Structural Optimization 3, 115–120 (1991)

Structural Optimization

Layup optimization against buckling of shear panels

J.L. Grenestedt*

Research Engineer, Asea Brown Boveri Research Center, CH-5405 Baden-Dättwil, Switzerland

Abstract. The object of the study was to optimize the shear buckling load of laminated composite plates. The laminates lacked coupling between bending and extension (E_{ij}=0) but had otherwise arbitrary selection of the ply angle variation through the thickness. The plates were rectangular and either simply supported or clamped on all edges. For orthotropic plates, it was seen that there is only one parameter necessary for finding the optimal design for different materials and plate aspect ratios. This parameter can be interpreted as the layup angle θ in a $(+/-\ \theta)$ orthotropic laminate. When bending-twisting coupling is present, the buckling strength depends on the direction of the applied load. A laminate with non-zero bending-twisting coupling stiffnesses can be described with four lamination parameters. The allowable region of these parameters was investigated, and an optimization of the buckling load within this region was performed. It was seen that even this is a one parameter problem. This parameter can be interpreted as the layup anlge θ in an off-axis unidirectional laminate (θ).

Notations

A_{ij}	in-plane stiffnesses of anisotropic plates, Tsai and Hahn (1980)
B_{ij}	coupling stiffnesses of anisotropic plates
D_{ij}	bending stiffnesses of anisotropic plates
D^*_{ij}	normalized bending stiffnesses
a, b, h	length, width and thickness of the plate
x, y	in-plane coordinates
z	through-the-thickness coordinate
z^*	normalized through-the-thickness coordinate
$w(x, y)$	out-of-plane deformation
N_{xy}	shear buckling load
W^*_1 to W^*_4	lamination parameters
U_1 to U_5	linear combinations of the on-axis moduli
$\theta(z)$	layup angle
f_k	functional of $\theta(z)$

1 Introduction

The task of optimizing the layup of laminated plates in order to maximize in-plane shear buckling loads has not been studied extensively. The first treatment of the problem was probably by Housner and Stein (1975), who optimized the angle in an orthotropic angle-ply laminate. Hirano (1979) allowed more freedom in the choice of layup, but still neglected anisotropy (bending-twisting coupling). The conclusion from this study was that angle-ply laminates are optimal, and the data presented by Housner and Stein remained valid.

Thielemann (1950) showed that non-zero bending-twisting coupling stiffnesses (D_{16}, D_{26}) results in the fact that a plate with an infinite aspect ratio gets a preferred direction of shear. In one direction, the shear buckling strength is higher than for the corresponding orthotropic laminate, in the other it is lower.

No investigation of the optimal layup for non-orthotropic laminates has been found, even though it is clear that such plates have higher shear buckling strengths (but only in one direction). The present article studies this problem. It is seen that shear buckling optimization is a one parameter problem. Similar results have been obtained for the optimization of vibration frequency, uniaxial compression buckling, and deflection under a constant pressure (Grenestedt 1990).

2 Plate and laminate configurations

The plates considered in this article are rectangular of the

* Currently at Japan National Aerospace Laboratory, Structural Mechanics, 7-44-1 Jindaiji Higashi-Machi, Chofu-shi, 182 Tokyo, Japan

size $a * b$ in the x and y directions, respectively, and with the constant thickness h. The edges are either all simply supported or clamped concerning the out-of-plane deformation. The materials used in the study are given in Table 1. The laminates considered are characterized by a full matrix of bending stiffnesses, but without coupling between bending and extension. Under these restrictions, any choice of ply angle variation through the thickness is allowed. Symmetric laminates fulfil these requirements, but there are also other laminates that fulfil them. The pre-buckling stresses are constant over the plate, so the in-plane stiffnesses A_{ij} do not affect the buckling stresses. Because of this and the lack of bending-extension coupling, the governing equilibrium differential equations can be decoupled, and the equation for the out-of-plane deformation is solved to give the shear buckling load.

The shear buckling equation is

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} +$$

$$+4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} - 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} = 0\,, \qquad (1)$$

according to e.g. Ashton and Whitney (1970).

By neglecting the D_{16} and D_{26} stiffnesses, the orthotropic equation is obtained.

The plates considered in the present paper are supposed to satisfy the Kirchoff-Love assumptions, i.e. the effect of transverse shear deformation is neglected. This might lead to significant errors if a characteristic in-plane length, e.g. the buckling wave length, is of the same order as the plate thickness. Furthermore, the effect of transverse shear deformation increases when the ratios between transverse shear moduli and in-plane moduli of the laminate decrease. For ordinary FRP materials such a ratio might be in the order of 1:50. Cohen (1982) presented an example where the classical plate theory overestimated the buckling load with 40% for a plate with a thickness-to-width ratio of only 0.05. Accordingly, since FRP are considered here, the present analysis should only be applied to laminates which are very thin compared to the characteristic in-plane lengths.

Table 1. Material constants

Material	U_1 (Gpa)	U_2 (Gpa)	U_3 (Gpa)	U_4 (Gpa)
Graphite/epoxy (T300-5208)	76.4	85.7	19.7	22.6
Aramid/epoxy (Teijin HM50/epoxy)	35.2	39.0	9.86	12.6
Glass/epoxy (Scotchply 1002)	20.4	15.4	3.33	5.51

3 Lamination parameters and their allowable region

The bending stiffnesses of the laminates considered can be described by the four lamination parameters W_1^* to W_4^* introduced by Tsai and Hahn (1980). In the present article these parameters have been normalized, so that

$$W^*_{[1,2,3,4]} = \frac{12}{h^3}\int_{-\frac{h}{2}}^{\frac{h}{2}} [\cos 2\theta,\ \cos 4\theta,\ \sin 2\theta,\ \sin 4\theta] z^2\,\mathrm{d}z =$$

$$= \frac{3}{2}\int_{-1}^{1} [\cos 2\theta,\ \cos 4\theta,\ \sin 2\theta,\ \sin 4\theta] z^{*2}\mathrm{d}z^* =$$

$$= \int_{-1}^{1} f_{[1,2,3,4]}(\theta) z^{*2}\mathrm{d}z^*\,, \qquad (2)$$

where $z^* = 2z/h$. The bending stiffnesses become

$$D^*_{11} = U_1 + U_2 W_1^* + U_3 W_2^*\,,$$

$$D^*_{22} = U_1 - U_2 W_1^* + U_3 W_2^*\,,$$

$$D^*_{12} = U_4 - U_3 W_2^*\,,$$

$$D^*_{66} = \frac{1}{2}(U_1 - U_4) - U_3 W_2^*\,,$$

$$D^*_{16} = \frac{1}{2}U_2 W_3^* + U_3 W_4^*\,,$$

$$D^*_{26} = \frac{1}{2}U_2 W_3^* - U_3 W_4^*\,, \qquad (3)$$

where $D^*_{ij} = 12D_{ij}/h^3$ and h is the thickness of the laminate, U_1 - U_5 are linear combinations of the on-axis moduli of a lamina and can be considered as material constants, Tsai and Hahn (1980).

For the four lamination parameters,

$$-1 \le W_1^*,\ W_2^*,\ W_3^*,\ W_4^* \le 1 \qquad (4)$$

is valid. Miki (1985) showed that the allowable region of W_1^* and W_2^* is

$$W_2^* \ge 2W_1^{*2} - 1\,, \qquad (5)$$

which is all that is needed for the study of orthotropic laminates.

When bending-twisting coupling is present, W_3^* and W_4^* are also needed, and the allowable four-dimensional space of the lamination parameters should be determined. However, we start by determining the projection of the allowable region on the six perpendicular, two-dimensional surfaces that each pair of W_1^* to W_4^* spans.

If θ is a continuous function of z^* (or z), variational methods can be used to determine the convex part of the allowable region for this special case. A θ that is a continuous function of z^* could be achieved by having infinitely many plies of infinitesimal thickness. Define a functional F_{ij},

$$F_{ij} = W_i^* + cW_j^* , \tag{6}$$

where $i \neq j$ (i and j take the values 1, 2, 3 and 4), and c is a constant. When F_{ij} is constant, (6) describes a straight line in the W_i^* - W_j^* plane, the slope of which is determined by c. By maximizing or minimizing F_{ij} the line is parallel translated until it tangents the allowable region. If this done for all c, the convex part of the allowable region when θ is a continuous function of z^* is found. The Euler equation for F_{ij} is

$$\left(\frac{\partial f_i}{\partial \theta} + c\frac{\partial f_j}{\partial \theta}\right) z^{*2} = 0 . \tag{7}$$

or

$$\left(\frac{\partial f_i}{\partial \theta} + c\frac{\partial f_j}{\partial \theta}\right) = 0 , \tag{8}$$

since $z^* = 0$ is nothing but a special point. Since f_k, defined through (2), is dependent of z^* only through the function θ, (8) states that θ is independent of z^*, i.e. all plies have the same orientation. If all plies have the same orientation, the dashed curves in Fig. 1 are obtained. The regions enclosed by these curves are completed with the dotted lines, corresponding to the following laminates:

In the W_1^* - W_2^*-plane: a symmetric laminate with two times two plies with the orientations $\theta_1 = 0$ and $\theta_2 = \pi/2$ and the normalized thicknesses α and $1-\alpha$. By varying α between 0 and 1 the line is obtained.

In the W_2^* - W_3^*-plane: as above, but with $\theta_1 = \pi/4$ and $\theta_2 = 3\pi/4$.

In the W_1^* - W_4^*-plane: as above, but with $\theta_1 = \pi/8$ and $\theta_2 = 5\pi/8$ for the top line, and $\theta_1 = 3\pi/8$ and $\theta_2 = 7\pi/8$ for the bottom line.

In the W_3^* - W_4^*-plane: as above, with $\theta_1 = \pi/8$ and $\theta_2 = 5\pi/8$ for the top line, and $\theta_1 = 3\pi/8$ and $\theta_2 = 7\pi/8$ for the bottom line.

To evaluate whether discontinuous θ will lead to laminates that fall outside these regions, a large number of laminates (4000) with a random number of plies with random thickness and random orientation were investigated and plotted also in Fig. 1. All laminates resulted in points falling inside the regions. These regions are the same also

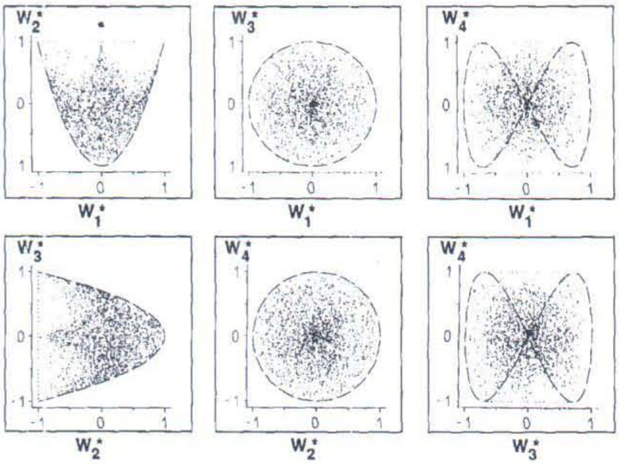

Fig. 1. Lamination parameters for 4000 non-symmetric random laminates, plotted on the projections of the allowable region of the lamination parameters W_1^* to W_4^*

if the laminates are symmetric. We now feel pretty convinced that the regions of Fig. 1 are the correct projections of the allowable region of W_1^* to W_4^*.

4 Method for calculating the buckling loads

For the calculation of the shear buckling loads, a numerical finite difference code was implemented, resulting in the matrix equation

$$(\mathbf{A} - N_{xy}\mathbf{B})\ \mathbf{w} = \mathbf{0} , \tag{9}$$

which is a generalized nonsymmetric eigenvalue problem. It was solved using standard numerical methods. Comparing the results of this approach with data presented in the literature for square plates revealed that the error was less than 1%. For other aspect ratios, the discretization was made finer, but supposedly the error for these aspect ratios was larger than for the square plates.

5 Optimization and results, orthotropic laminates

There are only two parameters, W_1^* and W_2^*, needed for describing the orthotropic laminates. For the aspect ratios $a/b = 1.0, 1.3, 1.7, 2.0, 2.5, 3.0, 4.0$, and infinity, the buckling load was plotted versus the lamination parameters. Figure 2 is an example of such a plot. Because of the symmetric boundary conditions, plates with aspect ratios a/b smaller than unity is equivalent to plates with the aspect ratio b/a. For the plate with infinite aspect ratio the data of Seydel (1933) was used. For each plot, 137 buckling loads for different values of the lamination parameters were calculated. As seen in the figure, the maximal buck-

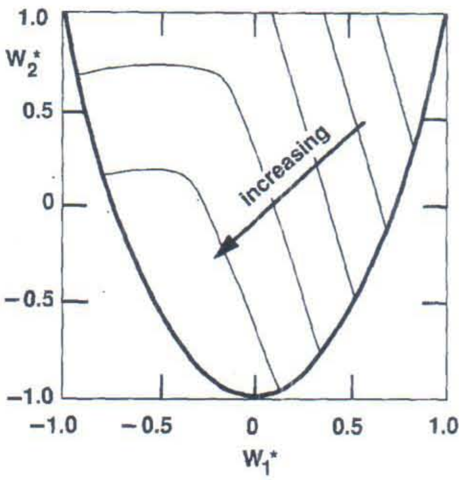

Fig. 2. Shear buckling load *vs.* W_1^* and W_2^* for a simply supported orthotropic aramid/epoxy laminate with the aspect ratio a/b=1.7

ling load is found on the border of the allowable region of the lamination parameters. The same was seen for all aspect ratios, all materials, for both simply supported and clamped edges. It appears to be the fact that there is only one parameter needed for the shear buckling optimization of orthotropic rectangular plates. This parameter can be interpreted as the layup angle θ in an orthotropic $(+/-\theta)$ laminate. This confirms the results of Hirano (1979), and the data presented by Housner and Stein (1975) remain valid.

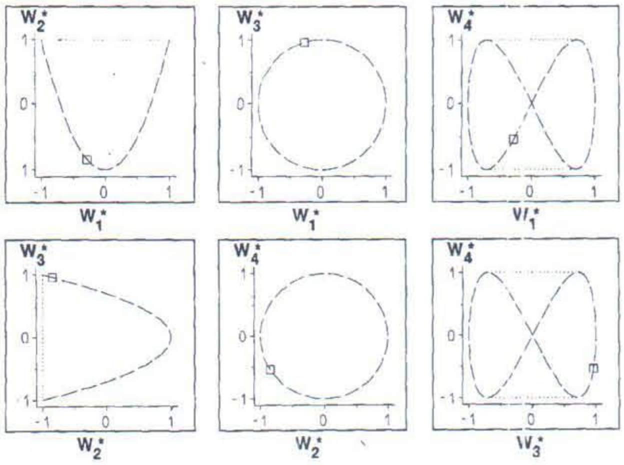

Fig. 3. The optimal point for a simply supported non-orthotropic aramid/epoxy laminate with the aspect ratio a/b=1.5, marked with a square on the projections of the allowable region of the lamination parameters W_1^* to W_4^*

6 Optimization and results, non-orthotropic laminates

When bending-twisting coupling is present in the laminate, all of the four lamination parameters are needed. A standard numerical routine was used for the task of finding the optimal design for the aspect ratios a/b=1.0, 1.2, 1.5, 1.7, 2.0, 2.5, 3.0 and 4.0, with the lamination parameters constrained by the six two-dimensional projections of the allowable region of W_1^* to W_4^*. This results in a too big region and it is possible that the optimal point found under such conditions cannot be realized physically. However, the optimization was carried out and it was seen that the optimal designs always were found very close to the curves corresponding to off-axis unidirectional laminates on the six projections of the allowable region. Figure 3 is an example of this. In Table 2 the optimal points for simply supported plates are compared with off-axis unidirectional laminates. The difference *might* be due to numerical errors – the vicinity of the optimal point is very flat so it is difficult to find the optimal point with good accuracy, even if the optimal buckling load could be determined with good precision – but it is suspected that the true optimum always is an off-axis unidirectional laminate. However, this has not been proved analytically.

The conjecture is that also the optimization of non-orthotropic plates subjected to shear buckling loads is a single parameter problem, or, *at least*, a very close to optimal design can be found by using only one parameter. This parameter can be interpreted as the layup angle θ in

Table 2. Comparison between lamination parameters found by the four parameter optimization (num. opt.), and off-axis unidirectional laminates (UD). Simply supported aramid/epoxy plates

a/b		W_1^*	W_2^*	W_3^*	W_4^*
1.0	num. opt.	−0.01	−1.00	1.00	0.00
	UD 45.0°	0.00	−1.00	1.00	0.00
1.2	num. opt.	−0.15	−0.95	0.99	−0.31
	UD 49.4°	−0.15	−0.95	0.99	−0.30
1.5	num. opt.	−0.33	−0.79	0.95	−0.62
	UD 54.5°	−0.33	−0.79	0.95	−0.62
1.7	num. opt.	−0.39	−0.69	0.92	−0.72
	UD 56.6°	−0.39	−0.69	0.92	−0.72
2.0	num. opt.	−0.44	−0.62	0.90	−0.79
	UD 58.0°	−0.44	−0.62	0.90	−0.79
2.5	num. opt.	−0.43	−0.63	0.90	−0.78
	UD 57.8°	−0.43	−0.63	0.90	−0.78
3.0	num. opt.	−0.49	−0.51	0.87	−0.86
	UD 59.8°	−0.49	−0.51	0.87	−0.86
4.0	num. opt.	−0.52	−0.46	0.85	−0.89
	UD 60.7°	−0.52	−0.46	0.85	−0.89

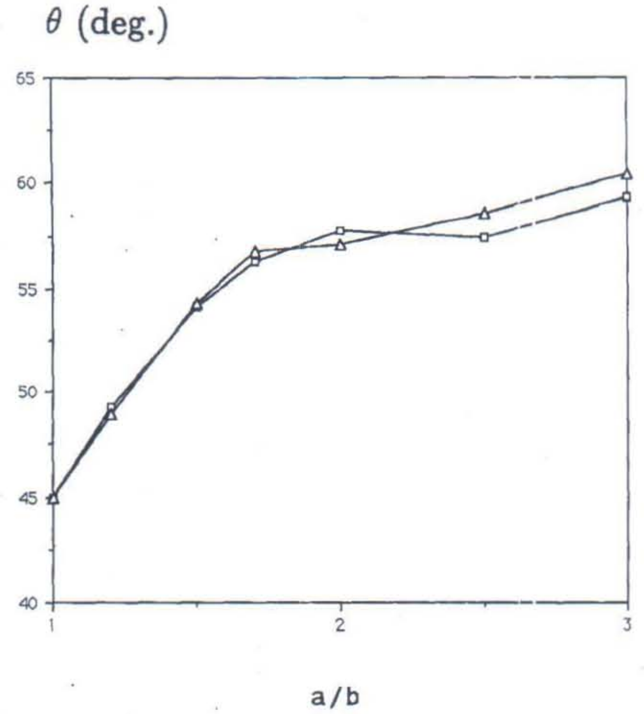

Fig. 4. Optimal layup angle θ vs. the aspect ratio a/b for simply supported aramid/epoxy laminates. Triangles: orthotropic laminate, squares; non-orthotropic laminate

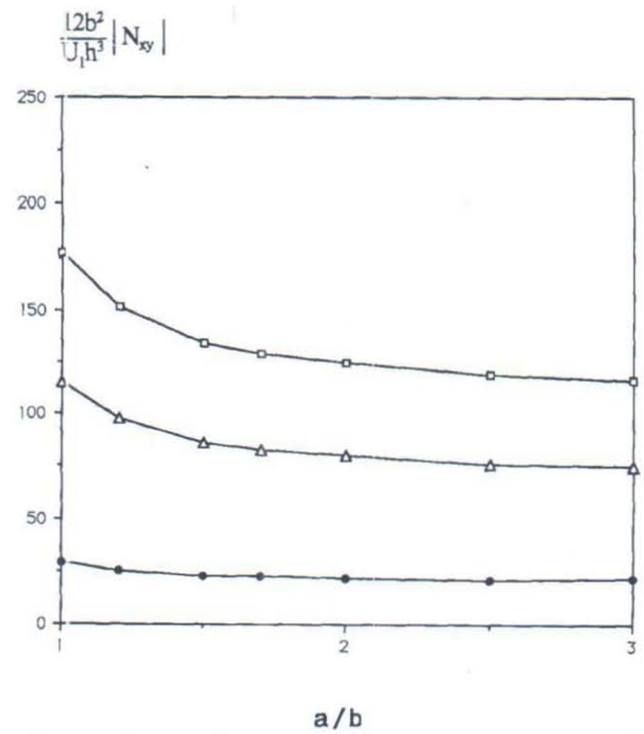

Fig. 6. Optimal shear buckling load vs. the aspect ratio a/b for simply supported aramid/epoxy laminates. Triangles: orthotropic laminate, squares: non-orthotropic laminate, optimal direction, dots: non-orthotropic laminate, reversed direction

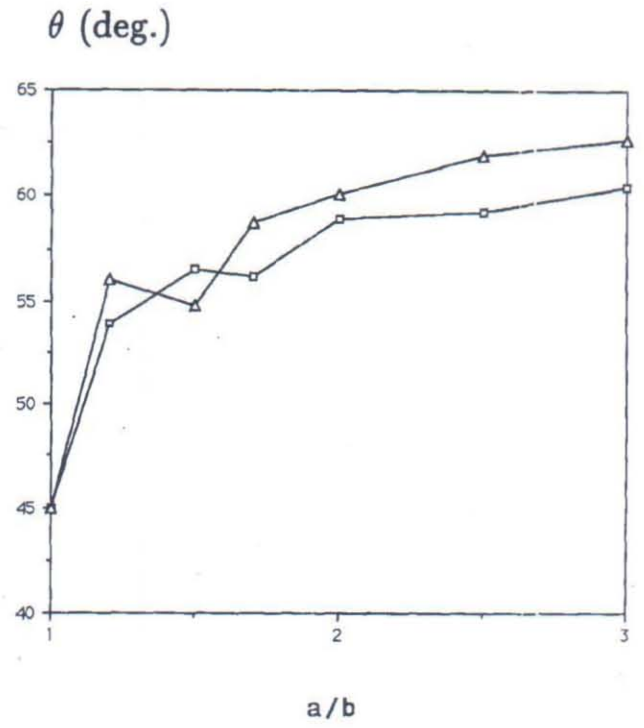

Fig. 5. Optimal layup angle θ vs. the aspect ratio a/b for clamped aramid/epoxy laminates. Triangles: orthotropic laminate, squares; non-orthotropic laminate

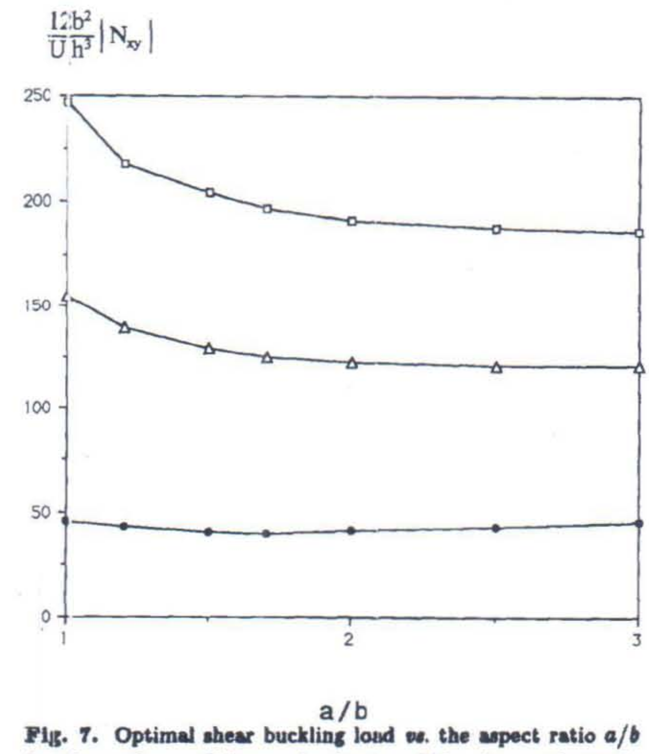

Fig. 7. Optimal shear buckling load vs. the aspect ratio a/b for clamped aramid/epoxy laminates. Triangles: orthotropic laminate, squares: non-orthotropic laminate, optimal direction, dots: non-orthotropic laminate, reversed direction