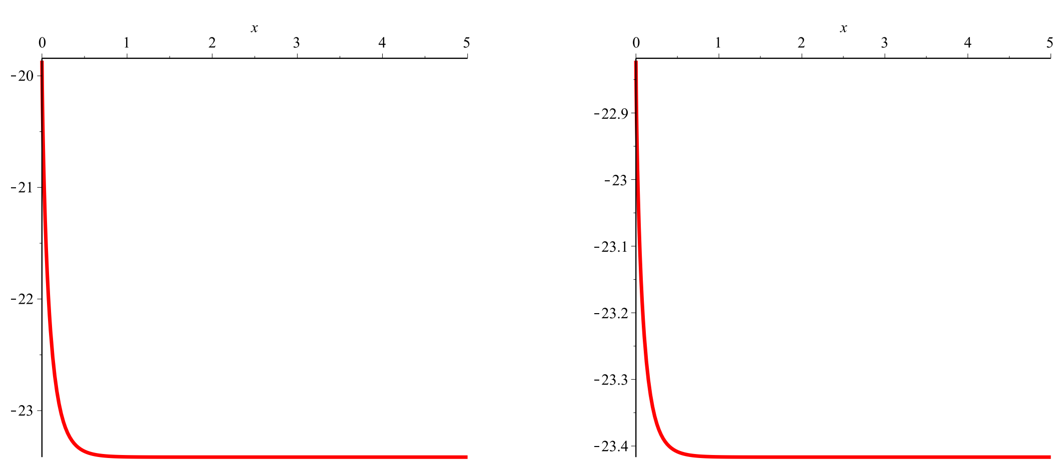

Fig. 4 3D graph of Eq. (4.21) with fractal dimension $\delta = 0.8$ at $t^{\delta} = 1$ and $t^{\delta} = 2$.

In summary, the shape of the fractal soliton wave depends on the fractal dimension, and the peak value is related to the selected parameters. When the fractal dimension and parameters are determined, the solitary wave propagates along the negative x-axis with increasing time.

6. Conclusion

In this work, we suggested fractal-fractional variational wave method to obtain the fractal soliton solutions of (3 + 1)-dimensional fractal-fractional shallow water wave equation. These obtained fractal solitary wave solutions are new types and have not appeared in other literature. The FFVWM is very simple, straightforward, and easy to implement. In addition, the characteristics of the fractal soliton solutions were observed by some 3D and 2D graphs. These obtained results are very helpful for ocean engineering, weather simulations with special conditions. The forthcoming work will be directed to the study the fractal soliton solutions of other types of fractional evolution equations involving different types of fractional derivatives.

Data Availability Statement: Not applicable.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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