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Multiple criteria linear programming model for portfolio selection

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The portfolio selection problem is usually considered as a bicriteria optimization problem where a reasonable trade-off between expected rate of return and risk is sought. In the classical Markowitz model the risk is measured with variance, thus generating a quadratic programming model. The Markowitz model is frequently criticized as not consistent with axiomatic models of preferences for choice under risk. Models consistent with the preference axioms are based on the relation of stochastic dominance or on expected utility theory. The former is quite easy to implement for pairwise comparisons of given portfolios whereas it does not offer any computational tool to analyze the portfolio selection problem. The latter, when used for the portfolio selection problem, is restrictive in modeling preferences of investors. In this paper, a multiple criteria linear programming model of the portfolio selection problem is developed. The model is based on the preference axioms for choice under risk. Nevertheless, it allows one to employ the standard multiple criteria procedures to analyze the portfolio selection problem. It is shown that the classical mean-risk approaches resulting in linear programming models correspond to specific solution techniques applied to our multiple criteria model.

Keywords: portfolio selection, multiple criteria, linear programming, equity

1. Introduction

The portfolio selection problem considered is based on a single period model of investment. At the beginning of the period, the investor allocates capital among various securities, assigning a nonnegative weight to each security. During the period, each security generates a random rate of return so that at the end of the period, the capital has been changed by the weighted average of the returns. In selecting security weights, the investor faces a set of linear constraints, one of which is that the weights must sum to one.

Following the seminal work by Markowitz [12], the portfolio selection problem is usually modeled as a bicriteria optimization problem where a reasonable trade-off between expected rate of return and risk is sought. The Markowitz model is frequently criticized as not consistent with axiomatic models of preferences for choice under risk (Bell and Raiffa [1]). Models consistent with the preference axioms are based on the relation of stochastic dominance or on expected utility theory (Levy [9]). The former is quite easy to implement for pairwise comparisons of given portfolios whereas it does not offer any computational recipe to analyze the portfolio selection problem. The latter when used for the portfolio selection problem is restrictive in modeling preferences of investors.

In the classical Markowitz model the risk is measured with variance thus generating a quadratic programming model. Following Sharpe [18], many attempts have been made to linearize the portfolio selection problem (cf. Speranza [19] and references therein). In this paper we develop a multiple criteria linear programming model of the classical portfolio selection problem where the finite set of securities is considered and for each security the expected return is defined with a finite discrete distribution (e.g., by historical data). The model is based on the preference axioms for the choice under risk.

Let $J = \{1, 2, \dots, n\}$ denote the set of securities in which one intends to invest a capital. We assume, as usual, that for each security $j \in J$ there is given a vector of data $(r_{ij})_{i=1,2,\dots,m}$, where r_{ij} is the observed (or forecasted) rate of return at event (time) i for security j (hereafter referred to as outcome). Thus we consider discrete distributions of return defined as m-dimensional lotteries, i.e., by the vectors of m outcomes corresponding to lots $i \in I = \{1, 2, \dots, m\}$. The data forms an $m \times n$ matrix $\mathbf{R} = (r_{ij})_{i=1,\dots,m;\ j=1,\dots,n}$ which columns correspond to securities while rows $\mathbf{r}_i = (r_{ij})_{j=1,2,\dots,n}$ correspond to outcomes. Further, let $\mathbf{x} = (x_j)_{j=1,2,\dots,n}$ denote the vector of decision variables (security weights) defining a portfolio. To represent a portfolio the decision variables must satisfy a set of constraints which define the feasible set Q. The simplest feasible set is defined by the requirement that the decision variables must sum to one, i.e.,

$$Q = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_m)^{\mathrm{T}} \colon \sum_{j=1}^{n} x_j = 1,\ x_j \geqslant 0 \text{ for } j = 1, 2, \dots, n \right\}.$$

The investor usually faces a set of additional side constraints. Hereafter we will assume that Q is a general LP feasible set given in the canonical form as a system of linear equations with nonnegative variables[1]

$$Q = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_m)^{\mathrm{T}} \colon \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqq \mathbf{0} \right\}, \tag{1}$$

where $\mathbf{A}$ is a given $p \times n$ matrix and $\mathbf{b} = (b_1, \dots, b_p)^{\mathrm{T}}$ is a given RHS vector. Hereafter we call vector $\mathbf{x} \in Q$ a *portfolio*.

[1] In the paper we use the following notation for vector inequalities:

$$\begin{aligned}
&\mathbf{x}' \geqq \mathbf{x}'' \Leftrightarrow x'_j \geqslant x''_j \text{ for } j = 1, 2, \dots, n,\\
&\mathbf{x}' \geqslant \mathbf{x}'' \Leftrightarrow \left(\mathbf{x}' \geqq \mathbf{x}'' \text{ and } \mathbf{x}'' \ngeqq \mathbf{x}'\right),\\
&\mathbf{x}' > \mathbf{x}'' \Leftrightarrow x'_j > x''_j \text{ for } j = 1, 2, \dots, n.
\end{aligned}$$

Each vector $\mathbf{x}$ generates a vector of outcomes $\mathbf{y} = \mathbf{R}\mathbf{x} = (\mathbf{r}_1\mathbf{x}, \mathbf{r}_2\mathbf{x}, \ldots, \mathbf{r}_m\mathbf{x})$. Vectors $\mathbf{y}$ we refer to as *achievement vectors*. An achievement vector $\mathbf{y}$ is attainable if it expresses outcomes of a portfolio $\mathbf{x} \in Q$ (i.e., $\mathbf{y} = \mathbf{R}\mathbf{x}$ for some $\mathbf{x} \in Q$).

The portfolio selection problem can be considered as an optimization problem with m uniform objective functions $f_i(\mathbf{x}) = \mathbf{r}_i\mathbf{x} = \sum_{j=1}^{n} r_{ij}x_j$. In the vector form it can be written as

$$\max\{\mathbf{R}\mathbf{x}\colon\ \mathbf{x} \in Q\}, \tag{2}$$

where Q denotes the feasible set (1). Model (2) only specifies that we are interested in maximization of all objective functions. In order to make it operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions. The standard multiple criteria optimization starts with an assumption that the criteria are incomparable. It leads to the concept of the efficient (Pareto-optimal) solutions. In our portfolio selection problem the objective functions are uniform and their values can be directly compared. In fact, we are interested in comparing distributions of outcomes within the achievement vectors rather than the achievement vectors themselves. Moreover, a solution concept should take into account risk aversion. Therefore, model (2) cannot be considered a standard multiple criteria optimization problem. Nevertheless, by utilizing the results concerning the ordering of achievement vectors and several related ideas, it is possible to obtain a linear multiple criteria optimization problem which serves in a surrogate role. That is, by seeking efficient solutions of this new problem, we find solutions of the portfolio selection problem (2) which are optimal with respect to various risk averse preferences consistent with the standard axioms for the choice under risk. It allows one to employ the standard multiple criteria procedures to solve the portfolio selection problem (2).

The paper is organized as follows. In the next section we use the standard axioms for the choice under risk to define the solution concept of equitably efficient solutions of the portfolio selection problem (2). We build a linear multiple criteria model such that its efficient solutions coincide with equitably efficient solutions of the portfolio selection problem. In section 3 we analyze the classical mean-risk approaches which lead to the linear programming models for the portfolio selection problem. We show that they can be viewed as specific solution techniques applied to our multiple criteria model. Further, in section 4 we analyze the ordered weighting approach which by varying the weights allows to identify any equitably efficient solution of the portfolio selection problem (2). This approach leads to linear programming problems with a large number of constraints. However, as shown in section 5 the corresponding dual problems can be effectively solved by the simplex method with the column generation technique.

2. The model

The solution concepts are defined by properties of the corresponding preference model. We assume that solution concepts depend only on evaluation of the achievement vectors (outcomes) while not taking into account any other solution properties not represented within the achievement vectors. Thus, we can limit our considerations to the preference model in the space of achievement vectors. The preference model is completely characterized by the relation of weak preference (Vincke [21]), denoted hereafter with $\succeq$. Namely, the corresponding relations of strict preference $\succ$ and indifference $\cong$ are defined by the following formulas:

$$\mathbf{y}' \succ \mathbf{y}'' \Leftrightarrow \big(\mathbf{y}' \succeq \mathbf{y}'' \text{ and } \mathbf{y}'' \not\succeq \mathbf{y}'\big),$$
$$\mathbf{y}' \cong \mathbf{y}'' \Leftrightarrow \big(\mathbf{y}' \succeq \mathbf{y}'' \text{ and } \mathbf{y}'' \succeq \mathbf{y}'\big).$$

The standard preference model related to the Pareto-optimal solution concept assumes that the preference relation $\succeq$ is *reflexive*:

$$\mathbf{y} \succeq \mathbf{y}, \tag{3}$$

transitive:

$$\big(\mathbf{y}' \succeq \mathbf{y}'' \text{ and } \mathbf{y}'' \succeq \mathbf{y}'''\big) \Rightarrow \mathbf{y}' \succeq \mathbf{y}''', \tag{4}$$

and *strictly monotonic*:

$$\mathbf{y} + \varepsilon \mathbf{e}_i \succ \mathbf{y} \quad \text{for } \varepsilon > 0,\ i = 1, 2, \dots, m, \tag{5}$$

where $\mathbf{e}_i$ denotes the ith unit vector in the criterion space. The last assumption expresses that for each individual objective function more is better (maximization). The preference relations satisfying axioms (3)–(5) are called hereafter *rational preference relations*. The rational preference relations allow us to formalize the Pareto-optimal solution concept with the following definitions. We say that achievement vector $\mathbf{y}'$ rationally dominates $\mathbf{y}''$ ($\mathbf{y}' \succ_r \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all rational preference relations $\succeq$. We say that feasible solution $\mathbf{x} \in Q$ is an efficient (Pareto-optimal) solution of the multiple criteria problem (2), iff $\mathbf{y} = \mathbf{R}\mathbf{x}$ is rationally nondominated.

The relation of weak rational dominance $\succeq_r$ may be expressed in terms of the vector inequality

$$\mathbf{y}' \succeq_r \mathbf{y}'' \Leftrightarrow \mathbf{y}' \geqq \mathbf{y}''.$$

As a consequence, we can state that a feasible solution $\mathbf{x}^0 \in Q$ is an efficient solution of the multiple criteria problem (2), if and only if, there does not exist $\mathbf{x} \in Q$ such that $\mathbf{R}\mathbf{x} \geqslant \mathbf{R}\mathbf{x}^0$. The latter refers to the commonly used definition of the efficient solutions as feasible solutions for which one cannot improve any criterion without worsening another (Chankong and Haimes [2], Steuer [20]). However, the axiomatic definition of the rational preference relation allows us to introduce additional properties of the preferences related to the principles of choice under risk.

While dealing with the uniform criteria, we assume that the preference model is *impartial* (anonymous, symmetric), i.e.,

$$(y_{\tau(1)}, y_{\tau(2)}, \dots, y_{\tau(m)}) \cong (y_1, y_2, \dots, y_m) \tag{6}$$

for any permutation τ of I. Further, to take into account that the investor is risk averse, the preference model should satisfy the Pigou–Dalton principle of transfers. The *principle of transfers* states that a transfer of small amount from an outcome to any relatively worse-off outcome results in a more preferred achievement vector, i.e.,

$$y_{i'} > y_{i''} \Rightarrow \mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \succ \mathbf{y} \quad \text{for } 0 < \varepsilon < y_{i'} - y_{i''},\ i', i'' \in I. \tag{7}$$

The preference relations satisfying all axioms (3)–(7) we will call hereafter equitable rational preference relations.

Requirement of impartiality (6) and the principle of transfers (7) do not contradict the multiple criteria optimization axioms (3)–(5). Therefore, we can consider equitable multiple criteria optimization (Ogryczak [14]) based on the preference model defined by axioms (3)–(7). The equitable rational preference relations allow us to define the concept of equitably efficient solution, similar to the standard efficient (Pareto-optimal) solution defined with the rational preference relations. We say that achievement vector $\mathbf{y}'$ *equitably dominates* $\mathbf{y}''$ ($\mathbf{y}' \succ_e \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all equitable rational preference relations $\succeq$. We say that a portfolio (feasible solution) $\mathbf{x} \in Q$ is *equitably efficient*, (is an equitably efficient solution of the multiple criteria problem (2)) if and only if there does not exist any $\mathbf{x}' \in Q$ such that $\mathbf{R}\mathbf{x}' \succ_e \mathbf{R}\mathbf{x}$. Note that each equitably efficient solution is also an efficient solution but not vice versa.

According to the theory of majorization (Marshall and Olkin [13]), the relation of weak equitable dominance $\succeq_e$ can be expressed in various ways with algebraic inequalities. Namely, the following proposition is valid.

Proposition 1. For $\mathbf{y}', \mathbf{y}'' \in R^m$ each of the three following conditions is equivalent to $\mathbf{y}' \succeq_e \mathbf{y}''$:

(1) for all $z \in R$

$$\sum_{i=1}^{m} \big(z - y'_i\big)_+ \leqslant \sum_{i=1}^{m} \big(z - y''_i\big)_+, \tag{8}$$

where the operator $(\cdot)_+$ denotes the nonnegative part of a number;

(2) for all continuous increasing concave functions u

$$\sum_{i=1}^{m} u\big(y'_i\big) \geqslant \sum_{i=1}^{m} u\big(y''_i\big); \tag{9}$$

(3) for $k = 1, 2, \ldots, m$

$$\sum_{i=1}^{k} \theta_i\big(\mathbf{y}'\big) \geqslant \sum_{i=1}^{k} \theta_i\big(\mathbf{y}''\big), \tag{10}$$

where $\theta_i(\mathbf{y})$ denote increasingly ordered coefficients of vector $\mathbf{y}$, i.e., $\theta_1(\mathbf{y}) \leqslant \theta_2(\mathbf{y}) \leqslant \cdots \leqslant \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$.

Condition (8) defines the (second degree) stochastic dominance relation (cf. Whitmore and Findlay [22]). Condition (9) is employed in the expected utility theory (cf. Fishburn [3], Levy [9]). Thus the relation of equitable dominance is completely consistent with the stochastic dominance and expected utility theory. This guarantees that by looking for various equitably efficient solutions of problem (2) we are able to identify optimal portfolios with respect to various risk averse preferences.

In this paper we focus on condition (10) which is related to the so-called dual theory of choice under risk (Yaari [24]). Using the cumulative ordering map $\bar{\Theta}(\mathbf{y}) = (\bar{\theta}_1(\mathbf{y}), \bar{\theta}_2(\mathbf{y}), \ldots, \bar{\theta}_m(\mathbf{y}))$, where

$$\bar{\theta}_i(\mathbf{y}) = \sum_{k=1}^{i} \theta_k(\mathbf{y}), \tag{11}$$

the condition (10) can be rewritten in terms of vector inequality

$$\mathbf{y}' \succeq_e \mathbf{y}'' \Leftrightarrow \bar{\Theta}(\mathbf{y}') \geqq \bar{\Theta}(\mathbf{y}''). \tag{12}$$

Thus equitable dominance is equivalent to rational dominance of achievement vectors transformed by the cumulative ordering map $\bar{\Theta}$. Hence, condition (10) allows us to express the portfolio selection problem (2) with the equitably rational preferences as a standard multiple criteria program with objective functions modified by the cumulative ordering map $\bar{\Theta}(\mathbf{Rx})$.

Corollary 1. Portfolio $\mathbf{x}$ is an equitably efficient solution of problem (2), if and only if it is an efficient solution of the multiple criteria problem

$$\max\left\{ \left(\bar{\theta}_1(\mathbf{Rx}), \bar{\theta}_2(\mathbf{Rx}), \ldots, \bar{\theta}_m(\mathbf{Rx})\right) \colon \ \mathbf{x} \in Q \right\}. \tag{13}$$

The objective functions in a multiple criteria problem can be divided by positive constants without affecting the set of efficient solutions. For better understanding of the multiple criteria problem (13) for portfolio selection, one may consider normalized objective functions $\bar{\theta}_i(\mathbf{y})/i$ for $i = 1, 2, \ldots, m$. Quantities $\bar{\theta}_i(\mathbf{y})/i$ define partial means of the first i coefficients in the ordered achievement vector $\Theta(\mathbf{y})$, i.e., the means of the i smallest outcomes in $\mathbf{y}$. Note that the first objective $\bar{\theta}_1(\mathbf{y})/1$ represents then the minimum outcome $y_{\min}$ and the last objective $\bar{\theta}_m(\mathbf{y})/m$ represents the expected (mean) outcome $\mu(\mathbf{y}) = \frac{1}{m}\sum_{i=1}^{m} y_i$. Thus the maximization of the expected return and the maximization of worst possible outcome are single objectives in our multiple criteria model. The complete set of m criteria allows us to model all risk averse preferences consistent with axioms (3)–(7).

In income economics, the Lorenz curve (cf. Kendall and Stuart [5], Gastwirth [4]) is a popular tool to explain inequalities. In the context of income distribution, the Lorenz curve is a cumulative population versus income curve. Exactly, all individuals are ranked by income, from poorest to richest. For each rank, we compute the proportion of income earned by all individuals at this rank and all ranks below this rank. The relationship between the proportions of population and income defines the Lorenz

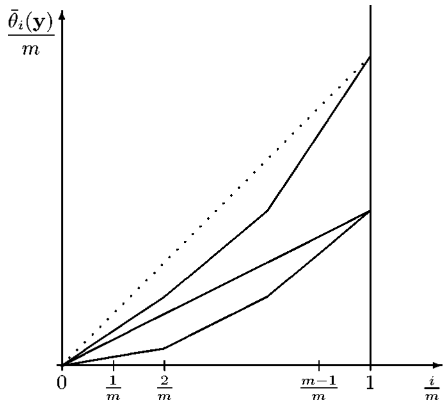

Figure 1. $\bar{\Theta}(\mathbf{y})$ and absolute Lorenz curves.

curve. A perfectly equal distribution of income has the diagonal line as the Lorenz curve. All other distributions generate convex Lorenz curves below the diagonal line. If the curve corresponding to distribution A falls below the curve corresponding to distribution B, then distribution A is considered as more unequal than the the latter one.

Note that the definition of values $\bar{\theta}_i(\mathbf{y})$ for $i = 1, 2, \dots, m$ is similar to the construction of the Lorenz curve for the population of m outcomes. Vector $\bar{\Theta}(\mathbf{y})$ can be viewed graphically with the Lorenz-type curve connecting point (0,0) and points $(i/m, \bar{\theta}_i(\mathbf{y})/m)$ for $i = 1, 2, \dots, m$. In the case of two achievement vectors $\mathbf{y}', \mathbf{y}'' \in Y$ with the same positive total of outcomes $\bar{\theta}_m(\mathbf{y}') = \bar{\theta}_m(\mathbf{y}'')$ (the same positive mean), the inequality $\bar{\Theta}(\mathbf{y}') \geqslant \bar{\Theta}(\mathbf{y}'')$ is equivalent to the dominance $\mathbf{y}'$ over $\mathbf{y}''$ in the sense of Lorenz curves. In the case of positive mean, the Lorenz curves may be considered the graphs of vectors $\frac{1}{\mu(\mathbf{y})}\bar{\Theta}(\mathbf{y})$. Graphs of vectors $\bar{\Theta}(\mathbf{y})$ take the form of unnormalized convex curves (figure 1), the *absolute Lorenz curves*. Note that in terms of the Lorenz curves no achievement vector can be better than the vector of equal outcomes. Relation (12) takes into account also values of outcomes. Vectors of equal outcomes are distinguished according to the value of their outcomes. They are graphically represented with various ascent lines in figure 1. With the preference relation (12), an achievement vector of large unequal outcomes may be preferred to an achievement vector with small equal outcomes.

The individual objective functions of problem (13) are concave piece-wise linear function of achievement vector $\mathbf{y} = \mathbf{Rx}$. They can be written in the form

$$\bar{\theta}_i(\mathbf{y}) = \min_{\tau \in \Pi} \left(\sum_{k=1}^{i} y_{\tau(k)} \right),$$

where Π denotes the set of all permutations τ of the index set I. Thus, our portfolio selection problem (13) can be expressed as the following multiple criteria linear program:

$$
\begin{aligned}
\text{maximize}\quad & (z_1, z_2, \dots, z_m) && (14)\\
\text{subject to}\quad & \mathbf{x} \in Q, && (15)\\
& y_i = \mathbf{r}_i\mathbf{x} \quad \text{for } i = 1, 2, \dots, m, && (16)\\
& z_i \leqslant \sum_{k=1}^{i} y_{\tau(k)} \quad \text{for } \tau \in \Pi,\ i = 1, 2, \dots, m. && (17)
\end{aligned}
$$

Multiple criteria linear program (14)–(17) is equivalent to problem (13) as stated in the following proposition (Kostreva and Ogryczak [8]).

Proposition 2. A triple $(\mathbf{x}^0, \mathbf{y}^0, \mathbf{z}^0)$ is an efficient solution of (14)–(17), if and only if, $\mathbf{y}^0 = \mathbf{R}\mathbf{x}^0$, $\mathbf{z}^0 = \bar{\Theta}(\mathbf{y}^0)$ and $\mathbf{x}^0$ is an efficient solution of problem (13).

Corollary 2. A triple $(\mathbf{x}^0, \mathbf{y}^0, \mathbf{z}^0)$ is an efficient solution of (14)–(17), if and only if, $\mathbf{y}^0 = \mathbf{R}\mathbf{x}^0$, $\mathbf{z}^0 = \bar{\Theta}(\mathbf{y}^0)$ and $\mathbf{x}^0$ is an equitably efficient solution of the portfolio selection problem (2).

Corollary 2 guarantees that by looking for various efficient solutions of the multiple criteria linear program (14)–(17), we are able to identify solutions of the portfolio selection problem (2) which are optimal with respect to various risk averse preferences. Thus the problem (14)–(17) is a linear multiple criteria model of the portfolio selection problem.

3. Bicriteria approaches

Efficient solutions of the multiple criteria problem (2) can be generated with simple scalarizations of the problem. Most of them are based on the sum of individual outcomes

$$
\max\left\{ \sum_{i=1}^{m} \mathbf{r}^i\mathbf{x}\colon\ \mathbf{x} \in Q \right\}, \tag{18}
$$

or on the maximin approach

$$
\max\left\{ \min_{i=1,\dots,m} \mathbf{r}^i\mathbf{x}\colon\ \mathbf{x} \in Q \right\}. \tag{19}
$$

Scalarization (18) always generates efficient solutions as the corresponding preference relation is a rational preference relation (it maintains the properties of reflexivity, transitivity and strict monotonicity). The maximin scalarization (19) generates efficient solutions except in the case of alternative optimal solutions. That means, the optimal solution of (19) can be (rationally) dominated only by another optimal solution. Thus the optimal set of (19) contains an efficient solution and the unique (in the criterion space) optimal solution is efficient. Scalarization (18) is equivalent to maximization of the expected outcome whereas scalarization (19) corresponds to maximization of worst outcome. Both the corresponding preference relations are impartial but they do not satisfy the principle of transfers. Therefore, scalarizations (18) and (19), in the general case, may generate solutions which are not equitably efficient.

Corollary 1 allows one to generate equitably efficient solutions of (2) as efficient solutions of problem (13). Note that scalarization (18), maximizing the expected outcome, corresponds to maximization of the last (mth) objective in problem (13). Similarly, the maximin scalarization (19) corresponds to maximization of the first objective in (13). Thus, in the case of $m = 2$, the set of equitably efficient solutions is equal to the set of efficient solutions of the bicriteria problem with objectives defined as the minimum and the sum of the original two objectives. Certainly it is not true in the portfolio selection problem where m is larger. In the general case of arbitrarily large m, the following corollary is valid.

Corollary 3. Except for portfolios with identical mean and worst outcome, every efficient solution to the bicriteria problem

$$\max\left\{\left(\min_{i=1,\dots,m} \mathbf{r}^i\mathbf{x}, \sum_{i=1}^{m} \mathbf{r}^i\mathbf{x}\right):\ \mathbf{x} \in Q\right\} \tag{20}$$

is an equitably efficient solution of the portfolio selection problem (2).

Bicriteria problem (20) may be considered a mean-risk approach with the risk measure $\varrho(\mathbf{y})$ defined as the maximum (downside) deviation (Young [28])

$$\Delta(\mathbf{y}) = \max_{i=1,\dots,m} \big(\mu(\mathbf{y}) - y_i\big) = \frac{1}{m}\sum_{i=1}^{m} y_i - \min_{i=1,\dots,m} y_i = \frac{1}{m}\bar{\theta}_m(\mathbf{y}) - \bar{\theta}_1(\mathbf{y}). \tag{21}$$

An important advantage of mean-risk approaches is the possibility of a pictorial trade-off analysis. Having assumed a trade-off coefficient λ between the risk an the mean, one may directly compare real values of $\mu(\mathbf{y}) - \lambda\varrho(\mathbf{y})$. The following proposition justifies such an analysis for risk defined as the maximum deviation (21).

Proposition 3. Except for portfolios with identical mean and maximum deviation, every portfolio $\mathbf{x} \in Q$ that is maximal by $\mu(\mathbf{Rx}) - \lambda\Delta(\mathbf{Rx})$ with $0 < \lambda < 1$ is an equitably efficient solution of the portfolio selection problem (2).

Proof. Let $0 < \lambda < 1$ and $\mathbf{x}^0 \in Q$ be maximal by $\mu(\mathbf{Rx}) - \lambda\Delta(\mathbf{Rx})$. Note that

$$\mu(\mathbf{Rx}) - \lambda\Delta(\mathbf{Rx}) = \lambda\bar{\theta}_1(\mathbf{Rx}) + \frac{1-\lambda}{m}\bar{\theta}_m(\mathbf{Rx}). \tag{22}$$

Hence, $\mathbf{x}^0$ is an efficient solution of the bicriteria problem (20) and, due to corollary 3, $\mathbf{x}^0$ is an equitably efficient solution of the portfolio selection problem (2). $\square$

The maximum deviation is a very pessimistic risk measure related to the worst case analysis. It is in some manner very "rough" as it does not take into account the distribution of outcomes other than the worst one which causes that only two objective functions $\bar{\theta}_i(\mathbf{y})$ from (13) are used. There are risk measures taking into account all the quantities $\bar{\theta}_i(\mathbf{y})$.

Konno and Yamazaki [7] introduced the mean-risk model using the absolute deviation

$$\delta(\mathbf{y}) = \frac{1}{2m}\sum_{i=1}^{m}\bigl|\mu(\mathbf{y}) - y_i\bigr| = \frac{1}{m}\sum_{i:y_i<\mu(\mathbf{y})}\bigl[\mu(\mathbf{y}) - y_i\bigr] \tag{23}$$

as the risk measure. The absolute deviation can be expressed in terms of $\bar{\theta}_i(\mathbf{y})$ as follows:

$$\delta(\mathbf{y}) = \frac{1}{m}\sum_{i:\theta_i(\mathbf{y})<\mu(\mathbf{y})}\bigl[\mu(\mathbf{y}) - \theta_i(\mathbf{y})\bigr] = \frac{1}{m}\max_{i=1,\dots,m-1}\left[\frac{i}{m}\bar{\theta}_m(\mathbf{y}) - \bar{\theta}_i(\mathbf{y})\right]. \tag{24}$$

It leads to the following assertion.

Proposition 4. Except for portfolios with identical mean and absolute deviation, every portfolio $\mathbf{x} \in Q$ that is maximal by $\mu(\mathbf{Rx}) - \lambda\delta(\mathbf{Rx})$ with $0 < \lambda < m/(m-1)$ is an equitably efficient solution of the portfolio selection problem (2).

Proof. Let $0 < \lambda < m/(m-1)$ and $\mathbf{x}^0 \in Q$ be maximal by $\mu(\mathbf{Rx}) - \lambda\delta(\mathbf{Rx})$. Note that, due to (24),

$$\begin{aligned}\mu(\mathbf{Rx}) - \lambda\delta(\mathbf{Rx}) &= \frac{1}{m}\bar{\theta}_m(\mathbf{Rx}) + \frac{\lambda}{m}\min_{i=1,\dots,m-1}\left[\bar{\theta}_i(\mathbf{Rx}) - \frac{i}{m}\bar{\theta}_m(\mathbf{Rx})\right]\\ &= \min_{i=1,\dots,m-1}\left[\frac{\lambda}{m}\bar{\theta}_i(\mathbf{Rx}) + \frac{m-i\lambda}{m^2}\bar{\theta}_m(\mathbf{Rx})\right].\end{aligned}$$

Thus, $\mathbf{x}^0$ is an optimal solution to the maximin scalarization of the multiple criteria problem:

$$\max\bigl\{\bigl(g_1(\mathbf{Rx}), g_2(\mathbf{Rx}), \dots, g_{m-1}(\mathbf{Rx})\bigr):\ \mathbf{x} \in Q\bigr\} \tag{25}$$

with $m-1$ objective functions g_i given by the formula:

$$g_i(\mathbf{y}) = \frac{\lambda}{m}\bar{\theta}_i(\mathbf{y}) + \frac{m-i\lambda}{m^2}\bar{\theta}_m(\mathbf{y}) \quad \text{for } i = 1, 2, \dots, m-1. \tag{26}$$

Moreover, both the coefficients in (26) are positive and therefore every efficient solution of (25) is also an efficient solution of problem (13).

Suppose there exists a portfolio $\mathbf{x}' \in Q$ which equitably dominates $\mathbf{x}^0$. Then $\bar{\Theta}(\mathbf{Rx}') \geqslant \bar{\Theta}(\mathbf{Rx}^0)$ and, due to positive coefficients in (26), $g_i(\mathbf{Rx}') \geqslant g_i(\mathbf{Rx}^0)$ for $i = 1, 2, \dots, m-1$. On the other hand, $\min_{i=1,\dots,m-1} g_i(\mathbf{Rx}') \leqslant \min_{i=1,\dots,m-1} g_i(\mathbf{Rx}^0)$.

Hence, there exists index i_0 such that $g_{i_0}(\mathbf{Rx}') = g_{i_0}(\mathbf{Rx}^0)$ and therefore $\bar{\theta}_m(\mathbf{Rx}') = \bar{\theta}_m(\mathbf{Rx}^0)$. Thus, $\mu(\mathbf{Rx}') = \mu(\mathbf{Rx}^0)$ and $\delta(\mathbf{Rx}') = \delta(\mathbf{Rx}^0)$, which completes the proof. □

Yitzhaki [27] introduced the mean-risk model using Gini's mean (absolute) difference

$$G(\mathbf{y}) = \frac{1}{2m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}|y_i - y_j| \tag{27}$$

as the risk measure. Gini's mean difference can be expressed in terms of $\bar{\theta}_i(\mathbf{y})$ as follows:

$$G(\mathbf{y}) = \frac{1}{m^2}\sum_{i=1}^{m-1}\bigl[i\bar{\theta}_{i+1}(\mathbf{y}) - (i+1)\bar{\theta}_i(\mathbf{y})\bigr] = \frac{m-1}{m^2}\bar{\theta}_m(\mathbf{y}) - \frac{2}{m^2}\sum_{i=1}^{m-1}\bar{\theta}_i(\mathbf{y}). \tag{28}$$

It leads to the following assertion.

Proposition 5. Every portfolio $\mathbf{x} \in Q$ that is maximal by $\mu(\mathbf{Rx}) - \lambda G(\mathbf{Rx})$ with $0 < \lambda < m/(m-1)$ is an equitably efficient solution of the portfolio selection problem (2).

Proof. Let $0 < \lambda < m/(m-1)$ and $\mathbf{x}^0 \in Q$ be maximal by $\mu(\mathbf{Rx}) - \lambda G(\mathbf{Rx})$. Note that, due to (28),

$$\mu(\mathbf{Rx}) - \lambda G(\mathbf{Rx}) = \frac{2\lambda}{m^2}\sum_{i=1}^{m-1}\bar{\theta}_i(\mathbf{Rx}) + \frac{m-\lambda(m-1)}{m^2}\bar{\theta}_m(\mathbf{Rx}). \tag{29}$$

Hence, in the case of $0 < \lambda < m/(m-1)$, function $\mu(\mathbf{Rx}) - \lambda G(\mathbf{Rx})$ is a linear combination with positive weights of the objective functions $\bar{\theta}_i(\mathbf{Rx})$ for $i = 1, 2, \ldots, m$. Therefore, $\mathbf{x}^0$ is an efficient solution of the multiple criteria problem (13) and, due to corollary 1, $\mathbf{x}^0$ is an equitably efficient solution of the portfolio selection problem (2). □

The three risk measures, we have considered, lead to parametric linear programming problems:

$$\max\bigl\{\mu(\mathbf{Rx}) - \lambda\varrho(\mathbf{Rx}):\ \mathbf{x} \in Q\bigr\}, \tag{30}$$

while looking for a mean-risk compromise. We have shown that in the case of $0 < \lambda < 1$ they can be considered scalarizations of the multiple criteria problem (13). It can be illustrated in the Lorenz-type diagram we considered in the previous section (figure 1). Recall that vector $\bar{\Theta}(\mathbf{y})$ can be viewed graphically with the absolute Lorenz curve connecting point (0,0) and points $(i/m, \bar{\theta}_i(\mathbf{y})/m)$ for $i = 1, 2, \ldots, m$, where the last point (for $i = m$) is $(1, \mu(\mathbf{y}))$. Note that in our model the risk-free achievement vector with mean value $\mu(\mathbf{y})$ has all the coefficients equal to $\mu(\mathbf{y})$ and its absolute Lorenz curve is the ascent line connecting points $(0, 0)$ and $(1, \mu(\mathbf{y}))$. Hence, the space between the absolute Lorenz curve $(i/m, \bar{\theta}_i(\mathbf{y})/m)$ and its ascent line represents the dispersion (and thereby the riskiness) of $\mathbf{y}$ in comparison to the deterministic outcome of $\mu(\mathbf{y})$.

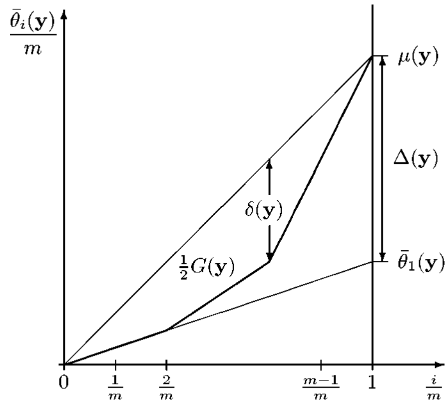

Figure 2. $\bar{\Theta}(\mathbf{y})$ and risk measures.

We shall call it the *dispersion space*. Both size and shape of the dispersion space are important for complete description of the riskiness. Nevertheless, it is quite natural to consider some size parameters as summary characteristics of riskiness. As shown in figure 2, all three risk measures, we have considered, represent some size parameters of the dispersion space. Note that vertical diameter of the dispersion space at point i/m is given as $\delta_i(\mathbf{y}) = \frac{i}{m^2}\bar{\theta}_m(\mathbf{y}) - \frac{1}{m}\bar{\theta}_i(\mathbf{y})$. Hence, for the absolute deviation, due to (24), we get $\delta(\mathbf{y}) = \max_{i=1,\dots,m} \delta_i(\mathbf{y})$. That means, $\delta(\mathbf{y})$ represents the largest vertical diameter of the dispersion space. Similarly, for the maximum deviation, due to (21), we get $\Delta(\mathbf{y}) = m\delta_1(\mathbf{y})$. Thus, $\Delta(\mathbf{y})$ represents the projection of $\delta_1(\mathbf{y})$ onto the vertical line at $i = m$ or the largest vertical diameter of the corresponding triangular envelope of the dispersion space. Gini's mean difference, due to (28), satisfies $G(\mathbf{y}) = \frac{2}{m}\sum_{i=1}^{m-1}\delta_i(\mathbf{y})$. That means, $G(\mathbf{y})$ is twice the area of the dispersion space. This explains why for this mean-risk approach we get the strongest result in the sense that every optimal solution of the corresponding problem (30) with $0 < \lambda \leqslant 1$ is equitably efficient (proposition 5). Similar strong result one can get using a combination of Gini's mean difference with other risk measures thus enriching the corresponding preference model. In particular, the following assertion follows from propositions 3 and 5.

Corollary 4. Every portfolio $\mathbf{x} \in Q$ that is maximal by $\mu(\mathbf{Rx}) - \lambda_1 G(\mathbf{Rx}) - \lambda_2\Delta(\mathbf{Rx})$ with $\lambda_1 > 0$, $\lambda_2 \geqslant 0$ and $\lambda_1 + \lambda_2 \leqslant 1$ is an equitably efficient solution of the portfolio selection problem (2).

4. Ordered weighted aggregation

In the case of efficiency, one may use the scalarization (18) with weighted objective functions to generate various efficient solutions. In fact, it provides a complete parameterization of the efficient set for multiple criteria linear programs. In the case of equitable multiple criteria programming, one cannot assign various weights to individual objective functions, as that violates the requirement of impartiality (6). However, due to corollary 1, the weighting approach can be applied to problem (13) resulting in the scalarization:

$$\max\left\{\sum_{i=1}^{m} w_i\bar{\theta}_i(\mathbf{Rx}):\ \mathbf{x}\in Q\right\}. \tag{31}$$

Note that, due to the definition of map $\bar{\Theta}$ with (11), the above problem can be expressed in the form with weights $\bar{w}_i = \sum_{j=i}^{m} w_j$ $(i = 1, 2, \ldots, m)$ allocated to coefficients of the ordered achievement vector $\Theta(\mathbf{Rx})$. Such an approach to multiple criteria optimization was introduced by Yager [25] as so-called Ordered Weighted Averaging (OWA). Since its introduction, the OWA aggregation has been applied to many fields as fuzzy logic controllers (Yager and Filev [26]) and in decision making under uncertainty (Segal [17]).

When applying the OWA aggregation to our portfolio selection problem (2) we get:

$$\max\left\{\sum_{i=1}^{m} w_i\theta_i(\mathbf{Rx}):\ \mathbf{x}\in Q\right\}. \tag{32}$$

Proposition 6. A portfolio $\mathbf{x}^0 \in Q$ is an equitably efficient solution of the portfolio selection problem (2), if and only if, there exist strictly decreasing and positive weights w_i, i.e.,

$$w_1 > w_2 > \cdots > w_{m-1} > w_m > 0, \tag{33}$$

such that $\mathbf{x}^0$ is an optimal solution of the corresponding OWA problem (32).

Proof. Problem (32) with weights w_i can be expressed in the form

$$\max\left\{\sum_{i=1}^{m} w_i'\bar{\theta}_i(\mathbf{Rx}):\ \mathbf{x}\in Q\right\},$$

where coefficients w_i' are defined as $w_m' = w_m$ and $w_i' = w_i - w_{i+1}$ for $i = 1, 2, \ldots, m-1$. If (33) holds, then $w_i' > 0$ for $i = 1, 2, \ldots, m$. Thus, due to corollary 1, each optimal solution of (32) is an equitably efficient solution of (2).

Further, we need to show that for each equitably efficient solution $\mathbf{x}^0 \in Q$ there exist strictly decreasing and positive weights w_i (i.e., weights satisfying (33)) such that $\mathbf{x}^0$ is an optimal solution of the corresponding OWA problem (32). Due to proposition 2, if $\mathbf{x}^0$ is an equitably efficient solution of (2), then $(\mathbf{x}^0, \mathbf{Rx}^0, \bar{\Theta}(\mathbf{Rx}^0))$ is an efficient solution of multiple criteria linear program (14)–(17). Thus, from the theory of multiple criteria linear optimization (Steuer [20]), there exist positive weights $\bar{w}_i$ $(i = 1, 2, \ldots, m)$ such that $(\mathbf{x}^0, \mathbf{Rx}^0, \bar{\Theta}(\mathbf{Rx}^0))$ is an optimal solution of the problem

$$\max\left\{\sum_{i=1}^{m} \bar{w}_i z_i\colon\ (15)\text{–}(17)\right\}.$$

Due to positive weights $\bar{w}_i$, the above problem is equivalent to the problem

$$\max\left\{\sum_{i=1}^{m} \bar{w}_i \bar{\theta}_i(\mathbf{Rx})\colon\ \mathbf{x} \in Q\right\}$$

which, by definition of the map $\bar{\Theta}$ with (11), can be expressed as the OWA problem (32) with weights $w_i = \sum_{j=i}^{m} \bar{w}_j$ $(i = 1, 2, \ldots, m)$. Moreover, weights w_i satisfy the requirement (33). Thus, there exist strictly decreasing and positive weights w_i such that $\mathbf{x}^0$ is an optimal solution of the corresponding OWA problem (32). $\Box$

From proposition 6 it follows that, by looking for the OWA optimal solutions for various decreasing and positive weights, we are able to identify various equitably efficient solutions of problem (2) and thereby to find portfolios optimal with respect to various risk averse preferences. Moreover, any portfolio optimal with respect to some risk averse preferences can be found as the optimal solution of the OWA problem (32) with some weights satisfying (33). Note that the mean-risk approach with the maximum deviation $\Delta(\mathbf{y})$ (21) as the risk measure, due to (22), may be viewed as the OWA aggregation (32) with weights: $w_1 = (1 + (m-1)\lambda)/m$ and $w_i = (1-\lambda)/m$ for $i = 2, \ldots, m$. Hence, for the trade-off coefficient $0 < \lambda < 1$ all the weights are positive but $w_2 = w_3 = \cdots = w_m$ which causes that not all optimal solutions are equitably efficient. Similar, the mean-risk approach with Gini's mean difference $G(\mathbf{y})$ (27) as the risk measure, due to (29), may be viewed as the OWA aggregation (32) with weights $w_i = (m + (m - 2i + 1)\lambda)/m^2$ for $i = 1, 2, \ldots, m$. Hence, for the trade-off coefficient $0 < \lambda < m/(m-1)$ the weights are positive and strictly decreasing (33) which causes that every optimal solution is equitably efficient. However, $w_i - w_{i+1} = 2\lambda/m^2$ for all $i = 1, 2, \ldots, m-1$. Thus this mean-risk approach, in terms of the OWA aggregation, considers only weights decreasing by a constant step. Therefore, not all equitably efficient solutions can be found in this way. In the next section we analyze in details a solution procedure for the OWA problems with arbitrary weights satisfying (33).

As the limiting case of the OWA problem (32), when the differences among weights w_i tend to infinity, we get the lexicographic problem:

$$\operatorname{lexmax}\left\{\left(\theta_1(\mathbf{Rx}), \theta_2(\mathbf{Rx}), \ldots, \theta_m(\mathbf{Rx})\right)\colon\ \mathbf{x} \in Q\right\}, \tag{34}$$

where first $\theta_1(\mathbf{Rx})$ is maximized, next $\theta_2(\mathbf{Rx})$ and so on. Problem (34) represents the lexicographic maximin approach to the original multiple criteria problem (2). It is a refinement (regularization) of the standard maximin scalarization (19), but in the former, in addition to the smallest outcome, we maximize also the second smallest outcome (provided that the smallest one remains as large as possible), maximize the third smallest (provided that the two smallest remain as large as possible), and so on. The lexicographic maximin solution is known in the game theory as the nucleolus of a matrix game (Potters and Tijs [16]). In the case of linear objective functions and convex feasible set, there exists a dominating objective function which is constant on the entire optimal set of the maximin problem. Therefore, similar to the nucleolus of a matrix game, the lexicographic maximin solution of problem (2) can easily be found by sequential optimization with elimination of the dominating functions. This approach has been recently used for linear programming problems related to multiperiod resource allocation (Klein et al. [6]) and for linear multiple criteria problems (Marchi and Oviedo [11]).

Due to (11), problem (34) is equivalent to the problem

$$\operatorname{lexmax}\big\{\big(\bar\theta_1(\mathbf{Rx}), \bar\theta_2(\mathbf{Rx}), \ldots, \bar\theta_m(\mathbf{Rx})\big)\colon\ \mathbf{x} \in Q\big\}$$

which can be considered the standard lexicographic optimization applied to problem (13). As the lexicographic optimization generates efficient solutions, thus due to corollary 1, we get the following assertion.

Corollary 5. The optimal solution of the lexicographic maximin problem (34) is an equitably efficient solution of the portfolio selection problem (2).

The lexicographic maximin solution is unique with respect to the ordered achievement vectors $\Theta(\mathbf{Rx})$. It can be considered in some sense the "most equitable solution" or "the most risk averse portfolio". Note that one may wish to consider the multiple criteria problem (13) as an equitable problem (with an equitable rational preference relation). In such a situation we should apply corollary 1 to problem (13). It results in the problem with doubly cumulative ordered criteria which again may be considered as equitable. As the limit of such an approach we get the lexicographic maximin problem (34). One may wish to look for the "least equitable solution" applying reverse lexicographic maximization to the problem (13), i.e., solving the lexicographic problem:

$$\operatorname{lexmax}\big\{\big(\bar\theta_m(\mathbf{Rx}), \bar\theta_{m-1}(\mathbf{Rx}), \ldots, \bar\theta_1(\mathbf{Rx})\big)\colon\ \mathbf{x} \in Q\big\}, \tag{35}$$

where first $\bar\theta_m(\mathbf{Rx})$ is maximized, next $\bar\theta_{m-1}(\mathbf{Rx})$ and so on. While the lexicographic maximin (34) is a refinement of the standard maximin approach (19), the problem (35) is a lexicographic refinement of the scalarization (18). Note, that in the lexicographic optimization problem dividing objectives by constants does not affect the solution and $\bar\theta_i(\mathbf{y})/i$ represents the mean of i largest coefficients in the achievement vector $\mathbf{y}$. Therefore, problem (35) is a refinement of maximization of expected return and we refer to it as the lexicographic mean problem. As the lexicographic optimization generates efficient solutions, from corollary 1, we get the following assertion.

Corollary 6. The optimal solution of the lexicographic mean problem (35) is an equitably efficient solution of the portfolio selection problem (2).

By using the basic properties of the lexicographic optimization and the equity $\theta_i(\mathbf{y}) = \theta_{m-i+1}(-\mathbf{y})$, the lexicographic mean problem can be rewritten as the lexicographic problem

$$\operatorname{lexmax}\left\{\left(\bar{\theta}_m(\mathbf{Rx}), \theta_1(-\mathbf{Rx}), \theta_2(-\mathbf{Rx}), \ldots, \theta_{m-1}(-\mathbf{Rx})\right):\ \mathbf{x} \in Q\right\}.$$

Hence, the lexicographic mean problem (35) can be implemented as the the lexicographic maximin approach to problem with negated outcomes and the feasible set defined by all the portfolios with the maximal expected return. Thus, similar to problem (34), the lexicographic mean solution of problem (2) can easily be found by sequential optimization.

5. Solution technique

The ordering operator Θ used in the OWA aggregation is nonlinear and, in general, it is hard to implement. Note, however, that for weights w_i satisfying (33), for any permutation τ of I the following inequality holds:

$$\sum_{i=1}^{m} w_{\tau(i)} y_i \geqslant \sum_{i=1}^{m} w_i \theta_i(\mathbf{y}). \tag{36}$$

Thus, the OWA aggregation is a concave piecewise linear function

$$\sum_{i=1}^{m} w_i \theta_i(\mathbf{y}) = \min_{\tau \in \Pi} \left(\sum_{i=1}^{m} w_{\tau(i)} y_i \right), \tag{37}$$

where Π denotes the set of all permutations τ of I. It leads us to the following sufficient and necessary optimality conditions for the OWA aggregations.

Proposition 7. If a portfolio $\mathbf{x}^0 \in Q$ is an optimal solution of the linear problem

$$\max\left\{ \sum_{i=1}^{m} w_i \mathbf{r}^{\bar{\tau}(i)} \mathbf{x}:\ \mathbf{x} \in Q \right\}, \tag{38}$$

where weights w_i satisfy (33) and $\bar{\tau}$ is such a permutation that

$$\mathbf{r}^{\bar{\tau}(1)}\mathbf{x}^0 \leqslant \mathbf{r}^{\bar{\tau}(2)}\mathbf{x}^0 \leqslant \cdots \leqslant \mathbf{r}^{\bar{\tau}(m)}\mathbf{x}^0, \tag{39}$$

then $\mathbf{x}^0$ is an optimal solution of the corresponding OWA problem (32).

Proof. If for $\mathbf{x}^0 \in Q$ satisfying (39) there exist strictly decreasing and positive weights w_i such that $\mathbf{x}^0$ is an optimal solution of the linear problem (38), then

$$\sum_{i=1}^{m} w_i \theta_i\left(\mathbf{Rx}^0\right) = \sum_{i=1}^{m} w_i \mathbf{r}^{\bar{\tau}(i)} \mathbf{x}^0 \geqslant \sum_{i=1}^{m} w_i \mathbf{r}^{\bar{\tau}(i)} \mathbf{x} \geqslant \sum_{i=1}^{m} w_i \theta_i(\mathbf{Rx})$$

for each $\mathbf{x} \in Q$. Thus $\mathbf{x}^0$ is an optimal solution of the corresponding OWA problem (32). $\square$

Proposition 8. A portfolio $\mathbf{x}^0 \in Q$ such that for some permutation $\bar{\tau}$

$$\mathbf{r}^{\bar{\tau}(1)}\mathbf{x}^0 < \mathbf{r}^{\bar{\tau}(2)}\mathbf{x}^0 < \cdots < \mathbf{r}^{\bar{\tau}(m)}\mathbf{x}^0 \tag{40}$$

is an optimal solution of the OWA problem (32) with strictly decreasing and positive weights w_i (i.e., weights satisfying (33)), if and only if, $\mathbf{x}^0$ is an optimal solution of the corresponding linear problem (38).

Proof. Sufficiency of the condition follows from proposition 7. Thus we only need to prove its necessity. Let $\mathbf{x}^0 \in Q$ be an optimal solution of the OWA problem (32) with some strictly decreasing and positive weights w_i (i.e., weights satisfying (33)). We will show that $\mathbf{x}^0$ satisfying (40) is also an optimal solution of the corresponding problem (38) with the same weights. If not, then there exists $\mathbf{x}^1 \in Q$ such that $\sum_{i=1}^m w_i \mathbf{r}^{\tau(i)}\mathbf{x}^1 > \sum_{i=1}^m w_i \mathbf{r}^{\tau(i)}\mathbf{x}^0$. Note that due to convexity of the feasible set Q, for any $0 < \varepsilon < 1$ vector $\mathbf{x}^\varepsilon = (1-\varepsilon)\mathbf{x}^0 + \varepsilon\mathbf{x}^1$ is a feasible solution and $\sum_{i=1}^m w_i \mathbf{r}^{\tau(i)}\mathbf{x}^\varepsilon > \sum_{i=1}^m w_i \mathbf{r}^{\tau(i)}\mathbf{x}^0$. Moreover, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$

$$\mathbf{r}^{\tau(1)}\mathbf{x}^\varepsilon < \mathbf{r}^{\tau(2)}\mathbf{x}^\varepsilon < \cdots < \mathbf{r}^{\tau(m)}\mathbf{x}^\varepsilon.$$

Hence, for sufficiently small positive ε

$$\sum_{i=1}^m w_i \theta_i\big(\mathbf{R}\mathbf{x}^\varepsilon\big) = \sum_{i=1}^m w_i \mathbf{r}^{\tau(i)}\mathbf{x}^\varepsilon > \sum_{i=1}^m w_i \mathbf{r}^{\tau(i)}\mathbf{x}^0 = \sum_{i=1}^m w_i \theta_i\big(\mathbf{R}\mathbf{x}^0\big),$$

which contradicts optimality of $\mathbf{x}^0$ for the OWA problem. $\square$

Recall that in our portfolio selection problem the feasible set Q is given in the canonical form as (1). Equation (37) allows us to express the corresponding OWA problem (32) as the following linear program:

$$\text{maximize} \quad z \tag{41}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \tag{42}$$

$$\mathbf{y} - \mathbf{R}\mathbf{x} = \mathbf{0}, \tag{43}$$

$$z - \sum_{i=1}^m w_{\tau(i)} y_i \leqslant 0 \quad \text{for } \tau \in \Pi, \tag{44}$$

$$x_j \geqslant 0 \quad \text{for } j = 1, 2, \ldots, n. \tag{45}$$

It is an LP problem with $n+m+1$ variables and $p+m+m!$ constraints. In problem (41)–(45) the ordering operator Θ is replaced with $m!$ linear inequalities (44). It generates a large number of constraints but all the inequalities (44) are defined by permutations of the single vector of weights w_i.

While solving an LP problem with the simplex method a smaller number of constraints than variables is preferred since it results in a smaller dimension of the basis and thereby in the lower computational complexity. Therefore, for the simplex approach it is much better to deal with the dual of (41)–(45) than the original problem. Introducing the dual variables: $\mathbf{u} = (u_1, \dots, u_p)$, $\mathbf{v} = (v_1, \dots, v_m)$ and $\mathbf{t} = (t_\tau)_{\tau \in \Pi}$ corresponding to the constraints (42), (43) and (44), respectively, we get the following dual:

$$\begin{aligned} \text{minimize} \quad & \mathbf{ub} && (46)\\ \text{subject to} \quad & \mathbf{uA} - \mathbf{vR} \geqq \mathbf{0}, && (47)\\ & v_i - \sum_{\tau \in \Pi} w_{\tau(i)} t_\tau = 0 \quad \text{for } i = 1, 2, \dots, m, && (48)\\ & \sum_{\tau \in \Pi} t_\tau = 1, && (49)\\ & t_\tau \geqslant 0 \quad \text{for } \tau \in \Pi. && (50) \end{aligned}$$

The dual problem (46)–(50) has $m!$ columns corresponding to variables t_τ. However, these columns can be handled implicitly with the column generation scheme. Note that each column corresponding to t_τ has the unit coefficient in row (49) and coefficients $-w_{\tau(i)}$ in rows (48). Thus there is no reason to keep them explicitly. We only need to identify the best column during the pricing and to generate the selected column for pivoting.

During the course of the simplex method, having the current basis **B** we have defined the current primal basic solution $(\mathbf{u}^0, \mathbf{v}^0, \mathbf{t}^0)$ and the current dual basic solution (the dual multipliers) $(\mathbf{x}^0, \mathbf{y}^0, z^0)$. The reduced cost for variable t_τ is given by the formula

$$d(t_\tau) = \sum_{i=1}^{m} w_{\tau(i)} y_i^0 - z^0 \quad \text{for } \tau \in \Pi.$$

Due to (36), the solution to the pricing problem $\min_{\tau \in \Pi} d(t_\tau)$ is given by permutation $\bar{\tau}$ such that its inverse $\bar{\tau}^{-1}$ nonincreasingly orders $\mathbf{y}^0$, i.e., $y^0_{\bar{\tau}^{-1}(1)} \leqslant y^0_{\bar{\tau}^{-1}(2)} \leqslant \cdots \leqslant y^0_{\bar{\tau}^{-1}(m)}$ where $\bar{\tau}^{-1}(\bar{\tau}(i)) = i$ for $i = 1, 2, \dots, m$. In the case of all different coefficients in vector $\mathbf{y}^0$, there is unique such permutation $\bar{\tau}$ and the uniquely defined incoming column. When some coefficients are equal, then we get a group of columns where the weights are permuted within the subsets of indices corresponding to equal coefficients y_i^0. We may take then a linear combination of these columns with positive scaling factors totaling to 1 (e.g., all equal). We are permitted to do it as such a combination column corresponds to the combination of inequalities (44) which can be added to the primal without affecting the solution.

We have run initial computational experiments using 1994 data from the Warsaw stock market. Exactly, we analyzed the set of 21 securities. With a rather straightforward implementation of the simplex method with column generation we easily solved problems for m varying from 10 to 20. In all the runs the number of simplex steps did not exceed 500 while in average it was close to 200. Further experiments on various data sets are necessary to justify if the simplex method can be used for medium-scale OWA problems. Certainly, the large-scale portfolio selection problems require another solution technique applied to the corresponding OWA problems or a different aggregation technique applied to the linear multiple criteria model.

6. Conclusions and further research

Following the pioneering work of Sharpe [18], many attempts have been made to linearize the portfolio selection problem. There were introduced several risk measures which lead to linear programming mean-risk models. In this paper we have developed a multiple criteria linear programming model of the portfolio selection problem. The classical linear programming mean-risk approaches turn out to be specific aggregation techniques applied to our multiple criteria model. The model is based on the preference axioms for the choice under risk. Therefore, by looking for various efficient solutions of the multiple criteria linear program, we are able to identify solutions of the portfolio selection problem which are optimal with respect to various risk averse preferences. Nevertheless, the model allows one to employ the variety of standard multiple criteria procedures to analyze the portfolio selection problem.

In the paper we have focused on the classical and widely known weighting approach to multiple criteria optimization. It results in linear programming problems with large number of constraints. However, the medium-size problems can be effectively solved by the simplex method with the column generation technique when applied to their duals. The weighting approach is a fundamental technique in the multiple criteria optimization. Nevertheless, it is not very effective for an interactive decision support (Steuer [20]). Therefore, further research is necessary on possible use of other multiple criteria approaches to the portfolio selection problem.

For the interactive decision support very useful are the so-called aspiration based techniques of multiple criteria optimization (Lewandowski and Wierzbicki [10]) originated from the reference point method (Wierzbicki [23], Steuer [20]). The reference point method, similar to goal programming, uses aspiration levels to define the decision maker preferences (Ogryczak and Lahoda [15]) but it is completely consistent with the rational model of preferences and therefore always generates an efficient solution. The reference point method, when applied to our multiple criteria model, results in an interactive technique of the reference distribution which seems to be a very attractive technique for decision support in the portfolio selection. The optimization problems to be solved for a specific reference distribution are very similar to those considered in the weighting approach.

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